Appendix A

Harmonic analysis of the velocity field of a gas disk

A.1 Introduction

This appendix is a more elaborate version of the original appendix of the article of Schoenmakers, Franx and de Zeeuw (1997), Chapter 2 of this thesis. In this appendix we derive all the equations used in that article in order to determine the elongation of the orbits in a filled gas disk by expanding the velocity field of that disk in harmonics: \[ \sum_{n=0}^{k} c_n \cos(n\psi) + s_n \sin(n\psi). \] In that way, we find the terms \( c_0 \) (the systemic velocity), \( c_1 \) (the circular velocity), and all other, non-circular, terms. We want to interpret the non-circular terms in the context of the orbital dynamics of the gas. In order to do that we shall take the following approach:

- Derive the gas orbits in a perturbed potential, where the perturbation is of a general nature, i.e. the perturbation has a radially dependent amplitude and a radially dependent phase.
- Determine the corresponding velocity field.
- Calculate the velocity field as it would appear to an observer looking at arbitrary viewing angles at the velocity field.
- Deproject the velocity field under the assumption of a circular velocity field, i.e. calculate the functional form of the observed harmonics after a tilted-ring fit has been made to the velocity field as seen on the sky.

The subsequent interpretation of the measured harmonic terms is presented in Chapters 2 (elongation) and 4 (kinematic lopsidedness).
A.2 Orbits in a non-axisymmetric potential

The first step is to derive the gas orbits in a perturbed, non-axisymmetric, potential. We shall follow the treatment given in Binney & Tremaine (1987, BT hereafter), §3.3.3(a) very closely. Slightly different notation is used throughout the derivation to avoid confusion in later paragraphs, since some of the variable names used by BT will appear in the following sections with a different meaning.

We make the following convention for the notation of radial and time derivatives, respectively: \( x' \equiv \left( \frac{\partial x}{\partial R} \right)_{R_0} \) and \( y \equiv \frac{\partial y}{\partial t} \).

We start with defining a weakly non-axisymmetric potential in the rest frame of the galaxy in polar coordinates \((R, \theta)\):

\[
V(R, \theta, t) = V_0(R) + \sum_m V_m(R) \cos \{m[\theta - \Omega_{p.m} t - \phi_m(R)]\}. \tag{1}
\]

Here, \( V_0(R) \) represents the unperturbed potential and \( V_m(R) \cos \{m[\theta - \Omega_{p.m} t - \phi_m(R)]\} \) is the \( m \)-th harmonic component of the perturbation, assumed to be small compared to \( V_0(R) \). The phase of the perturbation as a function of radius is denoted as \( \phi_m(R) \). Each harmonic component of the perturbation rotates with its own angular speed \( \Omega_{p.m} \). If the perturbation has non-zero pattern speed, or \( \phi_m(R) \) is not constant as a function of radius, then the choice of the line \( \theta = 0 \) is arbitrary. We focus on the velocity field as generated by a potential perturbed by a single harmonic component. The general case, equation (1), can be obtained by adding the results for individual harmonic components, since coupling between different harmonic components can be shown to be of second order only by simply re-calculating the equations in the next sections using the full summation in equation (1) and determining coupling terms. For a single harmonic component, the potential in a frame that corotates with the potential perturbation can be written as

\[
V(R, \phi) = V_0(R) + V_m(R) \cos \{m[\phi - \phi_m(R)]\}, \tag{2}
\]

where \( \phi = \theta - \Omega_{p.m} t \) and \( R \) is now the radial coordinate of a point on the closed loop orbit with guiding centre on \((R_0, \phi_0)\).

Now we want to solve for all the possible orbits in this potential. In order to do so, we write down the equations of motion for point particles in a potential in polar coordinates in a frame that corotates with the potential:

\[
\ddot{R} - R \dot{\phi}^2 = -\frac{\partial V}{\partial R} + 2R \dot{\phi} \Omega_{p.m} + \Omega_{p.m}^2 R, \tag{3}
\]

\[
R \ddot{\phi} + 2 \dot{R} \dot{\phi} = -\frac{1}{R} \frac{\partial V}{\partial \phi} - 2 \dot{R} \Omega_{p.m}. \tag{4}
\]

Note that the major difference between the treatment given here and in BT is the addition of a radially dependent phase in the perturbation, written here as \( \phi_m(R) \). For now, we shall assume that \( \phi_m(R) \) is an explicit function of time: \( \phi_m(R) = \phi_m(R, t) \), but we will come back to this point of time dependence later.
A.2. Orbits in a non-axisymmetric potential

To simplify notation, define $\eta = m(\phi - \phi_m(R))$. The equations of motion (eq. [3] and [4]) become in the potential [2]

$$\ddot{R} - R\dot{\phi}^2 = -\frac{\partial V_0}{\partial R} - \frac{\partial V_m}{\partial R} \cos \eta - m V_m \sin \eta \frac{\partial \phi_m}{\partial R} + 2 R \dot{\phi} \Omega_{p,m} + \Omega_m^2 R,$$  
(5)

$$R\ddot{\phi} + 2 \dot{R} \dot{\phi} = \frac{m V_m}{R} \sin \eta - 2 \dot{R} \Omega_{p,m}.$$  
(6)

