CHAPTER 4

REFINEMENTS OF THE CONCEPTS OF

PERFECT AND PROPER EQUILIBRIA

4.1 INTRODUCTION AND DEFINITIONS

The concept of perfect equilibrium (Selten (1975)) as defined in the previous section was refined by Myerson (1978) in the definition of proper equilibria. Whereas in the definition of perfect equilibrium the strategy spaces of the players are perturbed arbitrarily, the concept of proper equilibrium assumes a particular ordering of the perturbations.

By definition every proper equilibrium is also perfect. It is an open question whether a strictly perfect equilibrium, when it exists, is also proper. It is known, however, that not every minimal strictly perfect set contains a proper equilibrium (cf. example 4.3.3). Myerson showed that every game possesses a proper equilibrium.

Recently Garcia Jurado and Prada Sanchez (1989) defined equalized proper equilibria and gave a proof of existence. In this concept Myerson's definition is pushed further in the sense that further ordering of the perturbations is assumed. Besides the properness features it is assumed that each player puts equal probability on any two pure strategies that yield the same payoff, regardless of the actions of the opponent.

† whereas Selten interprets the perturbations of the strategy spaces as mistakes which the players could make, Myerson assumes that more costly mistakes are made with less probability
In this chapter we study the latter assumption more generally. First we provide another equivalent definition of a perfect equilibrium (cf. section 3.2).

**Proposition 4.1.1** Let \((A, B)\) be an \(m \times n\) bimatrix game and let \((\tilde{p}, \tilde{q}) \in \Delta_m \times \Delta_n\). Then \((\tilde{p}, \tilde{q})\) is a perfect equilibrium for \((A, B)\) if and only if there exist a sequence \(\{\epsilon_k\}_{k \in \mathbb{N}}\) of positive real numbers converging to zero and a sequence \(\{(p^k, q^k)\}_{k \in \mathbb{N}}\) of pairs of completely mixed strategies converging to \((\tilde{p}, \tilde{q})\), such that \((p^k, q^k)\) is \(\epsilon^k\)-perfect for \((A, B)\) for all \(k \in \mathbb{N}\). Here, with \(\epsilon > 0\), a pair \((p, q) \in \Delta_m \times \Delta_n\) is called \(\epsilon\)-perfect for \((A, B)\) if for all \(i, r \in M\) and \(j, s \in N\)

\[
\begin{align*}
\epsilon_i A q < \epsilon_r A q & \Rightarrow p_i \leq \epsilon \\
p B e_j < p B e_s & \Rightarrow q_j \leq \epsilon.
\end{align*}
\]

A proof of proposition 4.1.1 can be found in van Damme (1987) (theorem 2.2.5), but one may also adjust the proof of lemma 4.3.5 in this chapter to obtain the result above.

Proper equilibria are defined as follows:

An equilibrium \((\tilde{p}, \tilde{q})\) for \((A, B)\) is called *proper* if there exist a sequence \(\{\epsilon_k\}_{k \in \mathbb{N}}\) of positive real numbers converging to zero and a sequence \(\{(p^k, q^k)\}_{k \in \mathbb{N}}\) of pairs of completely mixed strategies converging to \((\tilde{p}, \tilde{q})\), such that \((p^k, q^k)\) is \(\epsilon^k\)-proper for \((A, B)\) for all \(k \in \mathbb{N}\). Here, with \(\epsilon > 0\), a pair \((p, q) \in \Delta_m \times \Delta_n\) is called \(\epsilon\)-proper for \((A, B)\) if for all \(i, r \in M\) and \(j, s \in N\)

\[
\begin{align*}
\epsilon_i A q < \epsilon_r A q & \Rightarrow p_i \leq \epsilon r \\
p B e_j < p B e_s & \Rightarrow q_j \leq \epsilon s.
\end{align*}
\]

In addition to the definition of an equalized proper equilibrium of Garcia Jurado and Prada Sanchez we give here also a definition of an equalized perfect equilibrium.

Let \((A, B)\) be an \(m \times n\) bimatrix game. A strategy pair \((p, q) \in \Delta_m \times \Delta_n\) is called *equalized perfect (equalized proper)* for \((A, B)\), or shortly *\(\epsilon\)-perfect (\(\epsilon\)-proper)*, if there exist a sequence \(\{\epsilon_k\}_{k \in \mathbb{N}}\) of positive real numbers converging to zero and a sequence \(\{(p^k, q^k)\}_{k \in \mathbb{N}}\) of pairs of completely mixed strategies converging to \((\tilde{p}, \tilde{q})\), such that \((p^k, q^k)\) is \(\epsilon^k\)-perfect (\(\epsilon^k\)-proper) and equalized for all \(k \in \mathbb{N}\). Here, a pair \((p, q) \in \Delta_m \times \Delta_n\) is called equalized if

\[
\begin{align*}
\epsilon_i A &= \epsilon_r A \Rightarrow p_i = p_r \quad (i, r \in M) \\
B e_j &= B e_s \Rightarrow q_j = q_s \quad (j, s \in N).
\end{align*} \tag{4.1.1}
\]

From this definition it is immediately clear that \(\epsilon\)-proper implies \(\epsilon\)-perfect and that \(\epsilon\)-proper (\(\epsilon\)-perfect) implies proper (perfect).
Chapter 4: Refinements of the concepts of perfect and proper equilibria

Garcia Jurado and Prada Sanchez showed existence of ε-proper equilibria. In the next section we provide a new proof of this fact.

