On the linear and non-linear evolution of dust density perturbations with MOND
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Conclusions

Several questions were raised in the introduction of this thesis related to cosmological evolution with MOND. This chapter summarizes the answer that this work gives to these questions and an outlook for future research. The main goal of the thesis was to test whether MOND can reproduce the success of the standard model in the context of post-recombination cosmological evolution, especially in the non-linear regime. The very few works that exist in this context (Nusser 2002; Knebe & Gibson 2004) show pessimistic results: the MOND effect is too strong on large scales. The consequence of this is that MOND predicts a normalization of the power spectrum at \( z = 0 \) which is too large. Nevertheless, these works assume as valid a set of untested hypotheses that could bias the results. The work by Nusser is based on the following approximations:

- It was assumed that an homogeneous and isotropic universe does not experience MOND effects. Thus, the expansion of the universe was taken as dictated by standard gravity.
- Gas dynamics and radiative cooling were neglected.
- The calculation of the MONDian forces was made through the non-conservative simple formula and hence curl field effects were neglected.
- The \( \mu \) function and \( a_0 \) were assumed as constant in space and time.
- The accelerations at high redshift were assumed to be in the Newtonian regime, which permits one to calculate the initial conditions using Newtonian linear theory.

Similar assumptions were made in Knebe & Gibson (2004), with the addition that a small amount of dark matter was included when generating the initial power spectrum. This gives a power law form to the spectrum and helps to generate the small structure that is not present in a purely baryonic universe.

In order to know if the problems found in these works are real or just a product of biases introduced by an oversimplified analysis, the underlying assumptions must be carefully check and relaxed if necessary. In first place, the simple MOND formula previously used, was replaced during this thesis by conservative versions of MOND.
Three different implementations were studied: the standard conservative version defined by the Bekenstein-Milgrom equation (Bekenstein & Milgrom 1984), the quMOND theory (Milgrom 2010b) and the twin matter prescription recently proposed in Milgrom (2010a).

The derivation of the cosmological field equations was made through a set of standard Newtonian arguments which is based on a change of coordinates to the so-called comoving frame and the “Jeans swindle”. The method is known to be valid in standard cosmology for regions smaller than the horizon. One of the strongest assumptions of this thesis is that this is also the case for MONDified theories. The standard mathematical description of the “Jeans swindle” is based on the linearity of the Poisson’s equation. As MOND is defined through a non-linear field equation, subtle differences with respect to the standard derivation had to be included. The expansion of the universe was assumed, as in the previous literature and with strong motivation from the theoretical side, to be given by the standard FRW metric.

Chapter 2 was focused on the effects that the curl field, responsible for the conservative properties of the MOND equation, has on the non-linear evolution. Instead of calculating the MONDian forces through the simple non-conservative MOND formula (as previous authors did), an new implementation of the solution of the Bekenstein-Milgrom (BM) equation was employed. With the aim of tackling only one problem at a time, the initial conditions for the simulations presented in this chapter were taken from the work by Knebe & Gibson (2004), which was taken also as a reference point for comparing results. The overall conclusion of the chapter is that the curl field drives the formation of structure on large scales and hence, to take into account its effects does not improve the result by Nusser (2002) that structure grows too rapidly.

Following the idea of improving the treatment of MONDian cosmology presented in previous works, chapter 5 was focused on the generation of the initial conditions for cosmological simulations. In first place, the assumption that the accelerations at high redshift are in the Newtonian regime was tested in an accurate way by solving the quMOND equation on a cubic box. Different cosmological models were studied and convergence tests related to the size of the box and the resolution were performed. The conclusion is that the accelerations that are present at the early universe are not high enough and that, even at decoupling, there are always pockets where the forces are in the MONDian regime. Thus, MONDian physics must be taken into account to generate the initial conditions for non-linear simulations.

