CHAPTER 7

SEMI-INFINITE BIMATRIX GAMES

7.1 INTRODUCTION AND DEFINITIONS

So far we have only considered finite bimatrix games, i.e. bimatrix games where both players have a finite number of pure strategies. For such games Nash (1951) showed that there is always an equilibrium. In general a two-person game for which there is an equilibrium, i.e. a prescribed pair of strategies from which unilateral deviation does not pay, is called determined.

If a game is not determined almost equilibria become important. In this chapter we discuss semi-infinite bimatrix games which are two-person games where one of the players (we will assume this is player 1) has a finite number of pure strategies and the other player (player 2) has a countably infinite pure strategy space. These games may not be determined (cf. example 7.1.1). In Lucchetti et al. (1986) different types of almost equilibria were defined for semi-infinite bimatrix games and they were used to define weak determinateness of these games.

Earlier Wald (1945) showed that every bounded semi-infinite matrix game is weakly determined, Tijs (1975) proved that also unbounded semi-infinite matrix games are weakly determined and weak determinateness of semi-infinite bimatrix games where the second player has an upper bounded payoff matrix was shown in Tijs (1975), Tijs (1977) and Tijs (1981). Lucchetti et al. dealt with semi-infinite bimatrix games with various boundedness restrictions guaranteeing weak determinateness. The problem of
weak determinateness of general semi-infinite bimatrix games is still unsolved.

In this chapter we show that semi-infinite bimatrix games where player 1 has two pure strategies, are weakly determined. This chapter is based on Jurg and Tijs (1993).

First we define general semi-infinite bimatrix games.

Let $A := [a_{ij}]_{i,j=1}^{m,\infty}$ and $B := [b_{ij}]_{i,j=1}^{m,\infty}$ be two real $m \times \infty$ matrices. By $(A, B)$ we denote the semi-infinite bimatrix game consisting of the payoff matrices $A$ and $B$ for player 1 and player 2 respectively and strategy spaces $\Delta_m$ for player 1 and $\Delta_\infty$ for player 2. In a play of this game player 1 chooses a $p \in \Delta_m$ and player 2 chooses a $q \in \Delta_\infty$. Subsequently player 1 obtains the payoff $pAq := \sum_{i=1}^{m} \sum_{j=1}^{\infty} p_ia_{ij}q_j$ and player 2 obtains $pBq := \sum_{i=1}^{m} \sum_{j=1}^{\infty} p_ib_{ij}q_j$.

In the following example we present a semi-infinite bimatrix game which is not determined.

**Example 7.1.1** Consider the $2 \times \infty$ bimatrix game $(A, B)$ where

$$A := \begin{bmatrix} 0 & 1 & 2 & 3 & \ldots \\ 1 & 0 & 0 & 0 & \ldots \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 1 & -1 & -2 & -3 & \ldots \\ 1 & 1 & 2 & 3 & \ldots \end{bmatrix}.$$  

Assume that $(\tilde{p}, \tilde{q}) \in \Delta_m \times \Delta_\infty$ is an equilibrium for $(A, B)$. Then we must have $\tilde{p}A\tilde{q} \geq pA\tilde{q}$ for all $p \in \Delta_m$ and $\tilde{p}B\tilde{q} \geq pB\tilde{q}$ for all $q \in \Delta_\infty$. If $\tilde{p} \in \text{conv} \left\{ (1, \frac{1}{2}, \frac{1}{2}), (\epsilon_2) \right\}$ we can make $\tilde{p}B\tilde{q}$ for $q \in \Delta_\infty$ arbitrary large by letting $q_j = 1$ for $j$ large. So $\tilde{p} \in \text{conv} \left\{ \epsilon_1, (\frac{1}{2}, \frac{1}{2}) \right\}$. Then $\sup_{q \in \Delta_\infty} pBq$ is attained uniquely by $\tilde{q} \in \Delta_\infty$ with $q_1 = 1$. However, then $\epsilon_1 A\tilde{q} < \epsilon_2 A\tilde{q}$, which implies $\tilde{p} = \epsilon_2 \tilde{q} \notin \text{conv} \left\{ \epsilon_1, (\frac{1}{2}, \frac{1}{2}) \right\}$. Since this is a contradiction, $(A, B)$ has no equilibrium. Hence $(A, B)$ is not determined.

From example 7.1.1 we learn that there is a need for the notion of weak determinateness.

