CHAPTER 6

A SYMMETRIZATION METHOD FOR

BIMATRIX GAMES

6.1 INTRODUCTION

This section is devoted to a symmetrization method for bimatrix games, which originates from a method for matrix games by Gale Kuhn and Tucker (1950).

Already in the fundamental paper of von Neumann (1928) attention is paid to symmetric matrix games and it is observed that these games have value zero. Brown and von Neumann (1950) gave a new proof of existence of a value for symmetric matrix games using a differential equation. By referring to a symmetrization method of Gale, Kuhn and Tucker (1950) they showed that their proof also applies to non-symmetric matrix games.

In a symmetric game both players have the same strategic possibilities and there is no discrimination in the payoffs. Therefore a symmetrization of a game is an extension to fair play. In fact, where in real-life situations people play games, they tend to symmetrize by means of tossing, exchanging roles in a second play, etc. This motivates the study of symmetrizations for finite two-person games that are not zero-sum. Furthermore symmetric games play an important role in the new field of sociobiology, founded by Maynard Smith (1982), where evolutionary stable strategies correspond to special symmetric equilibria.

Griesmer, Hoffman and Robinson (1963) propose a symmetrization method for bi-
matrix games. This method and a method of Brown and von Neumann (1950) are extensively studied in a paper of Jansen, Potters and Tijs (1986). Correspondences between equilibria and some refinements of equilibria for a bimatrix game and for its symmetrization are given with respect to either type. In this paper we deal with a similar correspondence, but now concerning the symmetrization method of Gale, Kuhn and Tucker. In section 6.2 we extend this method to the case of bimatrix games and we give a correspondence with respect to equilibria. In section 6.3 we test three types of refinements, i.e. quasi-strong, regular and perfect equilibria with respect to their behavior under the symmetrization. The results in this chapter are based upon Jurg et al. (1991).

The method described in this paper yields a one-to-one correspondence between pairs of equilibria for a game and pairs of equilibria for its symmetrization. A similar statement holds for each of the three refinements discussed here. Jansen, Potters and Tijs showed that such a nice correspondence does not exist for the method of Griesmer, Hoffman and Robinson. However, the method of Brown and von Neumann implies a similar one-to-one correspondence. But in this case the ‘size’ of the symmetrization of a game with $m$ and $n$ pure strategies for the players, respectively, is large. In this symmetrization both players have $m \cdot n$ pure strategies. For the symmetrization of Gale, Kuhn and Tucker this number is $m + n + 1$ (and $m + n$ for the symmetrization of Griesmer, Hoffman and Robinson). This makes the symmetrization that is discussed here more interesting for computational purposes.

### 6.2 The Gale, Kuhn and Tucker Symmetrization Method for Bimatrix Games

In this section we extend the method of Gale, Kuhn and Tucker for symmetrizing a matrix game to the case of a bimatrix game. We consider an $m \times n$ bimatrix game $(A, B)$ such that $A > 0$ and $B < 0$. In view of proposition 1.2.2 this is not a restriction. We will call the symmetric bimatrix game $(C, C^t)$, where $C$ is the $(m + n + 1) \times (m + n + 1)$ matrix

$$
C = \begin{bmatrix}
0 & A & -1_m \\
B^t & 0 & 1_n \\
1_m & -1_n & 0
\end{bmatrix},
$$

the *Gale, Kuhn and Tucker symmetrization of $(A, B)$*, in short *GKT-symmetrization*.

In order to show that there is a one-to-one correspondence between pairs of equilibria for the game $(A, B)$ and pairs of equilibria for the game $(C, C^t)$, we first describe how an equilibrium for $(C, C^t)$ yields two equilibria for $(A, B)$. Therefore we need a lemma and some notation.
For a strategy \( \tau \in \Delta_{m+n+1} \) we let \( \tau_x := (\tau_1, \ldots, \tau_m) \), \( \tau_y := (\tau_{m+1}, \ldots, \tau_{m+n}) \) and \( \tau_z := \tau_{m+n+1} \). Then \( \tau = (\tau_x, \tau_y, \tau_z) \).

**Lemma 6.2.1** Let \((C, C^t)\) be the GKT-symmetrization of an \(m \times n\) bimatrix game \((A, B)\) with \(A > 0\) and \(B < 0\). Let \((\rho, \sigma) \in \mathbf{E}(C, C^t)\). Then \(\rho_x \neq 0\), \(\rho_y \neq 0\), \(\rho_z \neq 0\), \(\sigma_x \neq 0\), \(\sigma_y \neq 0\) and \(\sigma_z \neq 0\).