The following example shows that a proper equilibrium need not be ε-perfect.

**Example 4.1.1** Consider the $2 \times 3$ bimatrix game $(A, B)$ given by

$$(A, B) = \begin{bmatrix}
(3, 6) & (0, 6) & (5, 5) \\
(0, 0) & (2, 0) & (5, 5)
\end{bmatrix}.$$  

Defining $e^k = \frac{1}{k+1}, q^k = \frac{k}{k+1}e_3 + \frac{1}{k+1}(\frac{3}{5}e_1 + \frac{2}{5}e_2)$ and $p^k = \frac{k}{k+1}e_2 + \frac{1}{k+1}e_1$, it is easily checked that $(p^k, q^k)$ is $e^k$-proper for all $k > 1$. Hence, $(e_2, e_3)$ is a proper equilibrium for $(A, B)$. However, $(e_2, e_3)$ is not ε-perfect because all equalized and ε-perfect strategy pairs $(p, q) \in \Delta_2 \times \Delta_3$ must satisfy $q_1 = q_2$, which implies that $e_2Ap < e_1Ap$ and so $p_2 \leq \varepsilon$. In fact the unique ε-perfect (ε-proper) equilibrium for $(A, B)$ is $(e_1, \frac{1}{2}e_1 + \frac{1}{2}e_2)$.

In section 4.2 we assign to each bimatrix game a so-called ‘equalized game’ and prove that the perfect and proper equilibria for this game correspond to the ε-perfect and ε-proper equilibria for the original game. In view of the results of Selten (1975) and Myerson (1978) this settles existence for both new concepts. In section 4.3 the equalized game also helps to show there always exists a persistent equilibrium that is ε-proper.

In section 4.2 we also define a stronger and a weaker version of ε-proper and ε-perfect equilibria, respectively called ‘iterated’ ε-proper and ε-perfect equilibria and ‘weakly’ ε-proper and ε-perfect equilibria. In section 4.3 we show that the stronger version does not nicely relate to persistent equilibria, but that the weaker version has an additional relation with respect to minimal strictly perfect sets.

Finally in section 4.4 some remarks concerning structure are made.

Most results in this chapter are based upon Jurg, Garcia Jurado and Borm (1989).

### 4.2 Equalized Games

We begin this section with some definitions.

For $S \subseteq \{1, \ldots, t\}$ the vector $e^S \in \mathbb{R}^t$ is defined by

$$e^S_k = \begin{cases} 
1 & \text{if } k \in S \\
0 & \text{otherwise}.
\end{cases}$$

Let $(A, B)$ be an $m \times n$ bimatrix game. In this game two pure strategies $e_i, e_r \in \Delta_m$ are called **payoff equivalent** for player $I$ if $e_iA = e_rA$ (cf. (4.1.1)). The equivalence
classes of payoff equivalent pure strategies induce a partition \( \{ M_1, M_2, \ldots, M_\pi \} \) of the set \( M \). Similarly one defines payoff equivalent pure strategies for player II and a partition \( \{ N_1, N_2, \ldots, N_\pi \} \) of \( N \). Furthermore, the \( \pi \times \pi \) equivalent game \((\mathcal{A}, \mathcal{B})\) corresponding to \((A, B)\) is defined by \( \mathcal{A} = \{ \bar{e}_\mu \}_{\mu=1}^{\pi} \) and \( \mathcal{B} = \{ \bar{f}_\nu \}_{\nu=1}^{\pi} \), where

\[
\bar{e}_\mu := \frac{e_{\mu \nu}}{[M_\mu]} \quad \text{and} \quad \bar{f}_\nu := \frac{e_{\mu \nu}}{[M_\nu]}.
\]

(4.2.1)

for all \( \mu \in \{1, \ldots, \pi\} \) and \( \nu \in \{1, \ldots, \pi\} \). Note that the pure strategy \( e_\mu \in \Delta_\pi \) of the equalized game \((\mathcal{A}, \mathcal{B})\) corresponds to the barycentre of the equivalence class of pure strategies corresponding to \( M_\mu \). Let the mapping \( f : \Delta_m \times \Delta_n \rightarrow \Delta_\pi \times \Delta_\pi \) be defined by

\[
f(p, q) = (\{ \sum_{i \in M_\mu} p_i \}_{\mu=1}^{\pi}, \{ \sum_{j \in N_\nu} q_j \}_{\nu=1}^{\pi}).
\]

(4.2.2)

Obviously \( f \) is continuous. Further we introduce the sets \( \Xi_m \subset \Delta_m \) and \( \Xi_n \subset \Delta_n \) by

\[
\Xi_m := \text{conv} \left( \{ \frac{e_{\mu \nu}}{[M_\mu]} \}_{\mu=1}^{\pi} \right) \quad \text{and} \quad \Xi_n := \text{conv} \left( \{ \frac{e_{\mu \nu}}{[N_\nu]} \}_{\nu=1}^{\pi} \right)
\]

(4.2.3)

so that all \( \epsilon \)-perfect equilibria for \((A, B)\) are contained in \( \Xi_m \times \Xi_n \). It is clear that the restriction \( \mathcal{f} \) of \( f \) to \( \Xi_m \times \Xi_n \) is a bijection and has a continuous inverse \( \mathcal{f}^{-1} \).