Under the light of this new results, the same chapter presented a method to generate initial conditions for N-body MONDian cosmological simulations. The method does not rely on a particular linear theory, but can be used with any kind of generalized growth equations; the information about the underlying linear theory is given through a set of power spectra at different redshifts. The method is based on the idea of starting the N-body simulation at a redshift close to recombination, using overdensity realizations of the input power spectra as source of the gravitational field equation. The method was tested with satisfactory results with Newtonian gravity, but its performance drops when it is applied to MOND in the context of the linear theory by Sanders (2001). The common feature is that the particles can recover the power spectrum given by Sanders on large scales, but there is always an underestimation at small scales. After discussing possible biases, simulations were run using these initial conditions. The results are comparable to those in Nusser (2002): there is an overprediction of the power on large scales. In addition to this, a new problem appeared. For first time, Silk dumping was taken into
account in the generation of the initial power spectrum. This gives a spectrum much steeper that the power law form used in previous works on non-linear evolution. Under these conditions, it was found that MOND can not bring the power on small scales to the observed levels at redshift \( z = 0 \). In short, **MOND over-predicts the power spectrum on large scales at redshift \( z = 0 \) and at the same time, when Silk dumping is taken into account to obtain the initial power spectrum at decoupling, it is unable to make the small scales to collapse.**

The possibility that the acceleration constant \( a_0 \) has a dependence with time was also investigated in this thesis. There are two motivations for this. In first place, to assume smaller \( a_0 \) values at early times can help to bring the accelerations at decoupling well into the Newtonian regime and thus, it permits the use of standard codes to calculate the initial power spectrum. Secondly, such a time dependence could help in solving the difficulties found on large scales by delaying the appearance of MOND effects. It was found that, in effect, there is an improvement of the behaviour of the theory on large scales, but at the same time, the problem on small scales is enhanced. **It is more difficult for MOND to form small-scale structure in the case of a time dependent \( a_0 \).**

In an effort to improve the performance of the method to generate the initial conditions and also to verify the viability of spherical collapse models in MOND, a self-consistent method to solve the standard growth equation coupled to the MOND field equation was proposed. The method relies on the fact that the growth equation is formally equivalent to the linearized geodesic equation and hence, it can be solved by means of a leap-frog integration scheme. The method proposed here uses a cubic box and solves the equations in a self consistent way, meaning that external field and curl field effects are taken into account as well as the coupling between different modes. Using spherical collapse techniques, Nusser (2002) found that the standard growth equation gives a power spectrum with a logarithmic slope of -1 when it is used in combination with MOND. The solution found with the new method is surprisingly different: the linear power spectrum develops a power law with a logarithmic slope of \(-4\). The solution was found to be very stable with respect to changes in the time dependence of \( a_0 \), resolution and box size. The reason for this difference in the solution is that the external field produced by the environment, which is taken into account only in the self consistent solution, decreases the MOND effect and hence gives a steeper power law. **In conclusion, spherical collapse models are not valid in MOND. The environment plays a very important role in MONDian cosmology and hence it must be taken into account when studying linear evolution. Furthermore, the linear evolution is not self similar in MOND.**

When generating initial conditions with this new linear solutions, a better agreement was found between the power spectrum given by the particles and the numerical solution of the growth equation. Owing to the steepness of the linear power spectrum, the problem found with the small scales at redshift \( z = 0 \) is actually greater. In other words, the better the calculations are made, the more serious is the disagreement with observations.

The introduction also proposed to study the performance of MOND when it is applied to the weak points of the ΛCDM model. This analysis is based on the data obtained from the simulations presented in chapter 2. The initial conditions for these simulations were calculated using Newtonian gravity and specially designed to reach the observed normalization at redshift \( z = 0 \). Naturally, for this to happens, an unrealistic power spectrum must be used at the initial redshift with respect to both its functional form and normalization. While this way of setting up the simulations seems unphysical, it
makes it possible to obtain a density distribution that is similar to the observed one at redshift $z = 0$, but that, at the same time, includes MONDian ingredients in its evolution. Through this kind of simulations, it was found that MOND performs better than $\Lambda$CDM in the context of collisional velocity of objects and the void phenomenon. Details on these results can be found in section 2.4 and chapter 3.