**Proof.** We prove the lemma by following the scheme below.

\[
\begin{align*}
\rho_x \neq 0 & \quad \implies [\sigma_y \neq 0] \quad \implies [\rho_z \neq 0] \\
\implies \quad \implies [\rho_y \neq 0] \quad \implies [\sigma_x \neq 0]
\end{align*}
\]

Note that, since \(\rho, \sigma \in \Delta_{m+n+1}\), we already have \(\rho \geq 0\), \(\sigma \geq 0\) and \(\rho \neq 0\), \(\sigma \neq 0\).

We start at the upper left corner of the scheme: (a) Assume \(\rho_x \neq 0\). Then there is an \(i_0 \in \{1, \ldots, m\}\) such that

\[
\epsilon_{i_0}C\sigma = \epsilon_{i_0}A\sigma_y - \sigma_z \geq \max_{i \in \{1, \ldots, m\}} \epsilon_i A\sigma_y - \sigma_z
\]

Suppose \(\sigma_y = 0\). Then \(\epsilon_{i_0}C\sigma = \epsilon_{i_0}A\sigma_y - \sigma_z = -\sigma_z \leq 0\). This yields \(0 \geq (\sigma_x, 1_m) - (\sigma_y, 1_n) = (\sigma_x, 1_m)\). So \(\sigma_x = 0\). This again implies \(0 \geq \max_{j \in \{1, \ldots, n\}} \sigma_x B\epsilon_j + \sigma_z = \sigma_z\).

(b) Assume \(\sigma_y \neq 0\). Then there is a \(j_0 \in \{m + 1, \ldots, m + n\}\) such that

\[
\rho C^t \epsilon_{j_0} = \rho_x B\epsilon_{j_0} - \rho_z \geq \max_{i \in \{1, \ldots, m\}} \epsilon_i A\rho_y - \rho_z
\]

Suppose \(\rho_z = \rho_x = 0\). Then \(0 \geq \max_{i \in \{1, \ldots, m\}} \epsilon_i A\rho_y\). Since \(A > 0\) this implies \(\rho_y = 0\). Consequently \(\rho = 0\), which is a contradiction. Now suppose \(\rho_z \neq 0\), \(\rho_x \neq 0\). Then, since \(B > 0\), we have \(0 > \rho_z B\epsilon_{j_0} - \rho_z = \max_{i \in \{1, \ldots, m\}} \epsilon_i A\rho_y\). This contradicts \(A > 0\). Consequently \(\rho_z \neq 0\).

(c) Assume \(\rho_z \neq 0\). Then

\[
\epsilon_{m+n+1} C\sigma = (\sigma_x, 1_m) - (\sigma_y, 1_n) \geq \max_{i \in \{1, \ldots, m\}} \epsilon_i A\sigma_y - \sigma_z
\]

This contradicts \(A > 0\). Consequently \(\rho_z \neq 0\).
Suppose $\sigma_x = 0$. Then $0 \geq -(\sigma_y, 1_n) \geq \max_{j \in \{1, \ldots, n\}} \sigma_x B e_j + \sigma_z = \sigma_z$. Hence $\sigma_z = 0$. Consequently $0 \geq -(-\sigma_y, 1_n) \geq \max_{i \in \{1, \ldots, m\}} \epsilon_i A \sigma_y$. Since $A > 0$, this implies $\sigma_y = 0$. Consequently $\sigma = 0$, which is a contradiction. Hence $\sigma_x \neq 0$.

The implications (d), (e) and (f) can be proved in a similar way. Since $\rho \neq 0 (\sigma \neq 0)$, at least one of the vectors $\rho_x, \rho_y$ and $\rho_z$ ($\sigma_x, \sigma_y$ and $\sigma_z$) has a positive coordinate. So we are in the situation of the scheme.

In view of lemma 6.2.1, for an equilibrium $(\rho, \sigma)$ for $(C, C^4)$, $\rho_x$ and $\rho_y$ can be normalized so that they become strategies in $\Delta_m$ and $\Delta_n$ respectively. These strategies are important in

**Theorem 6.2.2** Let $(C, C^4)$ be the GKT-symmetrization of an $m \times n$ bimatrix game $(A, B)$ with $A > 0$ and $B < 0$. Let $(\rho, \sigma) \in E(C, C^4)$. Then \((\rho_x(\rho_x, 1_m)^{-1}, \sigma_y(\sigma_y, 1_n)^{-1}) \in E(A, B)\) and \((\sigma_x(\sigma_x, 1_m)^{-1}, \rho_y(\rho_y, 1_n)^{-1}) \in E(A, B)\).