Now we want to divide the coordinates of the orbit into zero and first order terms, representing the unperturbed orbit and the perturbation respectively. Therefore we write the coordinates as:

$$R = R_0 + \delta R,$$
$$\phi = \phi_0 + \delta \phi,$$

where the subscript 0 denotes the unperturbed orbit and the $\delta$ denotes the perturbed part of the orbit. Substitute this in eq. [5] and [6]:

$$(R_0 + \delta R) - (R_0 + \delta R)(\phi + \delta \phi)^2 = -V_0' - V_m' \cos \eta - m V_m \sin \eta \phi_m +$$
$$2(R_0 + \delta R)(\phi_0 + \delta \phi)\Omega_{p,m} + \Omega_m^2 (R_0 + \delta R),$$

$$(R_0 + \delta R) - (\phi + \delta \phi) + 2(R_0 + \delta R)(\phi_0 + \delta \phi) = \frac{m V_m}{(R_0 + \delta R)} \sin \eta - 2(R_0 + \delta R)\Omega_{p,m}.$$  
(8)

The zeroth order equations simply become:

$$R_0 \ddot{\phi}_0^2 + 2 \Omega_{p,m} R_0 \dot{\phi}_0 + R_0 \Omega_{p,m}^2 = V_0',$$  
(9)

$$R_0 \ddot{\phi}_0 = 0.$$  
(10)

At this point, we define the circular frequency at $R$ in the potential $V_0$, cf. BT (3-111):

$$\Omega_0^2 \equiv \frac{V_0'}{R}.$$  

We see from eq. [10] that

$$\ddot{\phi}_0 = 0$$

Rewriting eq. [9] as $R_0(\dot{\phi}_0 + \Omega_{p,m})^2 = V_0'$, we see from the definition of $\Omega_0$ that

$$\dot{\phi}_0 = \Omega_0 - \Omega_{p,m},$$

the angular speed of the guiding centre $(R_0, \phi_0)$, BT (3-112). Now choose $t = 0$ such that

$$\phi_0 = (\Omega_0 - \Omega_{p,m})t.$$  

From eq. [7] and [8] and remembering that $\dot{R}_0 \equiv 0$ for a circular orbit, we find that the first order equations are:

$$\delta \ddot{R} = -V_m' \cos \eta - m V_m' \phi_m' \sin \eta + (\ddot{\phi}_0^2 + 2 \dot{\phi}_0 \Omega_{p,m} + \Omega_{p,m}^2 - V_0'')\delta R +$$
$$2(R_0 \Omega_{p,m} + 2 R_0 \dot{\phi}_0) \delta \phi,$$  
(11)
Substituting the zeroth order solutions for \( \dot{\phi}_0, \ddot{\phi}_0 \) in the first order equations, we get after re-arranging terms:

\[
\begin{align*}
\delta \ddot{R} &= -V'_m \cos \eta - m V_m \phi'_m \sin \eta + (\Omega_0^2 - V''_0) \delta R + 2 R_0 \Omega_0 \delta \dot{\phi}, \\
R_0 \delta \ddot{\phi} &= \frac{m V_m}{R_0} \sin \eta - 2 \Omega_0 \delta \dot{R}.
\end{align*}
\]  

(13)  

(14)

Integration with respect to \( t \) of equation [14] gives:

\[
2 \Omega_0 R_0 \delta \dot{\phi} + 4 \Omega_0^2 \delta R = \frac{-2 m V_m \Omega_0 \cos \eta}{R_0^2 (\Omega_0 - \Omega_p) - m \phi'_m (R)} + \text{const.}
\]

Now, we suppose that \( \phi_m (R) \neq \phi_m (R, t) \), so that \( \dot{\phi}_m (R) = 0 \), in other words, the phase of the potential does not change with time. This is a valid assumption, since the potential and its phase are supposed to be generated by the dark matter halo, which is not likely to change its phase-structure rapidly (note that rotation is allowed, this is handled by the pattern speed).

With this assumption we find:

\[
\delta \dot{\phi} = -\frac{2 \Omega_0 \delta R}{R_0} \cos \eta - \frac{V_m}{R_0^2 (\Omega_0 - \Omega_p)} + \text{const.}
\]

(15)

Substituting this in equation [13] gives:

\[
\delta \ddot{R} + \kappa_0^2 \delta R = \left[ -V'_m - \frac{2 \Omega_0 V_m}{R_0 (\Omega_0 - \Omega_p)} \right] \cos \eta - m V_m \phi'_m \sin \eta + \text{const},
\]

with \( \kappa_0^2 = 3 \Omega_0^2 + V''_0 \), which we recognize as the usual epicyclic frequency at \( R_0 \). The integration constant can be absorbed by applying a shift \( \delta R \to \delta R + \text{const.} \), as argued in BT. The general solution to this second order ordinary differential equation is:

\[
\delta R = \frac{-1}{\kappa_0^2 - m^2 (\Omega_0 - \Omega_p)^2} \left\{ \left[ \frac{2 \Omega_0 V_m}{R_0 (\Omega_0 - \Omega_p)} + V'_m \right] \cos \eta + m V_m \phi'_m \sin \eta \right\} + C_1 \cos(\kappa_0 t + \rho).
\]

(16)

We are only interested in closed orbits, since we are dealing with gas disks, which are dissipative. Non-closed orbits will cause shocks in the gas and will quickly disappear. In other words, we need to make the above solution for the orbit structure periodic. The only way to accomplish this periodicity is by setting \( C_1 = 0 \). In this case we find that the solution for closed loop orbits is

\[
\delta R = \frac{-1}{\kappa_0^2 - m^2 (\Omega_0 - \Omega_p)^2} \left\{ \left[ \frac{2 \Omega_0 V_m}{R_0 (\Omega_0 - \Omega_p)} + V'_m \right] \cos \eta + m V_m \phi'_m \sin \eta \right\}.
\]