First we recall the definitions of the four types of *almost equilibria* for semi-infinite bimatrix games occurring in Luccchetti et al. Let $(A, B)$ be an $m \times \infty$ bimatrix game and $k_1, k_2 \in \mathbb{R}$ and $\epsilon_1, \epsilon_2 \geq 0$. Then $(\tilde{p}, \tilde{q}) \in \Delta_m \times \Delta_\infty$ is called

an $(\epsilon_1, \epsilon_2)$-*equilibrium* if

$$\begin{align*}  
\tilde{p}A\tilde{q} + \epsilon_1 & \geq pA\tilde{q} & \text{for all } p \in \Delta_m \\
\tilde{p}B\tilde{q} + \epsilon_2 & \geq pB\tilde{q} & \text{for all } q \in \Delta_\infty
\end{align*}$$

(7.1.1)

(7.1.2)

an $(\epsilon_1, \epsilon_2)$-*equilibrium* if (7.1.1) holds and
Chapter 7: Semi-infinite bimatrix games 105

\[ \hat{p}B\hat{q} \geq k \]  
(7.1.3)

a \((k_1, k_2)\)-equilibrium if (7.1.2) holds and

\[ \hat{p}A\hat{q} \geq k_1 \]  
(7.1.4)

a \((k_1, k_2)\)-equilibrium if both (7.1.3) and (7.1.4) hold.

By \(E^{\epsilon_1, \epsilon_2}(A, B)\), \(E^{k_1, k_2}(A, B)\), \(E^{k_1, k_2}(A, B)\) and \(E^{k_1, k_2}(A, B)\) we denote the sets of \((\epsilon_1, \epsilon_2), (k_1, k_2), (k_1, k_2)\) and \((k_1, k_2)\) respectively.

The game \((A, B)\) is called weakly determined weak determined game if it has at least one of the following properties:

\begin{align*}
(WD1) &\quad E^{\epsilon_1, \epsilon_2}(A, B) \neq \emptyset \text{ for all } \epsilon_1, \epsilon_2 > 0 \\
(WD2) &\quad E^{k_1, k_2}(A, B) \neq \emptyset \text{ for all } k_1 \in \mathbb{R} \text{ and } \epsilon_2 > 0 \\
(WD3) &\quad E^{k_1, k_2}(A, B) \neq \emptyset \text{ for all } \epsilon_1 > 0 \text{ and } k_2 \in \mathbb{R}. \\
(WD4) &\quad E^{k_1, k_2}(A, B) \neq \emptyset \text{ for all } k_1, k_2 \in \mathbb{R}.
\end{align*}

Since the equilibrium conditions are equal to (7.1.1) and (7.1.2) with \(\epsilon_1 = \epsilon_2 = 0\), determinateness of \((A, B)\) implies (WD1).

The interpretation of an \((\epsilon_1, \epsilon_2)\)-equilibrium \((\hat{p}, \hat{q})\) for a semi-infinite bimatrix game is clear: if \(\epsilon_1\) and \(\epsilon_2\) are close to zero, then \((\hat{p}, \hat{q})\) is ‘almost’ an equilibrium, i.e. by unilateral deviation player 1 and player 2 can at most gain \(\epsilon_1\) and \(\epsilon_2\) respectively. This explains (WD1).

It only makes sense to look for an \((\epsilon_1, k_2)\)-equilibrium for an \(m \times \infty\) bimatrix game \((A, B)\) if \(B\) is not upper bounded. Here an \(m \times \infty\) matrix \(D := [d_{ij}]_{i=1}^{\infty} \) is called upper bounded (lower bounded) if there exists a \(k \in \mathbb{R}\) such that \(d_{ij} \leq k\) (\(d_{ij} \geq k\)) for all \(i \in M\) and \(j \in N\). If \(B\) is upper bounded, then clearly (WD3) cannot hold. If \(B\) is not upper bounded, then (WD3) means that player 2 can guarantee himself any payoff by choosing the an appropriate strategy, while player 1 is always able to play an almost (up to \(\epsilon_1\)) best reply to the strategy of player 2.

The interpretations of (WD2) and (WD4) are now obvious.

In the following example we show that the semi-infinite bimatrix game of example 7.1.1 is weakly determined.

**Example 7.1.2** Consider the \(2 \times \infty\) bimatrix game of example 7.1.1. We show that this game has the property (WD1). Let \(\epsilon_1, \epsilon_2 > 0\) and take \(\ell \in \mathbb{N}\) such that \(\ell \geq \frac{1}{\epsilon_2}\). Define \(\hat{p} := \left(\frac{1}{2}, \frac{1}{2}\right) \in \Delta_2\) and \(\hat{q} \in \Delta_\infty\) such that \(\hat{q}_1 = \frac{\ell - 1}{\ell}\) and \(\hat{q}_\ell = \frac{1}{\ell}\). Then \(\epsilon_1 \hat{A}\hat{q} = \epsilon_2 A\hat{q}\) so that \((\hat{p}, \hat{q})\) satisfies (7.1.1). Moreover \(\hat{p}B\hat{q} = \frac{\ell - 1}{\ell} \geq 1 - \epsilon_2 \geq \sup_{\hat{q} \in \Delta_\infty} \hat{p}B\hat{q} - \epsilon_2\).
Hence \((\hat{p}, \hat{q})\) satisfies (7.1.2).

The following theorem summarizes some results of Lucchetti et al.