**Proof.** Since $(\rho, \sigma) \in E(C, C^4)$ we have

\[
eq_i C \sigma = \begin{cases} 
\epsilon_i A \sigma_y - \sigma_z & \text{for } i \in \{1, \ldots, m\} \\
\sigma_x B \epsilon_i - \sigma_z & \text{for } i \in \{m + 1, \ldots, m + n\} \\
(\sigma_x, 1_m) - (\sigma_y, 1_n) & \text{for } i = m + n + 1
\end{cases} \tag{6.2.1}
\]

and

\[
\rho C^t e_j = \begin{cases} 
\epsilon_j A \rho_y - \rho_z & \text{for } j \in \{1, \ldots, m\} \\
\rho_x B \epsilon_j - \rho_z & \text{for } j \in \{m + 1, \ldots, m + n\} \\
(\rho_x, 1_m) - (\rho_y, 1_n) & \text{for } j = m + n + 1.
\end{cases} \tag{6.2.2}
\]

By lemma 6.2.1, $\rho_x \neq 0$ and $\sigma_y \neq 0$.

Suppose $(\rho_x)_i > 0$. Then, since $C(\rho) \subseteq PB(C, \sigma)$, we obtain

\[\epsilon_i A \sigma_y - \sigma_z = \max_{k \in \{1,\ldots,m\}} \{\epsilon_k A \sigma_y - \sigma_z\}. \quad \text{Hence } \epsilon_i A \sigma_y = \max_{k \in \{1,\ldots,m\}} \epsilon_k A \sigma_y.\]

Similarly, if $(\sigma_y)_j > 0$, then $\rho_x B \epsilon_j = \max_{i \in \{1,\ldots,n\}} \rho_x B \epsilon_i.$

This implies $C(\rho_x(\rho_x, 1_m)^{-1}) \subseteq PB(A, \sigma_y(\sigma_y, 1_n)^{-1})$, and

$C(\sigma_y(\sigma_y, 1_n)^{-1}) \subseteq PB(B, \rho_x(\rho_x, 1_m)^{-1}).$

Hence $(\rho_x(\rho_x, 1_m)^{-1}, \sigma_y(\sigma_y, 1_n)^{-1}) \in E(A, B)$.

Similarly one shows $(\sigma_x(\sigma_x, 1_m)^{-1}, \rho_y(\rho_y, 1_n)^{-1}) \in E(A, B).$ \hfill \(\square\)

If $(\rho, \sigma) \in E(C, C^4)$, then also $(\sigma, \rho) \in E(C, C^4)$. According to theorem 6.2.2 these two equilibria of $(C, C^4)$ yield the same two equilibria for $(A, B)$. We now concern ourselves with a converse statement. Therefore we need the following definition.

For $(p, q), (\bar{p}, \bar{q}) \in E(A, B)$, the **GKT-product** $(p, q) \ast (\bar{p}, \bar{q})$ is defined as

\[
\left(\frac{p}{2 - pBq}, \frac{q}{2 + pAq}, 1 - \frac{1}{2 - pBq} - \frac{1}{2 + pAq}\right) \ast \left(\frac{\bar{p}}{2 - \bar{p}B\bar{q}}, \frac{\bar{q}}{2 + \bar{p}A\bar{q}}, 1 - \frac{1}{2 - \bar{p}B\bar{q}} - \frac{1}{2 + \bar{p}A\bar{q}}\right) = \left(\frac{p}{2 - pBq}, \frac{q}{2 + pAq}, 1 - \frac{1}{2 - pBq} - \frac{1}{2 + pAq}\right).
\]
Note that the GKT-product $+$ is well-defined since $A > 0$ and $B < 0$.

In fact we have $(p, q) + (\tilde{p}, \tilde{q}) \in \Delta_{m+n+1} \times \Delta_{m+n+1}$. Furthermore, if $(\rho, \sigma) := (p, q) + (\tilde{p}, \tilde{q})$, then $(\sigma, \rho) = (\tilde{p}, \tilde{q}) + (p, q)$. The next theorem shows the relevance of the GKT-product.

**Theorem 6.2.3** Let $(C, C')$ be the GKT-symmetrization of an $m \times n$ bimatrix game $(A, B)$ with $A > 0$ and $B < 0$. Let $(p, q), (\tilde{p}, \tilde{q}) \in E(A, B)$. Then $(p, q) + (\tilde{p}, \tilde{q}) \in E(C, C')$.