With respect to Nash equilibria we now can formulate

**Theorem 4.2.1** Let \((A, B)\) be an \( m \times n \) bimatrix game and let \( f : \Delta_m \times \Delta_n \rightarrow \Delta_\pi \times \Delta_\pi \) be defined as in (4.2.2). Let \((p, q) \in \Xi_m \times \Xi_n \). Then \((p, q) \in E(A, B)\) if and only if \( f(p, q) \in E(\mathcal{A}, \mathcal{B}) \).

**Proof.** Let \( p = \sum_{\mu=1}^{\pi} c_\mu \frac{e_{\mu \nu}}{[M_\mu]} \) and \( q = \sum_{\nu=1}^{\pi} d_\nu \frac{e_{\mu \nu}}{[N_\nu]} \).

Define \((x, y) \in \Delta_\pi \times \Delta_\pi \) by \((x, y) := f(p, q)\). Let \( \mu \in \{1, \ldots, \pi\} \) be fixed. Then

\[
e_{\mu i} x_{\mu} \bar{y}_{\mu} = \sum_{\nu=1}^{\pi} \frac{e_{\mu \nu}}{[M_\mu]} \frac{e_{\nu i}}{[N_\nu]} \left( \sum_{\nu=1}^{\pi} d_\nu \frac{e_{\mu \nu}}{[N_\nu]} \right)
\]

\[
= \sum_{\nu=1}^{\pi} \frac{e_{\mu \nu}}{[M_\mu]} \frac{e_{\nu i}}{[N_\nu]} \left( \frac{e_{\nu i}}{[N_\nu]} \right)
\]

\[
= \frac{e_{\mu \nu}}{[M_\mu]} \frac{e_{\nu i}}{[N_\nu]} \quad \text{for all} \quad i \in M_\mu.
\]

(4.2.4)

It follows that \( \mu \in PB(\mathcal{A}, y) \) if and only if \( M_\mu \subset PB(A, q) \). Furthermore, since \( x_\mu = \sum_{i \in M_\mu} p_i = c_\mu \) we have that \( \mu \in C(x) \) if and only if \( M_\mu \subset C(p) \).
This proves that \( C(p) \subseteq PB(A, q) \) if and only if \( C(x) \subseteq PB(\overline{A}, y) \). Similarly one shows that \( C(q) \subseteq PB(B, p) \) if and only if \( C(y) \subseteq PB(\overline{B}, x) \).

The requirement in theorem 4.2.1 that \( (p, q) \in \Delta_m \times \Delta_n \) is a necessary one as the following example shows.

**Example 4.2.1** Consider the \( 2 \times 2 \) bimatrix game

\[
(A, B) = \begin{bmatrix}
(1,0) & (0,0) \\
(1,0) & (1,0)
\end{bmatrix}.
\]

Clearly \( \left( \frac{1}{4}e_1 + \frac{1}{2}e_2, e_1 \right) \) is an equilibrium for \((A, B)\). Since \((\overline{A}, \overline{B}) = \begin{bmatrix}
(\frac{1}{2}, 0) \\
(1, 0)
\end{bmatrix}\), it is also clear that \( E(\overline{A}, \overline{B}) = \{(e_2, e_1)\} \). Then \( f(\left( \frac{1}{4}e_1 + \frac{1}{2}e_2, e_1 \right)) = \left( \frac{1}{2}e_1 + \frac{1}{2}e_2, e_1 \right) \notin E(\overline{A}, \overline{B}) \).

With respect to \( \varepsilon \)-perfect and \( \varepsilon \)-proper equilibria we have

**Theorem 4.2.2** Let \((A, B)\) be an \( m \times n \) bimatrix game. Let \( f : \Delta_m \times \Delta_n \rightarrow \Delta_{\pi_m} \times \Delta_{\pi_n} \) be defined as in (4.2.2). Let \((p, q) \in \Delta_m \times \Delta_n \). Then

(i) \((p, q)\) is \( \varepsilon \)-perfect for \((A, B)\) if and only if \( f(p, q) \) is perfect for \((\overline{A}, \overline{B})\).

(ii) \((p, q)\) is \( \varepsilon \)-proper for \((A, B)\) if and only if \( f(p, q) \) is proper for \((\overline{A}, \overline{B})\).

**Proof.** We only prove (ii). First we demonstrate the only if part. Let \((p, q)\) be \( \varepsilon \)-proper for \((A, B)\). Then we can find sequences \( \{e^k\}_{k \in \mathbb{N}} \) of positive reals converging to zero and \( \{(p^k, q^k)\}_{k \in \mathbb{N}} \) of pairs of completely mixed strategies converging to \((p, q)\) such that \((p^k, q^k)\) is \( \varepsilon^k \)-proper and equalized for all \( k \in \mathbb{N} \). Defining \((x, y) := f(p, q)\) and \((x^k, y^k) := f(p^k, q^k)\) for all \( k \in \mathbb{N} \), it follows that \((x^k, y^k) \in \Delta_{\pi_m} \times \Delta_{\pi_n}\) for all \( k \in \mathbb{N} \) and, by continuity of \( f \), \((x^k, y^k)\) converges to \((x, y)\). Let, for all \( k \in \mathbb{N} \),

\[
\delta^k := \varepsilon^k \max \left\{ \max_{\mu \in \{1, \ldots, \pi_m\}} |M_\mu|, \max_{\nu \in \{1, \ldots, \pi_n\}} |N_\nu| \right\}
\]