(17)
A.3. The velocity in the orbit


\[
\delta \phi = \left\{ \frac{[V_m^2 \Omega_p^2 + m^2 (\Omega_0 - \Omega_p)^2]}{R_0 \left[ k_0^2 - m^2 (\Omega_0 - \Omega_p)^2 \right]} + 2 \Omega_0 V_m' \right\} \left[ \frac{m \Omega_0 \sin \eta}{R_0 \left[ k_0^2 - m^2 (\Omega_0 - \Omega_p)^2 \right]} - 2 \Omega_0 m V_m' \right] \sin \eta.
\]

Integrating this equation with respect to \( t \) gives:

\[
\delta \phi = \left\{ \frac{[V_m^2 \Omega_p^2 + m^2 (\Omega_0 - \Omega_p)^2]}{m (\Omega_0 - \Omega_p, m) R_0 \left[ k_0^2 - m^2 (\Omega_0 - \Omega_p)^2 \right]} + 2 \Omega_0 V_m' \right\} \sin \eta - 2 \Omega_0 m V_m \phi_m' \cos \eta.
\]

So this finally yields the orbits:

\[
R = R_0 + \delta R = R_0 \left( 1 - \frac{1}{2} a_{1m} \cos \eta - a_{2m} \sin \eta \right),
\]

\[
\phi = \phi_0 + \delta \phi = \phi_0 + \frac{1}{2 m} \left( a_{1m} + a_{3m} \right) \sin \eta - a_{4m} \cos \eta.
\]

with

\[
\eta = m [\phi - \phi_m(R)] \approx m [\phi_0 - \phi_m(R)] = m [(\Omega_0 - \Omega_p, m) t - \phi_m(R)],
\]

\[
a_{1m} = \frac{2}{\Delta_0} \left\{ \frac{2 \Omega_0 V_m}{R_0 (\Omega_0 - \Omega_p, m) + V_m'} \right\},
\]

\[
a_{2m} = \frac{m V_m \phi_m'(R)}{\Delta_0},
\]

\[
a_{3m} = \frac{2}{\Delta_0} \left\{ V_m \left[ (2 + m^2) \Omega_0^2 + 2 (1 - m^2) \Omega_0 \Omega_p, m + m^2 \Omega_p, m - k_0^2 \right] \right\} +
\]

\[
+ \frac{(\Omega_0 + \Omega_p, m)}{(\Omega_0 - \Omega_p, m)} V_m',
\]

\[
a_{4m} = \frac{2 \Omega_0 V_m \phi_m'(R)}{\Delta_0 (\Omega_0 - \Omega_p, m)}.
\]

with \( \Delta_0 = R_0 \left[ k_0^2 - m^2 (\Omega_0 - \Omega_p)^2 \right] \). These equations break down near resonances (at corotation where \( \Omega_0 = \Omega_p, m \), or when \( k_0^2 = m^2 (\Omega_0 - \Omega_p)^2 \), the Lindblad resonances).

A.3 The velocity in the orbit

Now that we have found the closed loop orbits in a perturbed potential, we can proceed and calculate the corresponding velocity structure of these orbits. The orbits were calculated in a frame that rotates with angular velocity \( \Omega_p, m \). This means that the velocities in a frame that is at rest with respect to the observer can be calculated from \( v_R = \dot{R} \) and \( v_{\phi} = \dot{R} (\phi + \Omega_p, m) \). Then:

\[
v_R = \dot{R} = m v_c (1 - \omega_m) \left\{ \frac{1}{2} a_{1m} \sin \eta - a_{2m} \cos \eta \right\},
\]

\[
v_{\phi} = \dot{\phi} = m v_c \left\{ \frac{1}{2} a_{1m} \sin \eta - a_{2m} \cos \eta \right\}.
\]
where \( \omega_m = \frac{\Omega_{m,0}}{\Omega_0} \) and \( v_c = R_0 \Omega_0 \), the circular velocity.

\[
v_\phi = R(\dot{\phi} + \Omega_{m,0}) = \\
= R_0 \left( 1 - \frac{1}{2} a_{1m} \cos \eta - a_{2m} \sin \eta \right) \cdot \\
[\Omega_0 + \frac{1}{2} (a_{1m} + a_{3m})(1 - \omega_m) \Omega_0 \cos \eta + a_{4m}m(1 - \omega_m) \Omega_0 \sin \eta] \\
= v_c \left\{ 1 - \frac{1}{2} a_{1m} \cos \eta - a_{2m} \sin \eta \right\} \cdot \\
[1 + \frac{1}{2} (1 - \omega_m)(a_{1m} + a_{3m}) \cos \eta + m(1 - \omega_m)a_{4m} \sin \eta] = \\
\text{Skipping all the terms of order } a_{xm}^2:
\]

\[
v_\phi = v_c \left\{ 1 + \frac{1}{2} (1 - \omega_m) a_{3m} - \omega_m a_{1m} \cos \eta + [m(1 - \omega_m)a_{4m} - a_{2m}] \sin \eta \right\}. \quad (28)
\]

So the velocities are:

\[
v_R = m v_c (1 - \omega_m) \left\{ \frac{1}{2} a_{1m} \sin \eta - a_{2m} \cos \eta \right\}, \quad (29)
\]

\[
v_\phi = v_c \left\{ 1 + \frac{1}{2} (1 - \omega_m) a_{3m} - \omega_m a_{1m} \cos \eta + [m(1 - \omega_m)a_{4m} - a_{2m}] \sin \eta \right\}. \quad (30)
\]

### A.4 The case of a gas disk

So far we have considered the case of a single orbit: the circular velocity \( v_c \) was a constant in all the equations. The case of multiple orbits, with a general rotation curve, is somewhat more complicated. The first thing we want to know is: what is the observed velocity if we take a look at the point \( (R_{map}, \phi_{map}) \) in the velocity field. What we have to do therefore, is to determine which orbit goes through \( (R_{map}, \phi_{map}) \). In other words, given \( (R_{map}, \phi_{map}) \), what is the corresponding \( (R_0, \phi_0) \)?