**Proof.** Let, for $(p, q), (\tilde{p}, \tilde{q}) \in E(A, B)$, $(\rho, \sigma) := (p, q) + (\tilde{p}, \tilde{q})$. Then $(\rho, \sigma) \in \Delta_{m+n+1} \times \Delta_{m+n+1}$ and $\rho_x \neq 0, \rho_y \neq 0, \rho_z \neq 0$, and $\sigma_x \neq 0, \sigma_y \neq 0, \sigma_z \neq 0$ by construction.

(a) For $i \in \{1, \ldots, m\}$ such that $\rho_i > 0$ we originally had $p_i > 0$. From $C(p) \subseteq PB(A, q)$ it then follows that $e_i A_q = \max_{k \in \{1, \ldots, m\}} e_k A_q$. Then we obtain from the GKT-product

$$e_i C\sigma = e_i A_q (2 + p Aq)^{-1} - (1 - (2 - \tilde{p} B \tilde{q})^{-1} - (2 + p Aq)^{-1})$$

$$= \max_{k \in \{1, \ldots, m\}} e_k C\sigma.$$

Since $\max_{k \in \{1, \ldots, m\}} e_k A_q = p A_q$, the last expression also equals $(2 - \tilde{p} B \tilde{q})^{-1} - (2 + p Aq)^{-1}$.

(b) Similarly, for $j \in \{m+1, \ldots, m+n\}$ with $\rho_j > 0$

$$e_j C\sigma = \max_{k \in \{m+1, \ldots, m+n\}} e_k C\sigma = (2 - \tilde{p} B \tilde{q})^{-1} - (2 + p Aq)^{-1}.$$

(c) Since $e_{m+n+1} C\sigma = (2 - \tilde{p} B \tilde{q})^{-1} - (2 + p Aq)^{-1}$, it follows from (a) and (b) that $\max_{k \in \{1, \ldots, m+n+1\}} e_k C\sigma = (2 - \tilde{p} B \tilde{q})^{-1} - (2 + p Aq)^{-1}$.

(d) Combining (a), (b) and (c) we obtain $C(\rho) \subseteq PB(C, \sigma)$. Similarly one shows $C(\sigma) \subseteq PB(C', \rho)$. Hence $(\rho, \sigma) \in E(C, C')$, or equivalently $(p, q) + (\tilde{p}, \tilde{q}) \in E(C, C')$. Similarly one proves

$$(\tilde{p}, \tilde{q}) + (p, q) \in E(C, C').$$

The next theorem shows that each equilibrium for the GKT-symmetrization of a bimatrix game is the GKT-product of two equilibria for this bimatrix game.

**Theorem 6.2.4** Let $(C, C')$ be the GKT-symmetrization of an $m \times n$ bimatrix game $(A, B)$ with $A > 0$ and $B < 0$. Let $(\rho, \sigma) \in E(C, C')$. Then there are equilibria $(p, q), (\tilde{p}, \tilde{q}) \in E(A, B)$ such that $(\rho, \sigma) = (p, q) + (\tilde{p}, \tilde{q})$. 

\(\square\)
Theorem 6.2.2 - 6.2.4 yield a one-to-one correspondence between pairs of equilibria for a bimatrix game, and (symmetric) pairs of equilibria for the GKT-symmetrization of this bimatrix game. In the last theorem of this section we show that there is a similar correspondence with respect to maximal Nash subsets for the two games.

In order to describe this correspondence, we define the GKT-product also for maximal Nash subsets.

Let \((A, B)\) be an \(m \times n\) bimatrix game and let \((C, C^\delta)\) be the GKT-symmetrization of \((A, B)\). Let \(S\) and \(T\) be maximal Nash subsets for \((A, B)\). Then the GKT-product of \(S\) and \(T\) is defined as

\[
S \ast T := \{(p, q) \ast (r, s) \mid (p, q) \in S \text{ and } (r, s) \in T\}.
\]

Theorem 6.2.5 Let \((C, C^\delta)\) be the GKT-symmetrization of an \(m \times n\) bimatrix game \((A, B)\) with \(A > 0\) and \(B < 0\).
(i) If $S$ and $T$ are maximal Nash subsets for $(A, B)$, then $S \ast T$ and $T \ast S$ are maximal Nash subsets for $(C, C^\ast)$.

(ii) If $U$ is a maximal Nash subset for $(C, C^\ast)$, then $U = S \ast T$ for some maximal Nash subsets $S$ and $T$ for $(A, B)$.