In order to find which of the two gravitational models studied in this thesis (Newtonian and MONDian gravity) gives a better representation of reality, it is important to find those observables quantities for which both theories give different predictions. The introduction stated this point as one of the goals of the thesis. The results presented in last paragraph gives one contribution in this direction. Another large difference that was found between the two theories is that the MONDian dynamical mass is not bounded on the lower side. Indeed, section 4.1, shows that the phantom density perturbations in voids can reach negative values as smaller as 10 times the mean baryonic density. To find through gravitational lensing such extreme under-dense regions with negative convergence will give an unmistakable signal of MOND.

The introduction also proposed to study the capability of MOND to explain the decoupling between peaks in visible matter and lensing signal that is observed in a number of clusters of galaxies. The standard explanation of this phenomenon invokes dark matter as responsible for the behavior of the lensing signal and a collision between clusters as responsible for the decoupling between baryonic and dark matter components. An alternative explanation from the MONDian side, which does not require dark matter, assumes the decoupling as a consequence of the non-linearity of the MOND equation. In chapter 4.2, it was proven that the effect may be present in idealistic situations, but the signal was not found in the simulation data. Many possible biases responsible for washing out the signal were discussed. The experiment should be repeated with better simulation data before rejecting the explanation. The peculiar lensing behavior of the Bullet Cluster remains an open question for MOND, although the high velocity of the cluster collision is problematic for $\Lambda$CDM.

Regarding improvements in the techniques made in this thesis, it was mentioned above that this thesis has proposed methods to generate initial conditions and to solve a particular class of generalized growth equations in a self consistent way. In addition to this, the multigrid techniques applied to the MOND equation were also refined. The success of the multigrid method to obtain solutions in a fast way is very sensitive to the quality of the initial guess which is employed for the iterations. A simple but effective method to obtain a powerful initial guess was proposed. Also, effective discretization formulas were presented for the calculation of the phantom term.

7.1 Discussion and outlook

The results presented in this thesis are sufficiently detailed to show that a Newtonian treatment (meaning cosmological field equations derived from the equations given on a Minkowsky background) of the original prescription for MOND can not explain structure formation. Many paths are open for future work. One can assume that the Newtonian treatment is a good approximation and that the problem is indeed in the theory. Appendix 6.A shows that, while very speculative, this could indeed be a valid route: non standard definitions of the $\mu$ function, with Newtonian regimes at high and low accel-
7.1: Discussion and outlook

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The main uncovered point of this thesis is the influence of gas and radiative physics on the cosmological evolution, especially of pressure and cooling effects. As the theory is non-linear, it is difficult to predict the consequences of including these two effects on the evolution (see for instance, the large difference found between the linear solutions obtained using spherical and self consistent analysis presented in section 5.2.4 as an example of the importance of non-linear MONDian effects). Self consistent simulations including these two effects must be run before accepting that these approximations are not the source of the problems. Nevertheless, the pressure is a local effect and hence, it is difficult to imagine how it can help to stop the extreme collapse found on large scales. Regarding cooling, it could help to solve the problems on small scales by decreasing the pressure and giving a stronger collapse.

Pre-recombination MONDian evolution should be also studied to reject the possibility that the problems found in this thesis originated because of ignoring MOND during that epoch. Chapter 5 has shown that the distribution of accelerations at high redshift presents always a tail towards MONDian accelerations. Furthermore, section 5.2.4, proved that this tail is enough to change very quickly the form of the power spectrum at high frequencies giving a power law form to an initially exponential spectrum (similar results were found by Sanders (2001)). If this apparently inoffensive tail in the distribution of accelerations has such a large effect on the evolution, one should ask the following question: can MOND act against the Silk dumping during the radiation dominated era and give a power law form to the power spectrum at decoupling even in the absence of dark matter? Self consistent linear simulations must be made in the radiation dominated era to answer this question.