We derived in the first section that (eqn. [20] and [21])

\[
R = R_0 \left\{ 1 - \frac{1}{2} a_{1m} \cos \eta - a_{2m} \sin \eta \right\}, \quad (31)
\]

\[
\phi = \phi_0 + \frac{1}{2m} (a_{1m} + a_{3m}) \sin \eta - a_{4m} \cos \eta. \quad (32)
\]

For simplicity of the following derivation we shall write these equations as

\[
R_{orbit}(t, R_0, \phi_0) = R_0 \left\{ 1 + r(R_0, \phi_0) \right\}, \quad (33)
\]

\[
\phi_{orbit}(t, R_0, \phi_0) = \phi_0 + p(R_0, \phi_0). \quad (34)
\]

The subscript "orbit" indicates that these coordinates describe the point on the closed loop orbit with guiding centre \( \alpha \) \( (R_0, \phi_0) \). So we have to solve \( (R_0, \phi_0) \) from

\[
R_{orbit}(t, R_0, \phi_0) = R_{obs} = R_0 \left\{ 1 + r(R_0, \phi_0) \right\}, \quad (35)
\]
A.5 Projection of the velocity field

\[ \phi_{\text{orbit}}(t, R_0, \phi_0) = \phi + p(R_0, \phi_0). \]  (36)

Making a Taylor expansion of \( r(R_0, \phi_0) \) and \( p(R_0, \phi_0) \) around \( (R_{\text{map}}, \phi_{\text{map}}) \), we find:

\[ r(R_0, \phi_0) = r(R_{\text{map}}, \phi_{\text{map}}) + (R_0 - R_{\text{map}}) r' + \cdots, \]  (37)

\[ p(R_0, \phi_0) = p(R_{\text{map}}, \phi_{\text{map}}) + (R_0 - R_{\text{map}}) p' + \cdots. \]  (38)

Since both \( R_0 - R_{\text{map}} \) and the derivatives of \( r \), respectively \( p \), are small, we shall take only the first term into account:

\[ r(R_0, \phi_0) = r(R_{\text{map}}, \phi_{\text{map}}), \]  (39)

\[ p(R_0, \phi_0) = p(R_{\text{map}}, \phi_{\text{map}}). \]  (40)

Inserting these equations in eqn.[35] and [36] we find that the coordinates for the guiding centre of the orbit going through \( (R_{\text{map}}, \phi_{\text{map}}) \) are

\[ R_0 = \frac{R_{\text{map}}}{1 + r(R_{\text{map}}, \phi_{\text{map}})}, \]  (41)

\[ \phi_0 = \phi_{\text{map}} - p(R_{\text{map}}, \phi_{\text{map}}). \]  (42)

Now we can write

\[ v_c(R_0) = v_c(R_{\text{map}})(1 - r \frac{d \ln v_c(R_{\text{map}})}{d \ln R_{\text{map}}}), \]

\[ \omega(R_0) = \omega(R_{\text{map}})(1 - r \frac{d \ln \omega(R_{\text{map}})}{d \ln R_{\text{map}}}). \]  (43)

So, to first order:

\[ v_R = m v_c(R)[1 - \omega_m(R)]\left[\frac{1}{2} a_{1m} \sin \eta - a_{2m} \cos \eta, \right], \]  (44)

\[ v_\phi = v_c(R) \left\{ 1 + \frac{1}{2} [(1 - \omega_m)a_{3m} + (\alpha - \omega)a_{1m}] \cos \eta \\
+ [m(1 - \omega_m)a_{4m} - (1 - \alpha)a_{2m}] \sin \eta \right\}. \]

with \( \alpha = d \ln v_c(R)/d \ln R \). We can therefore rewrite the epicyclic frequency \( \kappa_0 \) as \( \kappa_0^2 = 2(1 + \alpha)\Omega_0^2 \). Note that \( a_{xm}, \eta \) and \( \omega_m \) are evaluated at \( R_{\text{map}} \). We showed in equations [39] and [40] already that this is allowed.