**Proof.** Let $S$ and $T$ be maximal Nash subsets for $(A, B)$. We only show that $S \ast T$ is a maximal Nash subset for $(C, C^\ast)$.

(i) First we show that $S \ast T$ is a Nash subset. Take $(\rho_1, \sigma_1), (\rho_2, \sigma_2) \in S \ast T$. By definition $(\rho_1, \sigma_1) = (p_1, q_1) + (\tilde{p}_1, \tilde{q}_1)$ for some $(p_1, q_1) \in S$ and $(\tilde{p}_1, \tilde{q}_1) \in T$ and $(\rho_2, \sigma_2) = (p_2, q_2) + (\tilde{p}_2, \tilde{q}_2)$ for some $(p_2, q_2) \in S$ and $(\tilde{p}_2, \tilde{q}_2) \in T$. By theorem 6.2.3, $(\rho_1, \sigma_1), (\rho_2, \sigma_2) \in E(C, C^\ast)$.

If we show that $(\rho_1, \sigma_2) \in S \ast T$ and $(\rho_2, \sigma_1) \in S \ast T$, it follows that $S \ast T$ is a Nash subset. We only show $(\rho_1, \sigma_2) \in S \ast T$.

By construction
\[
\rho_1 = \left( \frac{p_1}{2 - p_1 B q_1}, \frac{\tilde{q}_1}{2 + p_1 A \tilde{q}_1}, 1 - (2 - p_1 B q_1)^{-1} - \left(2 + p_1 A \tilde{q}_1\right)^{-1} \right),
\]
\[
\sigma_2 = \left( \frac{\tilde{p}_2}{2 - \tilde{p}_2 B \tilde{q}_2}, \frac{q_2}{2 + p_2 A q_2}, 1 - (2 - \tilde{p}_2 B \tilde{q}_2)^{-1} - \left(2 + p_2 A q_2\right)^{-1} \right).
\]

Since $(p_1, q_1)$ and $(p_2, q_2)$ are elements of $S$, we find $p_1 B q_1 = p_1 B q_2 = \max_{q \in \Delta_a} p_1 B q$ and $p_2 A q_2 = p_1 A q_2$.

Similarly $\tilde{p}_1 A \tilde{q}_1 = \tilde{p}_2 A \tilde{q}_1$ and $\tilde{p}_2 B \tilde{q}_2 = \tilde{p}_2 B \tilde{q}_1$.

Consequently we find $(\rho_1, \sigma_2) \in S \ast T$. Hence $S \ast T$ is a Nash subset.

(ii) Now suppose $U$ is a maximal Nash subset for $(C, C^\ast)$ containing $S \ast T$. Let $(\tau_1, \omega_1), (\tau_2, \omega_2) \in U$.

From theorem 6.2.4 we obtain for $i = 1, 2$ that $(\tau_i, \omega_i) = (r_i, s_i) + (\tilde{r}_i, \tilde{s}_i)$, where $(r_i, s_i), (\tilde{r}_i, \tilde{s}_i) \in E(A, B)$.

Evidently also $(\tau_1, \omega_2)$ and $(\tau_2, \omega_1)$ are elements of $U$. From the expression for the GKT-products $(r_i, s_i) \ast (\tilde{r}_i, \tilde{s}_i)$ for $i = 1, 2$, we find
\[
\tau_1 = \left( \frac{r_1}{2 - r_1 B s_1}, \frac{s_1}{2 + r_1 A \tilde{s}_1}, 1 - (2 - r_1 B s_1)^{-1} - \left(2 + r_1 A \tilde{s}_1\right)^{-1} \right),
\]
\[
\omega_2 = \left( \frac{\tilde{r}_2}{2 - \tilde{r}_2 B \tilde{s}_2}, \frac{s_2}{2 + r_2 A s_2}, 1 - (2 - \tilde{r}_2 B \tilde{s}_2)^{-1} - \left(2 + r_2 A s_2\right)^{-1} \right).
\]

Since $(\tau_1, \omega_2) \in E(C, C^\ast)$, we obtain from theorem 3 that there are $(p, q), (\tilde{p}, \tilde{q}) \in E(A, B)$ such that $(\tau_1, \omega_2) = (p, q) \ast (\tilde{p}, \tilde{q})$.