This thesis also opens the door to changes in the observational techniques. For instance, chapter 2 gave indications that the interpretation of the internal dynamics of galaxies which is inferred from rotation curves could be biased when neglecting the environment in the context of the fully non-linear field equation. Predictions of rotation curves are made assuming that the simple MOND formula is valid, but this is only true in cases of high symmetry. Therefore, the observational results could be biased by asymmetries arising in the environment. This should not be considered as a problem, but in fact as a way to test the theory. Galaxies situated in the surroundings of clusters...
of galaxies should show the existence of a preference direction given by the location of the cluster. To confirm this unmistakable signal of MOND, the analysis presented in chapter 2 should be repeated including convergence tests for the size of the box (section 5.1.4 shows that this is an important quantity when calculating MONDian forces) and determining the effects of the asymmetries in the actual form of the rotation curves.

The time dependence in $a_0$ must be also studied from observations. The possibility of non-standard forms for the $\mu$ function must be also studied (especially in the very low accelerations regime), as well as the possibility that this is in fact a dynamical field, whose form is a function of space and time. This considerations require a specific relativistic theory to provide a context.

Further studies must be made with neutrinos. While they provide very convincing solutions to clusters and cmb problems, their effects on cosmological evolution is still and open question. Since the background expansion given by the presence of neutrinos is closer to the $\Lambda$CDM expansion, neutrinos will help to match the evolution at very large scales and prevent the extreme collapse that was found. Nevertheless, preliminary calculations made in this thesis show that while neutrinos give a correction in the right direction, it is not sufficient to avoid the problems. The problems are not in the expansion, but in the fact that giga parsec scales reach the MOND regime before redshift $z = 0$ and hence, give too much power at $k \lesssim 0.1$.

It is clear from the above discussion that this thesis represents only a beginning in the study of structure formation with MOND. There remains much work to do in many topics from both the relativistic side, to determine which are the equations that we really want to solve and the numerical side to propose methods to solve them. The impressive performance of the theory on galactic scales should be enough motivation to make the effort.
Appendix A

Calculation of the initial guess for non-linear MONDian solvers

An important issue that is present when solving non-linear partial differential equations by means of multigrid methods is related to the initial guess used for the iterations. In order to accelerate the convergence and to ensure that the final solution is the right one, it is important to set a proper initial guess for the potential. One possible method, which is known as the continuation method Brandt (1977), consist in solving the equation by changing a parameter of the theory that makes the equation to go for instance, from a linear equation to the non-linear version that one needs to solve. A different method is proposed here. In the case of the MOND equation (eq.1.64), there is a very good approximation for the solution, which is given by the solution of the simple version of MOND (eq.1.60). This equations can be easily solved using a Fourier method in the following way: in first place, the Newtonian potential $\phi_N$ is calculated by solving the Poisson’s equation (eq.1.52) with, for instance, a Fourier method. From this potential, one can compute the Laplacian of the simple MONDian forces as described in the Appendix B. This will give the source $\delta_{\text{simple}}$ of a new Poisson’s equation whose solution is the MONDian potential $\phi_{\text{simple}}$ in its simple version, which can be solved again with a Fourier method. Finally, this potential is the one that should be used as input for the multigrid method. Figure A.1 schematizes the complete procedure to go from the original over-density $\delta$ to the Bekenstein-Milgrom MONDian potential $\phi$.