**Theorem 7.1.1** Let \((A, B)\) be an \(m \times \infty\) bimatrix game. Then \((A, B)\) is weakly determined if one of the following assertions hold:

(i) \(B\) is upper bounded

(ii) both \(A\) and \(B\) are lower bounded

(iii) \(B\) is lower bounded and \(A\) is not upper bounded

(iv) \(m = 2\) and \(B\) is upper or lower bounded.

The \(2 \times \infty\) game of example 7.1.1 does not satisfy any of the conditions of theorem 7.1.1, since the matrix \(B\) is neither upper nor lower bounded. Nevertheless this game has the property (WD1) (cf. example 7.1.2).

In section 7.3 we prove that every \(2 \times \infty\) bimatrix game either has the property (WD1) or the property (WD3). In order to show this we use a labeling method which is typical for \(2 \times \infty\) bimatrix game. Other tools are the results in section 7.2 which concern an arbitrary convex set and nonnegative directions in \(\mathbb{R}^m\). We conclude this chapter with some remarks in section 7.4.

In view of the main result in this chapter we may entertain in a hope that every semi-infinite bimatrix game is determined. With respect to infinite bimatrix games, i.e. games where both players have countably infinitely many pure strategies, such a hope is not justified, since in Wald (1945) an example of an infinite matrix game is given which is not determined. For the convenience of the reader we discuss this example below.

**Example 7.1.3** Let \(A := [a_{ij}]_{i,j=1}^{\infty} \) be defined by

\[
    a_{ij} := \begin{cases} 
    1 & \text{if } i > j \\
    0 & \text{if } i = j \\
    -1 & \text{if } i < j.
    \end{cases}
\]

We consider the game \((A, -A)\) where both players have the strategy space \(\Delta_\infty\). Clearly for infinite bimatrix games properties similar to (WD1)-(WD4) can be defined and also upper and lower boundedness of the payoff matrices. Then the matrix \(A\) is both upper and lower bounded, so that we can only hope for (WD1) to hold for this game. Then if \((\hat{p}, \hat{q})\) is an \((\epsilon_1, \epsilon_2)\)-equilibrium for some \(\epsilon_1, \epsilon_2 \geq 0\), we must have \(\hat{p}A\hat{q} + \epsilon_1 \geq \sup_{\delta \in \Delta_\infty} \hat{p}A\delta = 1\) and \(\hat{p}A\hat{q} - \epsilon_2 \leq \inf_{\delta \in \Delta_\infty} \hat{p}A\delta = -1\). So that \(1 - \epsilon_1 \leq \hat{p}A\hat{q} \leq -1 + \epsilon_2\). This is not possible for \(\epsilon_1\) and \(\epsilon_2\) zero or close to zero.
7.2 TOOLS

Let $C$ be a convex set in $\mathbb{R}^m$. In this section we interpret $\Delta_m$ as the set of all (normalized) nonnegative directions in $\mathbb{R}^m$. We say that $C$ is upper bounded in the direction $p \in \Delta_m$ if $\sup_{x \in C} p \cdot x < \infty$. The set of all nonnegative directions in which $C$ is upper bounded is denoted by $UB(C)$, and its complement, $\Delta_m \setminus UB(C)$, by $NUB(C)$. Note that $UB(C)$ is convex.

Let $p \in UB(C)$ and $\epsilon \geq 0$. An element $\pi$ of $C$ is called $\epsilon$-optimal in direction $p$ if $p \cdot \pi + \epsilon \geq \sup_{x \in C} p \cdot x$. The set of all elements of $C$ that are $\epsilon$-optimal in direction $p$ is non-empty and convex. We denote it by $O_{p, \epsilon}$.

Now let $p \in NUB(C)$ and $k \in \mathbb{R}$. An element $\pi$ of $C$ is called $k$-guaranteeing in direction $p$ if $p \cdot \pi \geq k$. The set of all elements of $C$ that are $k$-guaranteeing in direction $p$ is denoted by $G_{p, k}$. Also this set is non-empty and convex.

Note that for $p \in UB(C)$ and $\epsilon_1 \geq \epsilon_2 \geq 0$ we have that $O_{p, \epsilon_1} \supseteq O_{p, \epsilon_2}$ and for $p \in NUB(C)$ and $k_1, k_2 \in \mathbb{R}$ with $k_1 \geq k_2$ we have that $G_{p, k_1} \subseteq G_{p, k_2}$.

We investigate these notions in the following example.

**Example 7.2.1** We let $B$ be as in example 7.1.1, and let $C$ be the set of all convex combinations of finitely many columns of $B$, i.e. $C = \{Bq \mid q \in \Delta_\infty\}$. Then $C \subseteq \mathbb{R}^2$ is the grey area in figure 7.2.1. We have $UB(C) = \text{conv} (\{(\frac{1}{2}, 0), \epsilon_2\})$ and $NUB(C) = \text{conv} (\{(\frac{1}{2}, 0), \epsilon_2\}) \setminus \{(\frac{1}{2}, 0)\}$.
In figure 7.2.1, $\hat{p} \in UB(C)$ and $O_{\hat{p}, \hat{\epsilon}}$ is the vertically shaded part of $C$. Further $G_{\epsilon, k}$ is the horizontally shaded part of $C$. Note that $O_{\hat{p}, \hat{\epsilon}} \cap G_{\epsilon, k} = \emptyset$ for large $k$. One easily checks that $O_{\hat{p}, \hat{\epsilon}} \cap G_{\epsilon, k} \neq \emptyset$ for all $\epsilon > 0$ and $k \in \mathbb{R}$.