Hence we have e.g. $\frac{\tilde{r}_2}{2 - r_1 B s_1} = \frac{r_1}{2 - r_1 B s_1}$. Using $\sum_{t=1}^{m} p_t = \sum_{t=1}^{m} (r_1)_t = 1$, we find
\[
\frac{1}{\frac{1}{p-Bq}} = \frac{1}{\frac{1}{r-Bs}}, \text{ and consequently } p = r_1.
\]
Similarly \( q = s_2, \tilde{p} = \tilde{r}_2 \) and \( \tilde{q} = \tilde{s}_1 \). So \((r_1, s_2) = (p, q) \in E(A, B)\) and \((\tilde{r}_2, \tilde{s}_1) = (\tilde{p}, \tilde{q}) \in E(A, B)\).

By considering \((\tau_1, \omega_1)\) in a similar way, we obtain \((r_2, s_1), (\tilde{r}_1, \tilde{s}_2) \in E(A, B)\). This implies that \{\((r_1, s_1), (r_2, s_2), (r_1, s_2), (r_2, s_1)\)\} and \{\((\tilde{r}_1, \tilde{s}_1), (\tilde{r}_2, \tilde{s}_2), (\tilde{r}_1, \tilde{s}_2), (\tilde{r}_2, \tilde{s}_1)\)\} are Nash subsets for \((A, B)\). Since the \((\tau_1, \omega_1)\) are chosen arbitrary in \(U\), we obtain that \(U\) is the GKT-product of two Nash subsets for \((A, B)\). So in view of (i) we obtain \(U = S \ast T\) and hence \(S \ast T\) is a maximal Nash subset.

(ii) Let \(U\) be a maximal Nash subset for \((C, C')\). Similar to the proof above, we obtain that \(U\) is the GKT-product of two Nash subsets for \((A, B)\). Each of these Nash subsets is contained in a maximal Nash subset for \((A, B)\), and from (i) we obtain that the GKT-product of these maximal Nash subsets is a maximal Nash subset \(\tilde{U}\) for \((C, C')\). Evidently \(U \subset \tilde{U}\). However, this implies \(U = \tilde{U}\) and \(U\) is the GKT-product of two maximal Nash subsets for \((C, C')\).

\[\textit{6.3 BEHAVIOUR OF REFINEMENTS}\]

In this section we investigate how three refinements of equilibria behave in the procedure of the GKT-symmetrization. First we deal with quasi-strong equilibria, which were introduced by Harsanyi (1973) (cf. section 5.3).

**Theorem 6.3.1** Let \((C, C')\) be the GKT-symmetrization of an \(m \times n\) bimatrix game \((A, B)\) with \(A > 0\) and \(B < 0\).

(i) If \((p, q), (\tilde{p}, \tilde{q}) \in E(A, B)\) are quasi-strong, then both \((p, q) \ast (\tilde{p}, \tilde{q}) \in E(C, C')\) and \((\tilde{p}, \tilde{q}) \ast (p, q) \in E(C, C')\) are quasi-strong.

(ii) If \((p, \sigma) \in E(C, C')\) is quasi-strong, then both \((\rho_x(p_x, 1_m^{-1}, \sigma_y(\sigma_y, 1_n)^{-1}) \in E(A, B)\) and \((\sigma_x(\sigma_x, 1_m)^{-1}, \rho_y(\rho_y, 1_n)^{-1}) \in E(A, B)\) are quasi-strong.

**Proof.** (i) Let \((p, q), (\tilde{p}, \tilde{q}) \in E(A, B)\) both be quasi-strong and define \((\rho, \sigma) \equiv (p, q) \ast (\tilde{p}, \tilde{q})\). Since, by theorem 6.2.3, \((\rho, \sigma) \in E(C, C')\), we obtain for \(i \in \{1, ..., m + n + 1\}\) that \(\rho_i > 0\) implies \(e_i C \sigma = \max_{k \in \{1, ..., m + n + 1\}} e_k C \sigma\).

Suppose that for \(i \in \{1, ..., m + n + 1\}\)

\[
e_i C \sigma = \max_{k \in \{1, ..., m + n + 1\}} e_k C \sigma. \tag{6.3.1}
\]

Since \(\rho_x = \frac{p}{1 - pBq}\), we find that \((\rho_x)_m > 0\) for at least one \(i_0 \in \{1, ..., m\}\).

Hence \(\max_{k \in \{1, ..., m + n + 1\}} e_k C \sigma = \max_{k \in \{1, ..., m\}} e_k C \sigma\).