During this thesis, the Poisson’s equation was solved using a standard Fourier method whose description can be found for instance in Hockney & Eastwood (1988). The Green’s function employed in the code Solve described in section 5.2.2 is given by:

$$G(k) = -\frac{B}{2n} \sin \left( \frac{B}{2n} \frac{2\pi}{B} \right).$$  \hspace{1cm} (A.1)

The FFTs were calculated with the open source code FFTW (Frigo & Johnson 2005).
Figure A.1: Complete method for the determination of the Bekenstein-Milgrom MONDian potential, including the calculation of the initial guess.
Appendix B

Laplacian of the simple MONDian forces from the Newtonian potential

As described in Appendix A, the Laplacian of the simple MONDian forces $\nabla^2 \phi_{\text{simple}}$ must be computed on a grid in order to calculate the initial guess for the solution of the Bekenstein-Milgrom equation (eq.2.20). The same quantity is also required to actualize the density on the leap-frog scheme described in section 5.2 in case that the simple version of MOND is employed. The most direct way to obtain this quantity on a given grid point is to calculate the simple MONDian forces by inverting eq.1.60 and to take its divergence by means of a discretization formula. The procedure could be appropriated to obtain initial guess for non-linear solvers, but as to make the derivative of a derivative on a given grid point implies the use of neighbors to neighbors, there is a lose of resolution that can have negative effects. For instance, experiments made with this algorithm to solve the growth equation as described in section 5.2.4, give an under-prediction of the power on small scales.

A more accurate way to calculate $\nabla^2 \phi_{\text{simple}}$ is by means of a formula that includes derivatives of only the Newtonian potential, without having the intermediate step of calculating the forces. The second derivatives of the simple MONDian potential can be written using eq.1.60 as:

$$\frac{\partial^2 \phi_{\text{simple}}}{\partial (x^i)^2} = \frac{\partial}{\partial x^i} \left[ g(|\nabla \phi_N|) \frac{\partial \phi_N}{\partial x^i} \frac{1}{|\nabla \phi_N|} \right],$$  \hspace{1cm} (B.1)

where the function $g$ is given by:

$$g(u) = \frac{u}{2} + \sqrt{aa_0u + \left(\frac{u}{2}\right)^2}$$  \hspace{1cm} (B.2)
for the simple \( \mu \) function. Making the derivative in equation B.1 gives:

\[
\frac{\partial^2 \phi_{\text{simple}}}{\partial x^j \partial x^i} = \frac{1}{\left| \nabla \phi_N \right|} \frac{\partial \phi_N}{\partial x^i} \frac{\partial \left| \nabla \phi_N \right|}{\partial x^j} \left( \frac{dg}{du} \right)_{\nabla \phi_N} - \frac{1}{\left| \nabla \phi_N \right|} \frac{\partial \phi_N}{\partial x^i} \frac{\partial \left| \nabla \phi_N \right|}{\partial x^j} \frac{1}{\left| \nabla \phi_N \right|} g(\left| \nabla \phi_N \right|) + \frac{1}{\left| \nabla \phi_N \right|} g(\left| \nabla \phi_N \right|) \frac{\partial^2 \phi_N}{\partial x^j \partial x^i}.
\] (B.3)

Furthermore, the derivative of the function \( g \) is:

\[
\frac{dg}{du} = \frac{1}{2} + \frac{1}{2} \frac{aa_0 + \frac{u}{2}}{\sqrt{aa_0 u + (\frac{u}{2})^2}}
\] (B.4)

and the one of the Newtonian accelerations is (only an example for the x coordinate is shown):

\[
\frac{\partial \left| \nabla \phi_N \right|}{\partial x} = \frac{1}{\left| \nabla \phi_N \right|} \left( \frac{\partial \phi_N}{\partial x} \frac{\partial^2 \phi_N}{\partial x^2} + \frac{\partial \phi_N}{\partial x} \frac{\partial^2 \phi_N}{\partial x \partial y} + \frac{\partial \phi_N}{\partial y} \frac{\partial^2 \phi_N}{\partial x \partial z} \right)
\] (B.5)