Lemma 7.2.2 below generalizes some of the observations made in the example above. For the proof of lemma 7.2.2 we need the following well-known lemma.

**Lemma 7.2.1** Let $C$ be a convex set in $\mathbb{R}^m$. Let $n^1, \ldots, n^s \in \mathbb{R}^m$ and $a^1, \ldots, a^s \in \mathbb{R}$. Define $\mathcal{P} := \cap_{i=1}^s H(n^i, a^i)$.

Suppose $\mathcal{P} \neq \emptyset$ and $C \cap \mathcal{P} = \emptyset$. Then there is an $n \in \text{conv} \{n^1, \ldots, n^s\}$ such that $x \cdot n \leq y \cdot n$ for all $x \in \mathcal{P}$ and $y \in C$.

**Proof.** From a well-known separation theorem for convex sets we obtain the existence of an $n \in \mathbb{R}^m \setminus \{0\}$ such that $x \cdot n \leq y \cdot n$ for all $x \in \mathcal{P}$ and $y \in C$. We have to show that there is a $\lambda \in \mathbb{R}^s$, $\lambda \geq 0$, such that $\sum_{k=1}^s \lambda_k n^k = n$. Suppose such a $\lambda$ does not exist. We show a contradiction. Then Farkas' lemma (lemma 1.4.1) implies the existence of a $x^0 \in \mathbb{R}^m$ such that $x^0 \cdot n > 0$ and $x^0 \cdot n^k \leq 0$ for all $k$.

Now take $x \in \mathcal{P}$. Then clearly $x + \mu x^0 \in \mathcal{P}$ for any $\mu > 0$. Take $y \in C$. Since $\mathcal{P} \subset H(n, y \cdot n)$, we have that $x + \mu x^0 \in H(n, y \cdot n)$ for any $\mu > 0$ and consequently $y \cdot n \geq (x + \mu x^0) \cdot n = x \cdot n + \mu x^0 \cdot n$. Since $x^0 \cdot n > 0$, this is a contradiction for large $\mu$. \[\square\]
Lemma 7.2.2 Let $C$ be a convex set in $\mathbb{R}^m$ and suppose $UB(C)$ is non-empty. Then for every $p \in NUB(C)$, $k \in \mathbb{R}$ and $\epsilon > 0$ there exists a $p' \in UB(C)$ such that $G_{p,k} \cap O_{p',\epsilon} \neq \emptyset$.

Proof. Take $p \in NUB(C)$, $k \in \mathbb{R}$ and $\epsilon > 0$. It suffices to show that $G_{p,k} \cap (\bigcup_{p' \in UB(C)} O_{p',\epsilon}) \neq \emptyset$. For a moment assume that this intersection is empty and take $x^0 \in G_{p,k}$. Then we show a contradiction in two steps.

(a) First we show that we can find an $x^1 \in C$ such that $x^1 \geq x^0 + \epsilon 1_m$. Assume such an $x^1$ does not exist. Then $C \cap \{y \in \mathbb{R}^m \mid y \geq x^0 + \epsilon 1_m\} = \emptyset$. Applying Lemma 7.2.1 we find that a $p \in \text{conv} \{e_1, \ldots, e_m\} = \Delta_m$ exists such that $p \cdot x \leq p \cdot y$ for all $x \in C$ and all $y \in \mathbb{R}^m$ satisfying $y \geq x^0 + \epsilon 1_m$. In particular this implies $p \cdot x \leq p \cdot x^0 + \epsilon$ for all $x \in C$. Hence $p \in UB(C)$ and $x^0 \in O_{p,\epsilon}$. However, this contradicts the assumption of the empty intersection.

(b) By part (a) we can find an $x^1 \in C$ such that $x^1 \geq x^0 + \epsilon 1_m$. This implies $p \cdot x^1 \geq p \cdot x^0 + \epsilon$ and consequently $x^1 \in G_{p,k}$. Then, by repetition of the arguments above, we can construct a sequence $x^0, x^1, x^2, \ldots$ of elements of $G_{p,k}$ such that $x^j \geq x^{j-1} + \epsilon 1_m \geq x^0 + n\epsilon 1_m$ for $n \in \mathbb{N}$. Take $p \in UB(C)$. Then for large $n$ we have $p \cdot x^n \geq p \cdot x^0 + n\epsilon > \sup_{x \in C} p \cdot x$, which is a contradiction.