We are finished if we can show that \((x^k, y^k)\) is \( \delta^k \)-proper for \((\overline{A}, \overline{B})\) for all \( k \in \mathbb{N} \). Let \( \mu, \nu \in \{1, \ldots, \pi_m\} \) be such that \( e^k \mu \overline{y}^k < e^k \nu \overline{y}^k \). Then \( e_i A q^k < e_i A q^k \) for all \( i \in M_\mu \) and \( r \in M_\nu \) (cf. (4.2.4)). Hence, \( p^k_i \leq \varepsilon^k p^k_{ir} \) for all \( i \in M_\mu \) and \( r \in M_\nu \). Consequently,

\[
x^k = \sum_{i \in M_\mu} p^k_i \leq \varepsilon^k |M_\mu| \sum_{r \in M_\nu} p^k_r = \varepsilon^k |M_\mu| |x^k| \leq \delta^k x^k.
\]

Similarly one finds that \( x^k \overline{B} e_\nu < x^k \overline{B} e_\sigma \) implies that \( y^k_\nu \leq \delta^k y^k_\sigma \) for all \( k \in \mathbb{N} \) and \( \nu, \sigma \in \{1, \ldots, \pi\} \).

Secondly we prove the if part. Let \((x, y) := f(p, q)\) be proper for \((\overline{A}, \overline{B})\). Let \( \{e^k\}_{k \in \mathbb{N}} \) and \( \{(x^k, y^k)\}_{k \in \mathbb{N}} \subseteq \Delta_{\pi_m} \times \Delta_{\pi_n} \) be sequences as required for the properness
of \((x, y)\). Defining \((p^k, q^k) := F^{-1}(x^k, y^k) \in \tilde{\Delta}_m \times \tilde{\Delta}_n\), the continuity of \(F^{-1}\) implies that \((p^k, q^k)\) converges to \(F^{-1}(x, y) = (p, q)\).

Let \(\delta^k\) be defined as in (4.2.5). By definition \((p^k, q^k)\) is equalized, so it suffices to show that \((p^k, q^k)\) is \(\delta^k\)-proper for all \(k \in \mathbb{N}\).

Let \(i, r \in \{1, \cdots, m\}\) be such that \(e_i\alpha q^k < e_i\alpha y^k\). For \(\mu, \rho \in \{1, \cdots, m\}\) with \(i \in M_\mu\) and \(r \in M_\rho\), (4.5) implies that \(e_\rho y^k \leq e_\rho y^k\) so \(x^k \leq e^k x^k\). Consequently,

\[
p^k = \left\lfloor \frac{x^k}{M_\mu} \right\rfloor \leq \frac{e^k x^k}{|M_\mu|} = \frac{e^k}{|M_\mu|} \left\lfloor \frac{x^k}{M_\mu} \right\rfloor \leq \frac{e^k |M_\mu| p^k}{|M_\mu|} \leq \delta^k p^k.
\]

Similarly one finds that \(p^k B e_j < p^k B e_s\) implies that \(q^k_j \leq \delta^k q^k_s\) for all \(k \in \mathbb{N}\) and \(j, s \in N\).

The existence of proper equilibria (Myerson (1978)) immediately implies the following

**Corollary 4.2.3** Every bimatrix game has at least one \( \epsilon \)-proper (and hence \( \epsilon \)-perfect) equilibrium.

The results so far are illustrated in

**Example 4.2.2** Consider the \(2 \times 3\) bimatrix game \((A, B)\) of example 4.1.1. Then \(\mathbb{M} = \mathbb{P} = 2, M_1 = \{1\}, M_2 = \{2\}, N_1 = \{1, 2\}\) and \(N_2 = \{3\}\). The equalized game \((\bar{A}, \bar{B})\) is given by

\[
(\bar{A}, \bar{B}) = \begin{bmatrix}
(1, 1) & (6, 5) \\
(1, 0) & (5, 5)
\end{bmatrix}
\]

It is easily checked that the unique perfect (proper) equilibrium for \((\bar{A}, \bar{B})\) is \((e_1, e_1)\).

Hence, \(\bar{F}^{-1}(e_1, e_1) = (e_1, \frac{1}{2} e_1 + \frac{1}{2} e_2) \in \Delta_2 \times \Delta_3\) is the unique \( \epsilon \)-perfect (\( \epsilon \)-proper) equilibrium for \((A, B)\).

In example 4.1.1 it was shown that \((e_2, e_3)\) is a proper equilibrium for \((A, B)\). By theorem 4.2.1 we have \(f(e_2, e_3) = (e_2, e_3) \in E(\bar{A}, \bar{B})\). However, \((e_2, e_2)\) is not proper.

Theorem 4.2.2 makes it possible to further refine the concepts of \( \epsilon \)-proper and \( \epsilon \)-perfect equilibria. This is shown in the following

**Example 4.2.3** Consider the \(2 \times 3\) bimatrix game \((A, B)\) given by

\[
(A, B) = \begin{bmatrix}
(0, 1) & (1, -1) & (2, 0) \\
(0, -1) & (1, 1) & (2, 0)
\end{bmatrix}
\]

The \(1 \times 3\) equalized game \((\bar{A}, \bar{B})\) is given by

\[
(\bar{A}, \bar{B}) = [0, 0] \begin{bmatrix} (1, 0) & (2, 0) \end{bmatrix}.
\]
Note that in \((\overline{A}, \overline{B})\) player II has three equivalent pure strategies. So one might consider the \(1 \times 1\) equalized game \((\overline{A}, \overline{B})\) corresponding to \((\overline{A}, \overline{B})\) given by
\[
(\overline{A}, \overline{B}) = [(1, 0)].
\]

Obviously \((\overline{A}, \overline{B})\) has only one (perfect and proper) equilibrium which corresponds to the \(\epsilon\)-proper equilibrium \((\frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2, \frac{1}{3}\epsilon_1 + \frac{1}{3}\epsilon_2 + \frac{1}{3}\epsilon_3)\) for \((A, B)\). However, using theorem 4.2.2, one finds that \((\frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2, q)\) is \(\epsilon\)-proper for \((A, B)\) for all \(q \in \Delta_3\).