Putting everything together, it is possible to write down the Laplacian of the MONDian potential on a given grid point, using only derivatives of the Newtonian potential at the same point, so everything can be calculated using just a stencil of 27 points. The derivatives of the Newtonian potential can be computed, for instance, using second order formulas given by:

\[
\left( \frac{\partial \phi_N}{\partial x} \right)_{i,j,k} = \frac{(\phi_N)_{i+1,j,k} - (\phi_N)_{i-1,j,k}}{2h}
\] (B.6)

\[
\left( \frac{\partial^2 \phi_N}{\partial x^2} \right)_{i,j,k} = \frac{(\phi_N)_{i+1,j,k} + (\phi_N)_{i-1,j,k} - 2(\phi_N)_{i,j,k}}{h^2}
\]

\[
\left( \frac{\partial^2 \phi_N}{\partial x \partial y} \right)_{i,j,k} = \frac{1}{4h^2} [(\phi_N)_{i+1,j+1,k} - (\phi_N)_{i+1,j-1,k} - (\phi_N)_{i-1,j+1,k} + (\phi_N)_{i-1,j-1,k}]
\]

where we one example of each type of derivative is shown.
Calculation of the overdensity power spectrum from a particle distribution

For the sake of clarity regarding normalization conventions and approximations made in the calculation of the power spectrum, this Appendix summarizes the main points related to the way in which the power spectrum was calculated during this thesis.

Given a distribution of particles, the first thing to do in order to calculate the power spectrum is to estimate the overdensity field on a grid. This calculation was made using a TSC kernel (e.g. Hockney & Eastwood 1988). The expression for the value of the density \( \rho_{i,j,k} \) on the node \((i,j,k)\) is:

\[
\rho_{i,j,k} = \frac{1}{h^3} \sum_n W_1(x^1_{i,j,k} \! - \! x^1_n)W_2(x^2_{i,j,k} \! - \! x^2_n)W_3(x^3_{i,j,k} \! - \! x^3_n)m_n,
\]

where \( m_n \) is the mass of the particle \( n \), \( x_n \) is the position of that particle, \( x_{i,j,k} \) is the position of the node, \( h \) is the size of the node and \( W \) is the smoothing kernel, that in the TSC case is given by:

\[
W(x) = \begin{cases} 
\frac{3}{4} - \left( \frac{|x|}{h} \right)^2 & \frac{h}{2} \leq |x| \\
\frac{1}{2} \left( \frac{3}{2} - \frac{|x|}{h} \right)^2 & \frac{h}{2} \leq |x| \leq \frac{3h}{2} \\
0 & \text{otherwise.}
\end{cases}
\]

The CIC smoothing was also used during the thesis, which has a kernel given by:

\[
W(x) = \begin{cases} 
1 - \frac{|x|}{h} & 0 \leq |x| \leq h \\
0 & \text{otherwise.}
\end{cases}
\]

The index in the summation in eq.C.1 runs over all the particles in the box. The mass of the particles is calculated assuming that the box has mean density \( \Omega_0 \rho_c \).

The power spectrum is defined as the mean value of the modulus squared of the Fourier transform of the overdensity:

\[
P(k) = \langle |\delta|^2 \rangle,
\]
where \( \hat{\delta} \) is the Fourier transform of the overdensity, that was estimated using the discrete Fourier transform given by:

\[
Y[k_1, k_2, k_3] = \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} \sum_{j_3=0}^{n-1} \delta[j_1, j_2, j_3] e^{2\pi j_1 k_1 \sqrt{-1}/n} e^{2\pi j_2 k_2 \sqrt{-1}/n} e^{2\pi j_3 k_3 \sqrt{-1}/n},
\]

where \( n \) is the number of cells per dimension and \( \delta[j_1, j_2, j_3] \) is the overdensity in the cell \([j_1, j_2, j_3]\). The summations in this equation were made using the code FFTW (Frigo & Johnson 2005), which gives the result as written above, without normalization. The mean value in eq.C.4 is taken on spherical shells in Fourier space and the normalization, which was made at the end of the calculations, is given by:

\[
\text{norm} = \frac{B^3}{n^6}.
\]

Note that this is the normalization of the power spectrum, not the Fourier transform.