Roughly speaking Lemma 7.2.2 relates for a convex set $C$ in $\mathbb{R}^m$ the sets $UB(C)$ and $NUB(C)$. For our purposes (cf. Section 4) this relation suffices. Now we focus on $UB(C)$. First we give a definition.

Let $C$ be a convex set in $\mathbb{R}^m$ and let $\epsilon > 0$. A subset $S$ of $C$ is said to $\epsilon$-dominate $C$ if for every $x \in C$ there is a $y \in S$ such that $y \geq x - \epsilon 1_m$.

Note that if $UB(C) = \emptyset$, there is no such subset for any $\epsilon > 0$.

Lemma 7.2.3 [Tijss 1977] and [Tijss 1981] Let $C$ be a convex set in $\mathbb{R}^m$ and suppose $\Delta_m = UB(C)$. Then for every $\epsilon > 0$ there exists a finite subset $S$ of $C$ such that $S$ $\epsilon$-dominates $C$.

Lemma 7.2.4 Let $C$ be a convex set in $\mathbb{R}^m$ and suppose $UB(C) \neq \emptyset$. Let $P := \text{conv} \{p^1, \ldots, p^k\}$ with $p^1, \ldots, p^k \in UB(C)$. Then for every $\epsilon > 0$ there is a finite subset $S$ of $C$ such that $S$ $\epsilon$-dominates $C$.

Proof. Let $\epsilon > 0$ and $\overline{C} := \{(p^1 \cdot x, p^2 \cdot x, \ldots, p^k \cdot x) \mid x \in C\}$. Then $\overline{C}$ is a convex set in $\mathbb{R}^\ell$ and since for any $\overline{p} \in \Delta_{\ell}$ we have $\sup_{x \in C} \overline{p} \cdot x = \sup_{x \in C} \sum_{i=1}^{k} \overline{p}_i (p^i \cdot x) < \infty$, we have $UB(\overline{C}) = \Delta_{\ell}$. Applying Lemma 7.2.3 to $\overline{C}$ and $\frac{1}{2} \epsilon$ we obtain a finite subset $\overline{S}$ of $\overline{C}$ such that $\overline{S} \frac{1}{2} \epsilon$-dominates $\overline{C}$. Let $S$ be a finite subset of $C$ such that $\overline{S} = \{(p^1 \cdot x, p^2 \cdot x, \ldots, p^k \cdot x) \mid x \in S\}$. Take an arbitrary $p \in P$ and $\hat{x} \in O_{p,\frac{1}{2} \epsilon}$. Then there is an $\overline{s} \in \overline{S}$ such that $\overline{s} \geq (p^1 \cdot \hat{x}, p^2 \cdot \hat{x}, \ldots, p^k \cdot \hat{x}) - \frac{1}{2} \epsilon 1_{\ell}$, or equivalently, there is an $s \in S$ such that $p^i \cdot s \geq p^i \cdot \hat{x} - \frac{1}{2} \epsilon$ for all $i \in \{1, \ldots, \ell\}$. Then also $p \cdot s \geq p \cdot \hat{x} - \frac{1}{2} \epsilon$. 


7.3 DETERMINATENESS OF $2 \times \infty$ BIMATRIX GAMES

In this section $(A, B)$ is a $2 \times \infty$ bimatrix game and $C$ is the set of all convex combinations of finitely many columns of $B$, i.e. $C = \{ B_q \mid q \in \Delta_\infty \} \subset \mathbb{R}^2$.

We define a labeling function $\lambda : \Delta_2 \rightarrow \{ 0, 1, 2 \}$ as follows: For $p \in UB(C)$:

$$
\lambda(p) = \begin{cases} 
1 & \text{if there is an } \epsilon > 0 \text{ such that } e_1 A_q > e_2 A_q \text{ for all } q \in \Delta_\infty \\
& \text{with } B_q \in O_p, \epsilon \\
2 & \text{if there is an } \epsilon > 0 \text{ such that } e_1 A_q < e_2 A_q \text{ for all } q \in \Delta_\infty \\
& \text{with } B_q \in O_p, \epsilon \\
0 & \text{else.}
\end{cases}
$$

and for $p \in NUB(C)$:

$$
\lambda(p) = \begin{cases} 
1 & \text{if there is a } k \in \mathbb{R} \text{ such that } e_1 A_q > e_2 A_q \text{ for all } q \in \Delta_\infty \\
& \text{with } B_q \in G_p, k \\
2 & \text{if there is a } k \in \mathbb{R} \text{ such that } e_1 A_q < e_2 A_q \text{ for all } q \in \Delta_\infty \\
& \text{with } B_q \in G_p, k \\
0 & \text{else.}
\end{cases}
$$

Accordingly we call $(A, B)$ 0-determined if there is a $p \in \Delta_2$ with $\lambda(p) = 0$, 1-determined if $\lambda(p) = 1$ for all $p \in \Delta_2$, 2-determined if $\lambda(p) = 2$ for all $p \in \Delta_2$ and quasi 0-determined if $UB(C) \neq \emptyset$ and for every $\epsilon > 0$ there are a $p \in UB(C)$ and a $q \in \Delta_\infty$ such that $B_q \in O_p, \epsilon$ and $e_1 A_q = e_2 A_q$.