A.5 Projection of the velocity field

We have now found the general form of a velocity field in a perturbed potential, where the velocities are know as a function of the measurable parameter \( R \). The next step is to calculate the harmonic expansion of the velocity field when it is projected onto the sky.
First, we have to calculate the line-of-sight velocity field \( v_{\text{los}} \):

\[
v_{\text{los}} = \left[ v_R \cos(\theta - \theta_{\text{obs}}) - v_\phi \sin(\theta - \theta_{\text{obs}}) \right] \sin i
= \left[ v_R \cos(\phi - \phi_{\text{obs}}) - v_\phi \sin(\phi - \phi_{\text{obs}}) \right] \sin i,
\]

where \( \phi_{\text{obs}} \) is the angle between the line \( \phi = 0 \) and the observer. The angle \( i \) is the inclination of the plane of the disk with respect to the observer. Introduce the angle \( \psi = \theta - \theta_{\text{obs}} + \frac{\pi}{2} = \phi - \phi_{\text{obs}} + \frac{\pi}{2} \). This angle, sometimes called the azimuthal angle, is zero on the line-of-nodes. Then:

\[
v_{\text{los}} = \left[ v_R \sin \psi + v_\phi \cos \psi \right] \sin i.
\]

Now replace \( \phi \rightarrow \psi + \phi_{\text{obs}} - \frac{\pi}{2} \) in the expressions for \( v_R \) and \( v_\phi \). Then expand the line-of-sight velocity in multiple angles of \( \psi \). This angle \( \psi \) is again a measurable parameter.

\[
v_{\text{los}} = v_c \left[ m(1 - \omega_m) \left\{ \frac{a_{1m}}{2} \sin(m[\phi + \phi_m(R)]) - a_{2m} \cos(m[\phi + \phi_m(R)]) \right\} \sin \psi +
+ \left\{ 1 + \frac{1}{2} [(1 - \omega_m)a_{3m} + (\alpha - \omega_m)a_{1m}] \cos(m[\phi + \phi_m(R)]) +
[ m(1 - \omega_m)a_{4m} - (1 - \alpha)a_{2m} \sin(m[\phi + \phi_m(R)]) \} \cos \psi \right\} \sin i. \]
\]

Say \( v_* = v_c \sin i \) and \( \varphi = \phi_{\text{obs}} - \frac{\pi}{2} - \phi_m(R) \). Then:

\[
v_{\text{los}} = v_* \left[ m(1 - \omega_m) \left\{ \frac{a_{1m}}{2} \sin(m[\psi + \varphi]) - a_{2m} \cos(m[\psi + \varphi]) \right\} \sin \psi +
+ \left\{ 1 + \frac{1}{2} [(1 - \omega_m)a_{3m} + (\alpha - \omega_m)a_{1m}] \cos(m[\psi + \varphi]) +
[ m(1 - \omega_m)a_{4m} - (1 - \alpha)a_{2m} \sin(m[\psi + \varphi]) \} \cos \psi \right\} \sin i. \]
\]

Using \( \cos(a + b) = \cos a \cos b - \sin a \sin b \) and \( \sin(a + b) = \sin a \cos b + \cos a \sin b \), we find:

\[
\sin mx \sin x = \frac{1}{2} \left[ \cos(m - 1)x - \cos(m + 1)x \right],
\sin mx \cos x = \frac{1}{2} \left[ \sin(m - 1)x + \sin(m + 1)x \right],
\cos mx \sin x = -\frac{1}{2} \left[ \sin(m - 1)x - \sin(m + 1)x \right],
\cos mx \cos x = \frac{1}{2} \left[ \cos(m - 1)x + \cos(m + 1)x \right].
\]

(48)

Using these rules in equation [47], we find:

\[
v_{\text{los}} = v_* \left[ m(1 - \omega_m) \frac{a_{1m}}{2} \left\{ \frac{1}{2} (\cos(m - 1)\psi - \cos(m + 1)\psi) \cos m\varphi - \right. \right.
\]
\[ \begin{aligned} &-\frac{1}{2}(\sin(m-1)\psi - \sin(m+1)\psi) \sin m\varphi \} - \\ &-m(1 - \omega_m)a_{2m} \left\{ -\frac{1}{2}(\sin(m-1)\psi - \sin(m+1)\psi) \cos m\varphi \right\} - \\ &-\frac{1}{2}(\cos(m-1)\psi - \cos(m+1)\psi) \sin m\varphi \} + \\ &+ \cos \psi + \frac{1}{2}[(1 - \omega_m)a_{3m} + (\alpha - \omega_m)a_{1m}] \\ &\left\{ \frac{1}{2}(\cos(m-1)\psi + \cos(m+1)\psi) \cos m\varphi - \\ &-\frac{1}{2}(\sin(m-1)\psi + \sin(m+1)\psi) \sin m\varphi \right\} + \\ &+[m(1 - \omega_m)a_{4m} + (\alpha - 1)a_{2m}] \\ &\left\{ \frac{1}{2}(\sin(m-1)\psi + \sin(m+1)\psi) \cos m\varphi + \\ &+\frac{1}{2}(\cos(m-1)\psi + \cos(m+1)\psi) \sin m\varphi \right\} \right] . \quad (49) \end{aligned} \]

Writing
\[ v_{\text{los}} = [c_1 \cos \psi + s_{m-1} \sin(m-1)\psi + c_{m-1} \cos(m-1)\psi + \\ +s_{m+1} \sin(m+1)\psi + c_{m+1} \cos(m+1)\psi], \quad (50) \]

we can identify the coefficients of the harmonic expansion as:

\[ \begin{aligned} c_1 &= v_* \\ s_{m-1} &= v_*(-\frac{1}{4}\{m-(m+1)\omega_m + \alpha]a_{1m} + (1-\omega_m)a_{3m}\} \sin m\varphi_m \\ &+\frac{1}{2}\{m(1 - \omega_m)a_{4m} + [m(1 - \omega_m)-1 + \alpha]a_{2m}\} \cos m\varphi_m), \\ c_{m-1} &= v_*(-\frac{1}{4}\{m-(m+1)\omega_m + \alpha]a_{1m} + (1-\omega_m)a_{3m}\} \cos m\varphi_m \\ &+\frac{1}{2}\{m(1 - \omega_m)a_{4m} + [m(1 - \omega_m)-1 + \alpha]a_{2m}\} \sin m\varphi_m), \\ s_{m+1} &= v_*(\frac{1}{4}\{m-(m-1)\omega_m - \alpha]a_{1m} - (1-\omega_m)a_{3m}\} \sin m\varphi_m \\ &+\frac{1}{2}\{m(1 - \omega_m)a_{4m} - [m(1 - \omega_m)+1 - \alpha]a_{2m}\} \cos m\varphi_m, \\ c_{m+1} &= v_*(-\frac{1}{4}\{m-(m-1)\omega_m - \alpha]a_{1m} - (1-\omega_m)a_{3m}\} \cos m\varphi_m \\ &+\frac{1}{2}\{m(1 - \omega_m)a_{4m} - [m(1 - \omega_m)+1 - \alpha]a_{2m}\} \sin m\varphi_m). \quad (51) \end{aligned} \]

So we see that if insert an \( m=5 \) potential perturbation for instance, only fourth and sixth order harmonic components will be seen in the projected velocity field. Note that these formulae are very general: if we take \( m = 2, \omega_m = 0 \) and \( \phi_m(R) = 0 \), we find the formulae for epicyclic orbits back as presented in FvGdZ (their equation A5). The above equations apply when one takes the correct viewing angles (position angle and inclination). The next step will be to derive what these coefficients will be when we deproject under the wrong viewing angles. With these new coefficients, we will be able to calculate what happens if we fit a circular velocity field to the observed
velocity field, in other words: what the measured harmonic coefficients in a galaxy will be.

A.6 Deprojection with incorrect parameters

In this section, we will derive how the harmonic terms measured from the line-of-sight velocity change if we deproject the velocity field with incorrect parameters (inclination, position angle and kinematic centre). We will need the results of this section if we want to calculate the effect of deprojecting a non-axisymmetric velocity field assuming that it is circular symmetric (i.e. calculate how a tilted-ring fit will influence the measured harmonics).

A.6.1 Deprojection with an incorrect inclination

Suppose we assume the wrong inclination ($i$) for the velocity field. Then the (inferred) axis ratio $q = \cos i$ will become $\hat{q} = \cos i = \cos(i + \delta i) = \cos i - \sin \delta i = q + \delta q$. Furthermore, the equations for a circular orbit projected with inclination $i$ are:

$$x'' = \dot{R} \cos \psi,$$

$$y'' = \dot{q} R \sin \psi.$$  \hspace{1cm} (52)

So that

$$\cos \psi = \frac{x''}{\sqrt{x''^2 + y''^2}}$$

$$= \frac{\cos \dot{\psi}}{\sqrt{\cos^2 \dot{\psi} + \frac{2}{q^2} \sin^2 \dot{\psi}}},$$

$$= \frac{\cos \dot{\psi}}{\sqrt{1 + \frac{2\delta q}{q} \sin^2 \dot{\psi}}}$$

$$= \cos \dot{\psi} (1 - \frac{\delta q}{q} \sin^2 \dot{\psi})$$

$$= \cos \dot{\psi} - \frac{\delta q}{q} (\cos \dot{\psi} - \cos^3 \dot{\psi}).$$

Finally, using $\cos^3 \psi = \frac{1}{4}(\cos 3\psi + 3 \cos \psi)$:

$$\cos \psi = \left(1 - \frac{\delta q}{4q}\right) \cos \dot{\psi} + \frac{\delta q}{4q} \cos 3\dot{\psi}. \hspace{1cm} (54)$$

In the same way we can derive the formula for $\sin \psi$:

$$\sin \psi = \frac{y''/q}{\sqrt{x''^2 + y''^2}}$$
A.6. Deprojection with incorrect parameters

\[
\begin{align*}
\frac{q}{q} \sin \psi &= \sqrt{\cos^2 \psi + \frac{q^2}{q^2} \sin^2 \psi} \\
&= \frac{q}{q} \sin \psi \left( 1 - \frac{dq}{q} \sin^2 \hat{\psi} \right) \\
&= \left( 1 + \frac{dq}{q} \right) \sin \hat{\psi} - \frac{dq}{q} \sin^3 \hat{\psi}.
\end{align*}
\]

Using \( \sin^3 \hat{\psi} = \frac{1}{4} (3 \sin \hat{\psi} - \sin 3\hat{\psi}) \) we find that

\[
\sin \psi = \left( 1 + \frac{dq}{4q} \right) \sin \hat{\psi} + \frac{dq}{4q} \sin 3\hat{\psi}. \tag{55}
\]

We want to express \( \cos a\psi \) and \( \sin a\psi \) as sums over harmonic terms \( \sum_n \cos n\psi \) and \( \sum_n \sin n\psi \). Realizing that the only terms we know is the case of \( a = 1 \) (the equations [54] and [55]), we must express \( \cos a\psi \) and \( \sin a\psi \) as functions of \( \cos \psi \) and \( \sin \psi \) only. According to Gradshteyn & Ryzhik (1965, formula 1.331.3; GR hereafter) we can write:

\[
\cos a \psi = 2^{a-1} \cos^a x - \frac{a}{1} 2^{a-3} \cos^{a-2} x + \frac{a}{2} \left( \frac{a-3}{1} \right) 2^{a-5} \cos^{a-4} x - \frac{a}{3} \left( \frac{a-4}{2} \right) 2^{a-7} \cos^{a-6} x + \ldots \tag{56}
\]