In order to correct for the discreteness effects of the particles, including the smoothing kernel that is used to estimate the overdensity field, the iterative method proposed in Jing (2005) was employed. As the method and the summations involved on it converge very quickly, only 1 to 2 iterations were used and 1 to 2 terms in the summations were taken into account.
Appendix D

The quMOND field equations in a cosmological context

A new formulation of MOND in its weak field limit was recently proposed by Milgrom (2010b). The field equations were derived from a classical lagrangian, so they have the same conserving properties that characterize the Bekenstein-Milgrom equation (eq.1.64). The difference with respect with the original formulation is that the original non-linear field equation is substituted by a set of two linear Poisson’s equations, where the sources are the real matter density and the phantom density:

\[ \nabla^2 \phi_N = 4\pi G \rho \]  
\[ \nabla^2 \phi = \nabla \cdot \left[ \nu \left( \frac{\left| \nabla \phi_N \right|}{a_0} \right) \nabla \phi_N \right] , \]  

where \( \phi_N \) is the standard Newtonian gravitational potential and \( \phi \) is the new MONDian potential that is responsible for the movement of matter. Note that the definition of the phantom term is different that the one presented in chapter 6, since here it is given in terms of the Newtonian potential. The advantages of this new equations are clear: as both are linear and only coupled through the source, standard FFT methods or even direct summation methods can be employed, with the consequent simplifications in the codes. The acronym of this new formulation is quMOND, which means quasi-linear MOND and make reference to the fact that the non-linearity is not present anymore in an explicit way in the operator of the field equation.

This equations were presented in Milgrom (2010b) as equations for perturbations over a Minkowsky metric. In order to be used in cosmology, the equations have to be translated into an expanding context. The methodology that will be employed to obtain this new equations is the same that was discussed in detail in section 2.1.1, where a cosmological version of the Bekenstein-Milgrom equation was derived. The same procedure was applied also in section 6.2 to derive equations in the context of the twin matter formalism. The background will be assumed to be unaffected by MOND and hence following an evolution dictated by the standard FRW equations. The equations for the pertubations on the
metric will be obtained by means of the following change of variables:

\[ x^i = \frac{r^i}{a}. \]  

As discussed before, to be able to use the new equations under periodic boundary conditions, the mean value of the sources must be zero. Taking this into account, the quMond equations become:

\[
\nabla^2 \phi_N = 4\pi G a^2 (\rho - \rho_0) \tag{D.4}
\]

\[
\nabla^2 \phi = \nabla \cdot \left[ \nu \left( \frac{|\nabla \phi_N|}{aa_0} \right) \nabla \phi_N \right] - \frac{1}{V} \int_V \nabla \cdot \left[ \nu \left( \frac{|\nabla \phi_N|}{aa_0} \right) \nabla \phi_N \right] d^3 x, \tag{D.5}
\]

where \( \rho_0 \) is the value of the unperturbed density and the integral in eq. D.5 is made over an arbitrary region of space, with a volume \( V \) large enough to see the effects of the cosmological principle.

Taking into account the usual way to write the source of the Poisson’s equation as a function of the overdensity (see section 2.1.1), the system can be written as:

\[
\nabla^2 \phi_N = \frac{3}{2} \Omega_0 H_0^2 \frac{\delta \rho}{a_0} \rho_0 \tag{D.6}
\]

\[
\nabla^2 \phi = \nabla \cdot \left[ \nu \left( \frac{|\nabla \phi_N|}{aa_0} \right) \nabla \phi_N \right] - \frac{1}{V} \int_V \nabla \cdot \left[ \nu \left( \frac{|\nabla \phi_N|}{aa_0} \right) \nabla \phi_N \right] d^3 x, \tag{D.7}
\]

which are the equations solved by the code presented in section 5.2.2 and that were used in section 5.1 to quantify the intensity of the gravitational field at high redshift. In this implementation, the integral in equation D.7 is approximated as a summation over all the nodes of the grid in which the potential \( \phi_N \) is given.