It is the aim of this section to show that every $2 \times \infty$ bimatrix game is 0-, 1-, 2- or quasi 0-determined. This ensures determinateness since

**Lemma 7.3.1** For a $2 \times \infty$ bimatrix game 0-, 1-, 2- and quasi 0-determinateness imply (WD1) or (WD3).

**Proof.** Let $(A, B)$ be a $2 \times \infty$ bimatrix game.

(a) Let $(A, B)$ be 0-determined. Then there is a $p \in \Delta_2$ such that $\lambda(p) = 0$. Suppose $p \in UB(C)$. Then for every $e_2 > 0$ there either is a $q \in \Delta_\infty$ such that $B_q \in O_p, e_2$ and $e_1 A_q = e_2 A_q$ or there are $q^1, q^2 \in \Delta_\infty$ such that $B_q^1, B_q^2 \in O_p, e_2$, $e_1 A_q^1 > e_2 A_q^1$ and $e_1 A_q^2 < e_2 A_q^2$. In the latter case let

$$
\lambda := \frac{e_1 A_q^1 - e_2 A_q^1}{e_1 A_q^1 - e_2 A_q^1 + e_2 A_q^2 - e_1 A_q^2} \in \Delta_2
$$

Since $x \in O_{p, \epsilon}$, this implies $p \cdot s \geq \sup_{x \in C} p \cdot x - \frac{1}{\epsilon} \epsilon - \frac{1}{\epsilon} \epsilon = \sup_{x \in C} p \cdot x - \epsilon$. Hence $s \in O_{p, \epsilon}$. This completes the proof. \qed
and define \( \tilde{q} := (1 - \lambda)q^1 + \lambda q^2 \). Then \( \tilde{q} \in \text{conv} \{ q^1, q^2 \} \) and \( e_1 A \tilde{q} = e_2 A \tilde{q} \). Since \( O_{p,\epsilon_2} \) is convex, also \( B \tilde{q} \in O_{p,\epsilon_2} \). So the latter case implies the first one. Hence for every \( \epsilon_2 > 0 \) we can find a \( q \in \Delta \infty \) such that \( Bq \in O_{p,\epsilon_2} \), i.e. \( pBq \geq \sup_{q \in \Delta \infty} pB y - \epsilon_2 \) and \( e_1 Aq = e_2 Aq \). The last equality implies that for every \( \epsilon_1 > 0 \) we have that \( pAq > \sup_{q \in \Delta \infty} xAQ - \epsilon_1 \). Hence \((A, B)\) has the property (WD1). Similarly one shows that \((A, B)\) has the property (WD3) if \( p \in NU B(C) \).

(b) Let \((A, B)\) be 1-determined. Then \( \lambda(\epsilon_1) = 1 \). Suppose \( \epsilon_1 \in NU B(C) \). Then there is a \( \bar{k} \in I R \) such that \( e_1Aq > e_2Aq \) for all \( q \in \Delta \infty \) with \( Bq \in G_{\epsilon_1, \bar{k}} \). Let \( \epsilon_1 > 0 \) and \( \epsilon_2 \in IR \). If \( \epsilon_2 > \bar{k} \), then take \( q \in \Delta \infty \) such that \( Bq \in G_{\epsilon_1, \epsilon_2} \). If \( \epsilon_2 < \bar{k} \), then take \( q \in \Delta \infty \) such that \( Bq \in G_{\epsilon_1, \bar{k}} \). In both cases \( e_1Aq > e_2Aq \), and hence \( e_1Aq > \sup_{q \in \Delta \infty} xAQ - \epsilon_1 \). Moreover, since also in both cases \( Bq \in G_{\epsilon_1, \epsilon_2} \) we have \( e_1 Bq \geq k_2 \). Hence \((\epsilon_1, q)\) is an \((\epsilon_1, k_2)\)-equilibrium in both cases. Since \( \epsilon_1 > 0 \) and \( \epsilon_2 \in IR \) were arbitrary, \((A, B)\) has the property (WD3). Similarly one shows that \((A, B)\) has the property (WD1) if \( \epsilon_1 \in UB(C) \).

(c) Similar to (b) one shows that 2-determinateness implies (WD1) \((\epsilon_2 \in UB(C))\) or (WD3) \((\epsilon_2 \in NU B(C))\).

(d) Let \((A, B)\) be quasi 0-determined. Let \( \epsilon_1, \epsilon_2 > 0 \). By definition \( UB(C) \neq \emptyset \) and we can find a pair \( (p, q) \in \Delta_2 \times \Delta_2 \) such that \( p \in UB(C) \), \( Bq \in O_{p,\epsilon_2} \) and \( e_1 Aq = e_2 Aq \). This implies that \( (p, q) \) is an \((\epsilon_1, \epsilon_2)\)-equilibrium. Hence quasi 0-determinateness implies (WD1).