For \(k \in \{1, ..., m\}\) we have \(e_k C \sigma = e_k Aq(2 + pAq)^{-1} - \sigma_z\). Consequently, if (6.3.1) is
satisfied for an \( i \in \{1, \ldots, m\} \), then
\[
e_{i}Aq(2 + pAq)^{-1} - \sigma_{z} = \max_{k \in \{1, \ldots, m+n+1\}} e_{k}C\sigma = \max_{k \in \{1, \ldots, m\}} e_{k}C\sigma
\]
\[= \max_{k \in \{1, \ldots, m\}} e_{k}C\sigma = \max_{k \in \{1, \ldots, m\}} e_{k}Aq(2 + pAq)^{-1} - \sigma_{z}.
\]
So \( e_{i}Aq = \max_{k \in \{1, \ldots, m\}} e_{k}Aq \). Since \( C(p) = PB(A, q) \), we obtain that \( p_{i} > 0 \), and consequently \( p_{i} > 0 \).
Similarly one proves that if (6.3.1) is satisfied for an \( i \in \{m+1, \ldots, m+n\} \), then also \( p_{i} > 0 \).
Finally, by lemma 1, \( \rho_{m+n+1} > 0 \). Hence we have that for \( i \in \{1, \ldots, m+n+1\} \), (6.3.1) implies \( p_{i} > 0 \). This implies \( C(p) = PB(C, \sigma) \). Similarly one shows \( C(\sigma) = PB(C^{t}, \rho) \). Therefore \( (\rho, \sigma) \) is a quasi-strong equilibrium for \((C, C^{t})\).
Similar arguments show that \((\sigma, \rho) = (\tilde{p}, \tilde{q}) \ast (p, q) \) is a quasi-strong equilibrium for \((C, C^{t})\).
(ii) The proof follows immediately from (6.2.1) and (6.2.2).

For a bimatrix game an isolated equilibrium is an equilibrium which is a maximal Nash subset itself. In view of theorem 6.2.5 we obtain that theorem 6.3.1 also holds if we replace the word quasi-strong by isolated. In Jansen (1987) a regularity concept for equilibria is introduced and it is proved that an equilibrium is regular if and only if it is isolated and quasi-strong. Hence we have

**Corollary 6.3.2** Theorem 6.3.1 also holds if quasi-strong is replaced by regular.

Next we prove a result on perfect equilibrium. Instead of using the original definition of perfectness we use in the proof the equivalent concept of undominatedness (cf. theorem 3.3.1).

**Theorem 6.3.3** Let \((C, C^{t})\) be the GKT-symmetrization of an \( m \times n \) bimatrix game \((A, B)\) with \( A > 0 \) and \( B < 0 \).
(i) If \((p, q), (\tilde{p}, \tilde{q}) \in E(A, B)\) are perfect, then both \((p, q) \ast (\tilde{p}, \tilde{q}) \in E(C, C^{t})\) and \((\tilde{p}, \tilde{q}) \ast (p, q) \in E(C, C^{t})\) are perfect.
(ii) If \((\rho, \sigma) \in E(C, C^{t})\) is perfect, then both \((\rho_{x}(\rho_{y}, 1_{m})^{-1}, \sigma_{y}(\sigma_{x}, 1_{n})^{-1}) \in E(A, B)\) and \((\sigma_{x}(\sigma_{y}, 1_{m})^{-1}, \rho_{y}(\rho_{y}, 1_{n})^{-1}) \in E(A, B)\) are perfect.

**Proof.** (i) Let \((p, q), (\tilde{p}, \tilde{q}) \in E(A, B)\) both be perfect and define \((\rho, \sigma) := (p, q) \ast (\tilde{p}, \tilde{q})\). Since, by theorem 6.2.3, \((\rho, \sigma) \in E(C, C^{t})\) we only have to show that \( \rho \) and \( \sigma \) are undominated strategies for the game \((C, C^{t})\). Suppose a \( \overline{p} \in \Delta_{m+n+1} \) exists such that \( \overline{p}C \geq \rho C \), or equivalently, using the GKT-product,
For proper equilibria it is possible to prove an analogue to theorem 6.3.3(ii).

\[
\begin{pmatrix}
B\bar{p}_y + \bar{p}_z, 1_m \\
\bar{p}_y A - \bar{p}_z, 1_m \\
(\bar{p}_y, 1_n) - (\bar{p}_x, 1_m)
\end{pmatrix} \geq \begin{pmatrix}
B\bar{q}(2 + \bar{p}A\bar{q})^{-1} + (1 - (2 + \bar{p}A\bar{q})^{-1} - (2 - pBq)^{-1})1_m \\
pA(2 - pBq)^{-1} - (1 - (2 + pAq)^{-1} - (2 - pBq)^{-1})1_n \\
(2 + pAq)^{-1} - (2 - pBq)^{-1}
\end{pmatrix}
\]

(6.3.2)

We consider two cases:

(a) \(\bar{p}_z \leq 1 - (2 + \bar{p}A\bar{q})^{-1} - (2 - pBq)^{-1}\)

(b) \(\bar{p}_z \geq 1 - (2 + \bar{p}A\bar{q})^{-1} - (2 - pBq)^{-1}\).