Example 4.2.3 motivates the following definitions. Let \(g\) be the map that assigns to a bimatrix game the equalized game corresponding to it. As we have seen in example 4.2.3 the map \(g\) can be applied iteratively more than once for a bimatrix game. If \(g\) is applied \(n\) times, then we denote this by \(g^n\). Clearly for each bimatrix game \((A, B)\) there exists a number \(t\) such that \(g^t(A, B) = g^{t+1}(A, B)\). Then we call \(g^t(A, B)\) the \(\text{iterated equalized game}\) corresponding to \((A, B)\). Strategy pairs in the original game \((A, B)\) will be called \(\text{iterated proper}\) for the iterated equalized game.

By definition, the existence of these equilibrium concepts is guaranteed and, clearly, iterated \(\epsilon\)-proper (iterated \(\epsilon\)-proper) implies \(\epsilon\)-perfect (\(\epsilon\)-proper). However, example 4.2.3 shows that converse of the last statement need not hold. Examples in the next section will show that for iterated \(\epsilon\)-proper and even iterated \(\epsilon\)-perfect equilibria there is not a nice relation with persistent equilibria.

Garcia Jurado (1989) introduced another modification of the properness concept, which can also be transfered to the perfectness concept:

Let \((A, B)\) be an \(m \times n\) bimatrix game. Then \((p, q) \in \Delta_m \times \Delta_n\) is called \(\text{weakly } \epsilon\)-proper \(\text{(weakly } \epsilon\text{-perfect)}\) for \((A, B)\) if there exist sequences \(\{\epsilon^k\}_{k \in \mathbb{N}}\) of positive real numbers converging to zero and \(\{(p^k, q^k)\}_{k \in \mathbb{N}}\) of pairs of completely mixed strategies converging to \((p, q)\) such that \((p^k, q^k)\) is \(\epsilon^k\)-proper \(\text{(}\epsilon^k\text{-perfect)}\) and weakly equalized for all \(k \in \mathbb{N}\).

Here a pair \((p, q) \in \Delta_m \times \Delta_n\) is called \(\text{weakly equalized}\) if for all \(i, r \in M \setminus P B(A, q)\), and all \(j, s \in N \setminus P B(B, p)\) we have
\[
\begin{cases}
\epsilon_i A = \epsilon_r A & \Rightarrow p_i = p_r \\
B \epsilon_j = B \epsilon_s & \Rightarrow q_j = q_s
\end{cases}
\]

By definition every \(\epsilon\)-perfect \(\epsilon\)-proper) equilibrium is also weakly \(\epsilon\)-perfect \(\epsilon\)-proper).

In the next we will see that there is a nice relation between minimal strictly perfect sets and weakly \(\epsilon\)-perfect equilibria which does not hold for \(\epsilon\)-perfect equilibria.
4.3 RELATIONS WITH OTHER REFINEMENTS

In this section we use equalized games to show relations between \( e \)-proper and persistent equilibria. Further we show a relation between weakly \( e \)-perfect equilibria and minimal strictly perfect sets.

Persistent equilibria were introduced by Kalai and Samet (1984). These are equilibria contained in a persistent retract which, in turn, is a minimal absorbing retract. For definitions of these notions we refer to section 3.3. Here we include two results from Kalai and Samet.

**Lemma 4.3.1** [Kalai and Samet (1984)]

(i) Every absorbing retract contains a persistent retract.

(ii) Every persistent retract contains a proper equilibrium.

For our purposes we also need

**Lemma 4.3.2** Let \((A, B)\) be an \(m \times n\) bimatrix game and let \(f : \Delta_m \times \Delta_n \rightarrow \Delta_{m'} \times \Delta_{n'}\) be defined as in (4.2.2). Let \(R \subseteq \overline{\Delta}_m \times \overline{\Delta}_n\) be an absorbing retract for \((A, B)\). Then \(f(R)\) is an absorbing retract for \((\overline{A}, \overline{B})\).

**Proof.** It is easily checked that \(f(R) = T_1 \times T_2\) for two convex and closed sets \(T_1 \subseteq \Delta_{m'}\) and \(T_2 \subseteq \Delta_{n'}\). Since \(R\) is an absorbing retract for \((A, B)\) there exists an open neighbourhood \(V \subseteq \Delta_m \times \Delta_n\), \(V \supseteq R\), such that for all \((p, q) \in V\) there exists a pair \((\bar{p}, \bar{q}) \in R\) with \(C(\bar{p}) \subseteq PB(A, q)\) and \(C(\bar{q}) \subseteq PB(B, p)\).

Define \(\overline{V} := V \cap (\overline{\Delta}_m \times \overline{\Delta}_n)\). Then \(f(\overline{V}) = \overline{f(V)}\) is an open set in \(\Delta_{m'} \times \Delta_{n'}\) because \(\overline{f}^{-1}\) is continuous. Since \(\overline{V} \supseteq R\) we have \(f(\overline{V}) \supseteq f(R)\). Let \((x, y) \in f(\overline{V})\). Defining \((p, q) := \overline{f}^{-1}(x, y) \in \overline{V}\), there exists a pair \((\bar{p}, \bar{q}) \in R\) such that \(C(\bar{p}) \subseteq PB(A, q)\) and \(C(\bar{q}) \subseteq PB(B, p)\).