The algebraic manipulation computer package Mathematica is used to substitute equation [54] in equation [56]. According to Mathematica the following relation holds to first order in \( dq \):

\[
\cos a\psi = \cos a\hat{\psi} + \frac{adq}{4q} \left( \cos(a + 2)\hat{\psi} - \cos(a - 2)\hat{\psi} \right) + O(dq^2). \tag{57}
\]

The same procedure can be applied to \( \sin a\psi \). The formula for \( \sin ax \) as a function of \( \cos x \) and \( \sin x \) is (after writing formula 1.331.1 of GR as a series)

\[
\sin ax = \sin x \left\{ 2^{a-1} \cos^{a-1} x + \sum_{k=2}^{\text{int}(\frac{a+1}{2})} (-1)^{k-1} \binom{a-k}{k-1} 2^{a-2k+1} \cos^{a-2k+1} x \right\} \tag{58}
\]

Doing the same here as we did for the cosine term (so substituting equations [54] and [55] in equation [58] and retaining only first order terms), we find:

\[
\sin a\psi = \sin a\hat{\psi} + \frac{adq}{4q} \left[ \sin(a + 2)\hat{\psi} - \sin(a - 2)\hat{\psi} \right] + O(dq^2). \tag{59}
\]

We can also derive the relation between \( R \) and \( \hat{R} \):

\[
\hat{R} \cos \hat{\psi} = R \cos \psi \Leftrightarrow
\]
\[ \hat{R} \cos \hat{\psi} = R \left( 1 - \frac{\delta q}{4q} \right) \cos \hat{\psi} + \frac{\delta q}{4q} \cos 3\hat{\psi} \leftrightarrow \]

\[ R = \hat{R} \left[ 1 - \frac{\delta q}{4q} \right] + \frac{\delta q}{4q} \cos 3\hat{\psi} \left( \cos \hat{\psi} \right)^{-1} \]

\[ \approx \hat{R} \left[ 1 + \frac{\delta q}{4q} \right] - \frac{\delta q \cos 3\hat{\psi}}{4q \cos \hat{\psi}} \]

\[ = \hat{R} \left[ 1 + \frac{\delta q}{4q} \right] - \frac{\delta q}{4q} (-1 + 2 \cos 2\hat{\psi}) \]

(60)

yielding:

\[ R = \hat{R} \left[ 1 + \frac{\delta q}{2q} (1 - \cos 2\hat{\psi}) \right]. \]

(62)

If we look at equation [51], we see that all terms are first order, except the \( c_1 \). Since the corrections due to an incorrect inclination are first order as well, only the deprojection of the \( \cos \psi \) term will give rise to first order contributions to the line-of-sight velocity.

### A.6.2 Deprojection with an incorrect position angle

Suppose we take an incorrect position angle \( \Gamma = \Gamma + \delta \Gamma \), where the deviation from the true position angle is \( \delta \Gamma \). Then

\[ x'' = \hat{x} \cos \delta \Gamma - \hat{y} \sin \delta \Gamma \approx \hat{x} - \hat{y} \delta \Gamma, \]

\[ y'' = \hat{x} \sin \delta \Gamma + \hat{y} \cos \delta \Gamma \approx \hat{y} + \hat{x} \delta \Gamma. \]

Again the equation for a projected circle is:

\[ x'' = \hat{R} \cos \hat{\psi}, \]

\[ y'' = \hat{q} \hat{R} \sin \hat{\psi}. \]

(63)

(64)

We find for the \( \sin \psi \) and \( \cos \psi \) in exactly the same way as was done for the inclination:

\[ \sin \psi = \sin \hat{\psi} + \frac{\delta \Gamma}{4} \left[ (q + \frac{3}{q}) \cos \hat{\psi} + \left( \frac{1}{q} - q \right) \cos 3\hat{\psi} \right], \]

\[ \cos \psi = \cos \hat{\psi} - \frac{\delta \Gamma}{4} \left[ (3q + \frac{1}{q}) \sin \hat{\psi} + \left( \frac{1}{q} - q \right) \sin 3\hat{\psi} \right]. \]

(65)

For \( \sin a \psi \) and \( \cos a \psi \) we find to order \( O(\delta \Gamma^2) \), again using eqn. [58] & [56]:

\[ \sin a \psi = \sin \hat{\psi} + \frac{a \delta \Gamma}{2q} \left\{ \frac{(1 - q^2)}{2} \left[ \cos((a - 2)\hat{\psi}) \cos((a + 2)\hat{\psi}) \right] + (1 + q^2) \cos a\hat{\psi} \right\}, \]

\[ \cos a \psi = \cos \hat{\psi} + \frac{a \delta \Gamma}{2q} \left\{ \frac{(q^2 - 1)}{2} \left[ \sin((a - 2)\hat{\psi}) \sin((a + 2)\hat{\psi}) \right] - (1 + q^2) \sin a\hat{\psi} \right\}. \]

(66)
A.6. Deprojection with incorrect parameters

For the relation between \( R \) and \( \dot{R} \) we find:

\[
R = \dot{R}[1 + \frac{1}{2} \left( \frac{1}{q} - q \right) \sin 2\dot{\psi} \delta \Gamma].
\] (67)

Again, only deprojection of the \( \cos \psi \) term will give rise to first order effects.