In order to prove the final result of this section, we need two more lemmas.

**Lemma 7.3.2** Let \((A, B)\) be a \( 2 \times \infty \) bimatrix game and let \( C := \{ Bq \mid q \in \Delta_\infty \} \). Suppose \( UB(C) = \emptyset \). Then \((A, B)\) is either 1- or 2-determined if \((A, B)\) is not 0-determined.

**Proof.** Let \((A, B)\) not be 0-determined. Then \( \lambda(p) \in \{1, 2\} \) for all \( p \in \Delta_2 = NU B(C) \). Suppose \( p^1, p^2 \in \Delta_2 \) exist such that \( \lambda(p^1) = 1 \) and \( \lambda(p^2) = 2 \). Then \( k_1, k_2 \in IR \) exist such that \( e_1 Aq^1 > e_2 Aq^1 \) for all \( q^1 \in \Delta \infty \) with \( Bq^1 \in G_{p^1, k_1} \) and \( e_1 Aq^2 < e_2 Aq^2 \) for all \( q^2 \in \Delta \infty \) with \( Bq^2 \in G_{p^2, k_2} \). This implies \( \lambda = \{ x \in IR^2 \mid p^1 \cdot x \geq k_1 \) and \( p^2 \cdot x \geq k_2 \} \). By applying lemma 7.2.1 we obtain a \( p \in \text{conv} \{ p^1, p^2 \} \) such that \( px \leq py \) for all \( x \in C \) and all \( y \in IR^m \) satisfying \( p^1 y \geq k_1 \) and \( p^2 y \geq k_2 \). Let \( y^0 \) be the solution of the system of equations \( p^1 y^0 = k_1 \) and \( p^2 y^0 = k_2 \). Then \( px \leq py^0 \) for all \( x \in C \). Hence \( \sup_{x \in C} px < \infty \). This contradicts \( UB(C) = \emptyset \). So either \( \lambda(p) = 1 \) for all \( p \in \Delta_2 \) or \( \lambda(p) = 2 \) for all \( p \in \Delta_2 \), which proves the lemma.

**Lemma 7.3.3** Let \((A, B)\) be a \( 2 \times \infty \) bimatrix game and let \( C := \{ Bq \mid q \in \Delta_\infty \} \). Suppose \( UB(C) \neq \emptyset \). Then \( \lambda \) is constant on \( UB(C) \), if \((A, B)\) is neither 0- nor quasi 0-determined.

**Proof.** Let \((A, B)\) be neither 0- nor quasi 0-determined. Then there exists an \( \epsilon_0 > 0 \) such that for every \( p \in UB(C) \), \( \lambda(p) = 1 \) implies \( e_1 Aq > e_2 Aq \) for all \( q \in \Delta \infty \) with
$Bq \in O_{p,t_0}$ and $\lambda(p) = 2$ implies $\epsilon_1 Aq < \epsilon_2 Aq$ for all $q \in \Delta_\infty$ with $Bq \in O_{p,t_0}$.

Take $p^1, p^2 \in UB(C)$. Let $p(1), p(2), \ldots$ be a sequence of elements in $\text{conv } \{p^1, p^2\}$ with $\lambda(p(n)) = 1$ for all $n \in \mathbb{N}$, which converges to $p$. Clearly $p \in \text{conv } \{\{p^1, p^2\}\}$. We show that $\lambda(p) = 1$. By lemma 7.2.4 there is a finite subset $S$ of $C$ such that $S \cap O_{p(\overline{t_0})} \neq \emptyset$ for all $p \in \text{conv } \{p^1, p^2\}$. So for each $n \in \mathbb{N}$ we can take $q(n) \in \Delta_\infty$ such that $Bq(n) \in S \cap O_{p(n),\frac{1}{2}t_0}$. Consequently $\epsilon_1 Aq(n) > \epsilon_2 Aq(n)$. Since $S$ is a finite set we may assume, without loss of generality that $q(n) = q$ for all $n$. Take $q^0 \in \Delta_\infty$ such that $Bq^0 \in O_{p,\frac{1}{2}t_0}$. Then for each $n \in \mathbb{N}$ we have $p(n)Bq^0 \geq \sup_{x \in C} p(n) \cdot x -\frac{1}{2}t_0 \geq -\frac{1}{2}t_0$. This implies that $pBq^0 \geq pBq^0 -\frac{1}{2}t_0 \geq \sup_{x \in C} p \cdot x -\frac{1}{2}t_0 -\frac{1}{2}t_0$, so that $Bq^0 \in O_{p,\frac{1}{2}t_0}$. Since $\epsilon_1 Aq > \epsilon_2 Aq$, this implies $\epsilon_1 Aq > \epsilon_2 Aq$ for all $q \in \Delta_\infty$ with $Bq^0 \in O_{p,\frac{1}{2}t_0}$. Hence $\lambda(p) = 1$. So the set of all $p \in \text{conv } \{p^1, p^2\}$ having $\lambda(p) = 1$ is closed. Similarly the set of $p \in \text{conv } \{p^1, p^2\}$ having $\lambda(p) = 2$ is closed. Since $(A, B)$ is not 0-determined, $\text{conv } \{p^1, p^2\}$ is the union of these two sets. But then $\lambda$ is constant on $\text{conv } \{p^1, p^2\}$. Since $p^1, p^2 \in UB(C)$ were taken arbitrary, we can conclude that $\lambda$ is constant on $UB(C)$.