With \((\bar{x}, \bar{y}) := f(\bar{p}, \bar{q}) \in f(R)\) this implies (cf. (4.2.4)) that \(C(\bar{x}) \subseteq PB(\overline{A}, y)\) and \(C(y) \subseteq PB(\overline{B}, x)\). Hence, \(R\) is an absorbing retract for \((\overline{A}, \overline{B})\). \(\square\)

Now we can provide a new proof for the following theorem of García Jurado (1989).

**Theorem 4.3.3** For a bimatrix game there is a persistent retract which contains an \(e\)-proper equilibrium.

**Proof.** Let \((A, B)\) be an \(m \times n\) bimatrix game. Let \(R := \overline{\Delta}_m \times \overline{\Delta}_n\). Clearly \(R\) is an absorbing retract for \((A, B)\). Using lemma 4.3.1(i), \(R\) contains a persistent retract \(P\) for \((A, B)\) and so, by lemma 4.3.2, \(f(P)\) is an absorbing retract for the equalized game \((\overline{A}, \overline{B})\). Lemma 4.3.1(ii) implies that \(f(P)\) contains a proper equilibrium \((x, y)\) for \((\overline{A}, \overline{B})\). Since \(P \subseteq \overline{\Delta}_m \times \overline{\Delta}_n\) we have that \(f^{-1}(x, y) \in P\) and, by theorem 4.2.2,
$\mathcal{F}^{-1}(x, y)$ is an $\varepsilon$-proper equilibrium for $(A, B)$.  

In some cases theorem 4.3.3 helps to find a proper and persistent equilibrium. For example in

**Example 4.3.1** Consider the $3 \times 3$ bimatrix game

\[
(A, B) := \begin{bmatrix}
(1,1) & (0,2) & (2,2) \\
(1,3) & (0,0) & (2,0) \\
(1,0) & (2,1) & (1,1)
\end{bmatrix}.
\]

Then

\[
(A, B) = \begin{bmatrix}
(1,2) & (1,1) \\
(1,0) & (\frac{3}{2},1)
\end{bmatrix}
\]

and the only undominated and hence only perfect equilibrium (cf. proposition 3.4) for this game is $(e_2, e_2)$. So the unique $\varepsilon$-perfect and $\varepsilon$-proper equilibrium for $(A, B)$ is $(e_3, \frac{1}{2}e_2 + \frac{1}{2}e_3)$. By theorem 4.3.3 the uniqueness of this equilibrium implies that it is persistent.

A direct consequence of theorem 4.3.3 is

**Corollary 4.3.4** Each bimatrix game has a persistent equilibrium which is also $\varepsilon$-proper.

The following example shows there need not be a persistent retract which contains an iterated $\varepsilon$-proper, or even iterated $\varepsilon$-perfect equilibrium.

**Example 4.3.2** Consider the $2 \times 3$ bimatrix game $(A, B)$ of example 4.2.3. The unique iterated $\varepsilon$-proper and unique iterated $\varepsilon$-perfect equilibrium for this game is $(\frac{1}{2}e_1 + \frac{1}{2}e_2, \frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{3}e_3)$, whereas the persistent retracts for $(A, B)$ are the sets

\[
\text{conv } \{e_2, \frac{1}{2}e_1 + \frac{1}{2}e_2 \} \times \{e_2\} \setminus \{(\frac{1}{2}e_1 + \frac{1}{2}e_2, e_2)\}
\]

\[
\text{conv } \{e_1, \frac{1}{2}e_1 + \frac{1}{2}e_2 \} \times \{e_1\} \setminus \{(\frac{1}{2}e_1 + \frac{1}{2}e_2, e_2)\}
\]

and

\[
\{\frac{1}{2}e_1 + \frac{1}{2}e_2\} \times \text{conv } \{e_1, e_2\}.
\]

Garcia Jurado (1989) showed that every persistent retract contains a weakly $\varepsilon$-proper equilibrium.

We now turn to examining the relations with minimal strictly perfect sets. Lemma 4.3.1(ii) and theorem 3.4.4 may give hope for the following result to be true: 'Every minimal strictly perfect set contains a proper equilibrium'. The following example shows that this is not the case.
**Example 4.3.3** Consider the $2 \times 3$ bimatrix game
\[
(A, B) = \begin{bmatrix}
(2, -4) & (0, 2) & (0, 4) \\
(0, 4) & (0, 2) & (2, -4)
\end{bmatrix}.
\]
Then $(\frac{1}{4}e_1 + \frac{1}{3}e_2, e_2)$ is the only proper equilibrium for $(A, B)$ and the unique minimal strictly perfect set is $\{(\frac{1}{4}e_1 + \frac{3}{4}e_2, e_2), (\frac{3}{4}e_1 + \frac{1}{4}e_2, e_2)\}$. This example also rules out a nice relation between $\epsilon$-proper, iterated and weakly $\epsilon$-proper equilibria on one side and persistent equilibria on the other side. However, in section 3.2 we already noted that every equilibrium contained in a minimal strictly perfect set is perfect, so that one may wonder whether every minimal strictly perfect set contains an $\epsilon$-perfect equilibrium. The following example gives a negative answer.