(a) Suppose \(\bar{p}_y = 0\). Then the third line of (6.3.2) yields

\[-(\bar{p}_x, 1_m) \geq (2 + \bar{p}A\bar{q})^{-1} - (2 - pBq)^{-1}.

Since \(\bar{p}_z = 1 - (\bar{p}_x, 1_m)\), we obtain

\[1 + (2 + \bar{p}A\bar{q})^{-1} - (2 - pBq)^{-1} \leq \bar{p}_z \leq 1 - (2 + \bar{p}A\bar{q})^{-1} - (2 - pBq)^{-1}\]

This implies \((2 + \bar{p}A\bar{q})^{-1} \leq 0\), which contradicts \(A > 0\). So \(\bar{p}_y \neq 0\).

The first line of the inequality (6.3.2) yields \(B\bar{p}_y \geq B\bar{q}(2 + \bar{p}A\bar{q})^{-1}\), or equivalently,

\[\begin{pmatrix}
\bar{p}_y, 1_n \end{pmatrix} B\bar{p}_y \geq B\bar{q}(2 + \bar{p}A\bar{q})^{-1}
\]

Since \(\bar{p}_y, 1\) is dominated and \(B < 0\), we obtain from the last inequality that \((\bar{p}_y, 1_n) \leq (2 + \bar{p}A\bar{q})^{-1}\). Then the third line of (6.3.2) yields \((\bar{p}_x, 1_n) \leq (2 - pBq)^{-1}\). Since \(\bar{p}_z = 1 - (\bar{p}_x, 1_m)\), the last two inequalities yield \(\bar{p}_z \geq 1 - (2 + \bar{p}A\bar{q})^{-1} - (2 - pBq)^{-1}\). Thus \(\bar{p}_z = 1 - (2 + \bar{p}A\bar{q})^{-1} - (2 - pBq)^{-1}\). This implies \((\bar{p}_x, 1_m) = (2 - pBq)^{-1}\) and \((\bar{p}_y, 1_n) = (2 + \bar{p}A\bar{q})^{-1}\). Then the dominatedness of \(p\) and \(\bar{q}\) for \((A, B)\) implies \(\bar{p}C = \rho C\). Consequently \(\rho\) is dominated.

(b) A similar proof shows that also in this case \(\rho\) is dominated.

We can also distinguish two cases to show that \(\sigma\) is dominated. Hence \((\rho, \sigma)\) is a perfect equilibrium for the game \((C, C')\).

It is easily verified that then also \((\sigma, \rho) = (\bar{p}, \bar{q}) + (p, q)\) is an undominated and hence perfect equilibrium for \((C, C')\).

(ii) Let \((\rho, \sigma)\) be a perfect equilibrium for \((C, C')\). We only show that \((\rho_x(\rho_x, 1_m)^{-1}, \sigma_y(\sigma_y, 1_n)^{-1})\) is a perfect equilibrium for \((A, B)\). Suppose a \(p \in \Delta_m\) exists such that \(pA \geq \rho_x(\rho_x, 1_m)^{-1}A\). Define \(\bar{p} := (p(\rho_x, 1_m), \rho_y, \rho_z) \in \Delta_{m+n+1}\). Then

\[pC = (B\rho_y + \rho_z, 1_m, (\rho_x, 1_m)pA - \rho_z, 1_n, (\rho_y, 1_n) - (\rho_x, 1_m)) \geq \rho C.

Since \(\rho\) is dominated, this implies \(pC = \rho C\), and in particular \(pA = \rho_x(\rho_x, 1_m)^{-1}A\).

Hence \(\rho_x(\rho_x, 1_m)^{-1}\) is undominated. Similarly one shows that also \(\sigma_y(\sigma_y, 1_n)^{-1}\) is undominated. Hence \((\rho_x(\rho_x, 1_m)^{-1}, \sigma_y(\sigma_y, 1_n)^{-1})\) is a perfect equilibrium for \((A, B)\).

A similar proof shows that \((\sigma_x(\sigma_x, 1_m)^{-1}, \rho_y(\rho_y, 1_n)^{-1})\) is also a perfect equilibrium for \((A, B)\). 

For proper equilibria it is possible to prove an analogue to theorem 6.3.3 (ii).