We are now in a position to prove

**Theorem 7.3.4** Every $2 \times \infty$ bimatrix game is $\emptyset$, $1$-, $2$- or quasi $\emptyset$-determined.

**Proof.** Let $(A, B)$ be a $2 \times \infty$ bimatrix game and let $C := \{Bq \mid q \in \Delta_\infty\}$. Suppose $(A, B)$ is neither $\emptyset$- nor quasi 0-determined. We show that $(A, B)$ is either 1- or 2-determined. In view of lemmas 7.3.2 and 7.3.3 we may suppose $UB(C) \neq \emptyset$ and $NUB(C) \neq \emptyset$. By lemma 7.3.3 $\lambda$ is constant on $UB(C)$ and since $(A, B)$ is not 0-determined, we can assume without loss of generality that $\lambda(p) = 1$ for all $p \in UB(C)$. Since $(A, B)$ is not quasi 0-determined we can find an $e_0 > 0$ such that for all $p \in UB(C)$ and $q \in \Delta_\infty$ with $Bq \in O_{p,t_0}$ we have $\epsilon_1 Aq > \epsilon_2 Aq$. Now take $p \in NUB(C)$ and $k \in \mathbb{R}$. Then, by lemma 7.2.2, there is a $p^1 \in UB(C)$ such that $G_{p,k} \cap O_{p^1,t_0} \neq \emptyset$. Take $q \in \Delta_\infty$ such that $Bq \in G_{p,k} \cap O_{p^1,t_0}$. Since $\epsilon_1 Aq > \epsilon_2 Aq$ and since $(A, B)$ is not 0-determined, this implies that $\lambda(p) = 1$. So we have showed that $\lambda(p) = 1$ for all $p \in \Delta_\infty$ and hence that $(A, B)$ is 1-determined. Similarly one shows that $(A, B)$ is 2-determined if $\lambda(p) = 2$ for all $p \in UB(C)$.

As we have mentioned before we have in view of lemma 7.3.1

**Corollary 7.3.5** Every $2 \times \infty$ bimatrix game has the property (WD1) or (WD3).

### 7.4 Remarks

Carefully looking at the proof of lemma 7.3.1 one will find that a $2 \times \infty$ bimatrix
game actually has stronger properties than (WD1) or (WD3), namely for every $2 \times \infty$ bimatrix game we have:

(i) there is a $(0, \varepsilon)$-equilibrium for every $\varepsilon > 0$

or

(ii) there is a $(0, k)$-equilibrium for every $k \in \mathbb{R}$.

For $m \times \infty$ bimatrix games where $m \geq 3$, our labeling method fails. The actual problem is the following: in the proof of proposition 7.6 we used that, if there are $q^1, q^2 \in \Delta_\infty$ such that $\epsilon_1 A q^1 > \epsilon_2 A q^1$ and $\epsilon_1 A q^2 < \epsilon_2 A q^2$, then there is a $q \in \text{conv} \{q^1, q^2\}$ such that $\epsilon_1 A q = \epsilon_2 A q$. This property fails to hold already for $m = 3$.

Let $(A, B)$ be a $3 \times \infty$ bimatrix game such that the first column of $A$ is

\[
\begin{pmatrix}
4 \\
0 \\
3
\end{pmatrix}
\]

and

the second is

\[
\begin{pmatrix}
0 \\
4 \\
3
\end{pmatrix}
\].

Let $q^1 = (1, 0, 0, \ldots)$ and $q^2 = (0, 1, 0, \ldots)$. Then $\epsilon_1 A q^1 > \epsilon_3 A q^1$ and $\epsilon_2 A q^2 > \epsilon_3 A q^2 > \epsilon_1 A q^2$, but there is no $q \in \text{conv} \{q^1, q^2\}$ with $\epsilon_1 A q^2 = \epsilon_2 A q^2 \geq \epsilon_3 A q^2$. A similar problem occurred in Borm et al. (1989).

Finally we remark that the results stated in this chapter also hold for mixed extensions of two-person games $\langle X, Y, K_1, K_2 \rangle$, where $|X| = 2$, $Y$ is arbitrary and $K_1, K_2 := X \times Y \to \mathbb{R}$ are the payoff functions to player 1 and 2 respectively.