**Example 4.3.4** Consider the $2 \times 5$ bimatrix game
\[
(A, B) = \begin{bmatrix}
(1, 1) & (1, 3) & (0, 4) & (6, 2) & (0, 2) \\
(0, 3) & (0, 1) & (0, -2) & (2, 0) & (4, 2)
\end{bmatrix}.
\]
Then $E(A, B)$ is the union of the maximal Nash subsets $T_1 := \{\frac{1}{4}e_1 + \frac{1}{2}e_2\} \times \text{conv} \{\frac{1}{4}e_1 + \frac{1}{2}e_5, \frac{1}{2}e_2 + \frac{1}{2}e_5, \frac{3}{2}e_4 + \frac{3}{2}e_5\}$ and $T_2 := \text{conv} \{\frac{3}{4}e_1 + \frac{1}{2}e_1\} \times \{e_3\}$.
All minimal strictly perfect sets consist of one point and lie within $T_1$. Since $(A, B)$ lies in $T_2$.

In order to prove a relation between weakly $\epsilon$-perfect equilibria and minimal strictly perfect sets we first show that there is an alternative definition for these equilibria in terms of mistake vectors.

**Lemma 4.3.5** Let $(A, B)$ be an $m \times n$ bimatrix game. Then $(\hat{p}, \hat{q})$ is a weakly $\epsilon$-perfect equilibrium for $(A, B)$ if and only if there are a sequence of mistake vectors $\{(\epsilon^k, \delta^k)\}_{k \in \mathbb{N}}$ converging to zero and a sequence $\{(h^k, q^k)\}_{k \in \mathbb{N}}$ converging to $(\hat{p}, \hat{q})$ such that, for every $k \in \mathbb{N}$, $(h^k, q^k) \in E(A(\delta^k), B(\epsilon^k))$ and moreover for $i, r \in M$ and $j, s \in N$
\[
\begin{align*}
\epsilon_{i}A & = \epsilon_{r}A \Rightarrow \epsilon_{i}^{k} = \epsilon_{r}^{k} \\
B\epsilon_{j} & = B\epsilon_{s} \Rightarrow \delta_{j}^{k} = \delta_{s}^{k}.
\end{align*}
\]

**Proof.** (a) Suppose $(\hat{p}, \hat{q})$ is a weakly $\epsilon$-perfect equilibrium for $(A, B)$. Then there is a sequence $\{\eta^k\}_{k \in \mathbb{N}}$ of positive real numbers converging to zero and a sequence
{((\tilde{p}^k, \tilde{q}^k))_{k \in \mathbb{N}} \subset \tilde{\Delta}_m \times \tilde{\Delta}_n} converging to \((\check{p}, \check{q})\) such that, for every \(k \in \mathbb{N}\), \((\tilde{p}^k, \tilde{q}^k)\) is \(\eta^k\)-perfect and weakly equalized. For \(k \in \mathbb{N}\) define \((\check{e}^k, \check{\delta}^k) \in \mathbb{R}^m \times \mathbb{R}^n\) by

\[
\check{e}^k_i := \begin{cases} \tilde{p}^k_i & \text{if } i \not\in PB(A, \tilde{q}^k) \\ \min_{r \in M} \tilde{p}^k_r & \text{if } i \in PB(A, \tilde{q}^k) 
\end{cases}
\]

and

\[
\check{\delta}^k_j := \begin{cases} \tilde{q}^k_j & \text{if } j \not\in PB(B, \tilde{p}^k) \\ \min_{s \in N} \tilde{q}^k_s & \text{if } j \in PB(B, \tilde{p}^k) 
\end{cases}
\]

Then the \(\eta^k\)-perfectness of \((\tilde{p}^k, \tilde{q}^k)\) implies that \((\check{e}^k, \check{\delta}^k)\) is a mistake vector for large \(k\) and that \({((\check{e}^k, \check{\delta}^k))_{k \in \mathbb{N}}}\) converges to zero. Moreover, since \((\tilde{p}^k, \tilde{q}^k)\) is also weakly equalized for every \(k \in \mathbb{N}\), we find that \(\check{e}^k_i = \check{e}^k\) whenever \(\epsilon_i A = \epsilon_i A\) for \(i, r \in M\) and \(\check{\delta}^k_j = \check{\delta}^k\) whenever \(B \epsilon_j = B \epsilon_j\) for \(j, s \in N\). So we are left with the construction of a sequence \({((\tilde{p}^k, \tilde{q}^k))_{k \in \mathbb{N}}}\) such that \((\tilde{p}^k, \tilde{q}^k) \in E(A(\check{\delta}^k), B(\check{\delta}^k))\) (where \((A(\check{\delta}^k))\) and \(B(\check{\delta}^k)\) are defined as in section 3.2) for every \(k \in \mathbb{N}\) which converges to \((\check{p}, \check{q})\). In order to achieve this we define, for \(k \in \mathbb{N}\), \((\tilde{p}^k, \tilde{q}^k)\) by

\[
\tilde{p}^k := (1 - \sum_r \check{e}^k_r)^{-1}(\check{p}^k - \check{e}^k)
\]

and

\[
\tilde{q}^k := (1 - \sum_s \check{\delta}^k_s)^{-1}(\check{q}^k - \check{\delta}^k).
\]

Clearly \({((\tilde{p}^k, \tilde{q}^k))_{k \in \mathbb{N}}}\) is a subset of \(\Delta_m \times \Delta_n\) and converges to \((\check{p}, \check{q})\). Let \(k \in \mathbb{N}\) and take \(i \in C(\tilde{p}^k)\). Then \(\tilde{p}^k_i > 0\), so that, by definition of \(\tilde{p}^k\), \(\tilde{p}^k_i > \check{e}^k_i\). By the definition of \(\check{e}^k\) this implies that \(i \in PB(A, \tilde{q}^k)\). One easily checks that the definition of \(\tilde{q}^k\) implies that \(PB(A(\check{\delta}^k), \tilde{q}^k) = PB(A, \tilde{q}^k)\). Hence we find \(i \in PB(A(\check{\delta}^k), \tilde{q}^k)\). So we have proved that \(C(\tilde{p}^k) \subset PB(A(\check{\delta}^k), \tilde{q}^k)\). Similarly one shows that \(C(\tilde{q}^k) \subset PB(B(\check{\delta}^k), \tilde{p}^k)\).

(b) Now suppose that we can find sequences \({((\check{e}^k, \check{\delta}^k))_{k \in \mathbb{N}}}\) of mistake vectors converging to zero and \({((\tilde{p}^k, \tilde{q}^k))_{k \in \mathbb{N}}}\) satisfying all conditions in the theorem. For \(k \in \mathbb{N}\) let \(\eta^k := \max\{\max_{i \in M} \check{e}^k_i, \max_{j \in N} \check{\delta}^k_j\}\). Then \({\eta^k}_{k \in \mathbb{N}}\) is a sequence of positive real numbers converging to zero. Moreover, for \(k \in \mathbb{N}\), let \(p^k := p^k(\check{e}^k)\) and \(q^k := q^k(\check{\delta}^k)\). Then \({((p^k, q^k))_{k \in \mathbb{N}}} \subset \Delta_m \times \Delta_n\) converges to \((\check{p}, \check{q})\).

Now let \(k \in \mathbb{N}\). Suppose for \(i, r \in M\) we have \(\epsilon_i A q^k < \epsilon_r A q^k\). By the definition of \(q^k\) this is equivalent to \(\epsilon_i A(\check{\delta}^k)q^k < \epsilon_r A(\check{\delta}^k)q^k\). Hence \(p^k_i = 0\), so that \(p^k_i = p^k(\check{e}^k)i = \check{e}^k_i \leq \eta^k\). Similarly if \(\epsilon_i A q^k = \epsilon_r A q^k\) and \(i, r \not\in PB(A, q^k)\), then \(p^k_i = e^k_i = \check{e}^k_i = p^k_i\).

The other implications needed for the \(\eta^k\)-perfectness of \((\tilde{p}^k, \tilde{q}^k)\) follow similarly.

**Theorem 4.3.6** For a bimatrix game every minimal strictly perfect set contains a weakly e-perfect equilibrium.
**Proof.** Let \((A, B)\) be an \(m \times n\) bimatrix game and let \(S\) be minimal strictly perfect set for \((A, B)\). We can construct a sequence \(\{(e^k, \delta^k)\}_{k \in \mathbb{N}}\) of mistake vectors converging to zero such that for \(k \in \mathbb{N}\) for \(i, r \in M\) and \(j, s \in N\)

\[
e^i_A = e^r_A \Rightarrow e^k_i = e^k_r \quad \text{and} \quad B e^j = B e^s \Rightarrow \delta^k_j = \delta^k_s.
\]

Since \(\Delta_m \times \Delta_n\) is compact, any sequence \(\{(p^k, q^k)\}_{k \in \mathbb{N}}\) such that \((p^k, q^k) \in E(A(\delta^k), B(e^k))\) for every \(k \in \mathbb{N}\) has a limit point in \(\Delta_m \times \Delta_n\). By lemma 4.3.5 such a limit point is a weakly \(\epsilon\)-perfect equilibrium and since \(S\) is a minimal strictly perfect set, at least one of these limit points is an element of \(S\).

\[\square\]

### 4.4 The Structure of the Sets of \(\epsilon\)-Perfect, Iterated and Weakly \(\epsilon\)-Perfect Equilibria

We end this chapter with some remarks on structure problems. With respect to the structure of the set of \(\epsilon\)-perfect equilibria we obtain from theorem 4.2.2(i) and theorem 3.5.5 the following result.

**Theorem 4.1.1** The set of \(\epsilon\)-perfect equilibria for a bimatrix game is the union of finitely many polytopes.

The reader should note that the set of \(\epsilon\)-perfect equilibria is generally not the finite union of faces of maximal Nash subsets. Examples confirming this are e.g. examples 4.1.1, 4.2.1 and 4.3.1. It will be clear that theorem 4.4.1 also holds if ‘\(\epsilon\)-perfect’ is replaced by ‘iterated \(\epsilon\)-perfect’. The set of iterated \(\epsilon\)-perfect equilibria is contained in the set of \(\epsilon\)-perfect equilibria.

The set of \(\epsilon\)-perfect equilibria is contained in the set of weakly \(\epsilon\)-perfect equilibria. The latter set has a structure that resembles the structure of the set of perfect equilibria. Using lemma 4.3.5 and following, mutatis mutandis, the proof of theorem 3.5.5 one can show the following result.

**Theorem 4.3.9** The set of weakly \(\epsilon\)-perfect equilibria is the union of finitely many faces of maximal Nash subsets.

A consequence of theorem 4.3.9 is that, since \(\epsilon\)-perfectness implies weakly \(\epsilon\)-perfectness, all equilibria in a face of a maximal Nash subset of which the relative interior contains an \(\epsilon\)-perfect equilibrium, are weakly \(\epsilon\)-perfect.