A.6.3 Deprojection with an incorrect kinematic centre

Suppose that we choose the point \((\delta x, \delta y)\) to be the kinematic center for the deprojection, instead of the true kinematic center at \((x, y) = (0, 0)\). Here, the \(X-Y\) axis system is defined such that the major axis of the projected orbits fall along the X-axis (i.e. the line-of-nodes) and the minor axis along the Y-axis. Again, an elliptical orbit in the \(X-Y\) coordinate system is described by

\[
x'' = R \cos \dot{\psi},
\]
\[
y'' = q R \sin \dot{\psi}.
\]

So:

\[
\ddot{x} = R \cos \dot{\psi} - \delta x,
\]
\[
\ddot{y} = q R \sin \dot{\psi} - \delta y.
\] (68)

We then find:

\[
\cos \psi = \cos \dot{\psi} + \frac{\delta x}{2R} (1 - \cos 2\dot{\psi}) - \frac{\delta y}{2qR} \sin 2\dot{\psi}.
\] (69)

In the same way as we did for the inclination and position angle, we can derive the more general form of this equation:

\[
\cos \alpha \psi = \cos \alpha \dot{\psi} + \frac{a \delta x}{2R} [\cos (a - 1) \dot{\psi} - \cos (a + 1) \dot{\psi}] + \frac{a \delta y}{2qR} \left[ \sin (a - 1) \dot{\psi} - \sin (a + 1) \dot{\psi} \right] + \mathcal{O}(\delta x^2, \delta y^2).
\] (70)

The relation between \( R \) and \( \dot{R} \) is now

\[
R = \dot{R}[1 + \frac{\delta x}{R} \cos \dot{\psi} + \frac{\delta y}{qR} \sin \dot{\psi}].
\] (71)

Here too only the deprojection of the \( \cos \psi \) term is important.

A.6.4 The line-of-sight velocity

Now we are able to calculate the form of the line-of-sight velocity field if incorrect parameters are used to deproject it. Let us define

\[
dR = R - \dot{R} = \dot{R} \left[ \frac{\delta y}{2q} (1 - \cos 2\dot{\psi}) + \frac{1}{2} \left( \frac{1}{q} - q \right) \delta \Gamma \sin 2\dot{\psi} + \frac{\delta x}{R} \cos \dot{\psi} + \frac{\delta y}{qR} \sin \dot{\psi} \right].
\]
Now we can substitute the relations between $\psi$ and $\dot{\psi}$ (equations [54], [65] and [691]) and between $R$ and $\dot{R}$ (equations [62], [67] and [71]) into the expansion of $v_{\text{los}}$ (equation [50]). With regard to the expansion of the amplitudes of the harmonics, we note that:

$$c_i(R) = c_i(\dot{R} + d\dot{R}) = c_i(\dot{R}) + dR(c_i(\dot{R})/dR).$$

term is $dR c_i(\dot{R})/dR$, the shape of the rotation curve (so this contribution is zero in the case of a flat rotation curve). Noting that $R dR c_i(\dot{R})/dR = v_*(\dot{R})\alpha$ we can expand the line-of-sight velocity in the case of incorrect viewing angles and centre as:

$$v_{\text{los}} = v_*(\dot{R}) \left\{(1 + \alpha) \frac{\delta x}{2R} - (1 - \alpha) \frac{\delta q}{4q} \cos \psi \frac{\delta \Gamma}{4} \left[ (3q + \frac{1}{q}) - \alpha \left( \frac{1}{q} - q \right) \right] \sin \dot{\psi} 
- (1 - \alpha) \frac{\delta x}{2R} \cos 2\psi - (1 - \alpha) \frac{\delta y}{2qR} \sin 2\psi + (1 - \alpha) \frac{\delta q}{4q} \cos 3\psi 
- (1 - \alpha) \frac{\delta \Gamma}{4} \left( \frac{1}{q} - q \right) \sin 3\psi \right\} + \sum_{i \geq 0} c_i(\dot{R}) \cos i\psi + s_i(\dot{R}) \sin i\psi.
$$

If this is expanded into new harmonics, $v_{\text{los}} = \sum c_i(\dot{R}) \cos i\psi + s_i(\dot{R}) \sin i\psi$, then $s_i = s_i, c_i = c_i$, except for $i \leq 3$.

Now we are ready to tackle the problem in which we deproject the velocity fields under the assumption of a circular velocity field.

### A.7 Deprojection assuming a circular velocity field

Now we will calculate the errors in $\Gamma, i$ and centre if the general velocity field is deprojected under the assumption of a circular velocity field $v_{\text{los}} = v_* \cos \psi$. The $\chi^2$ deviation from the best fitting circular velocity field is:

$$\chi^2(\delta q, \delta \Gamma, \delta x, \delta y) \approx \sum_{i \geq 0} (c_i^2 + s_i^2) - c_1^2,$$

where $s_i$ and $c_i$ are expressed in terms of the $\delta q, \delta \Gamma, \delta x$ and $\delta y$ in equation (72):

$$\chi^2 = \left[ c_0 - (1 + \alpha) \frac{\delta x}{2R} \right]^2 + \left( s_1 - \frac{1}{4} \delta \Gamma \left[ 3q + \frac{1}{q} - \alpha \left( \frac{1}{q} - q \right) \right] c_1 \right)^2 + \left[ c_2 - (1 - \alpha) \frac{\delta x}{2R} c_1 \right]^2 + \left[ s_2 - (1 - \alpha) \frac{\delta y}{2qR} \right]^2 + \left[ s_3 - (1 - \alpha) \frac{1}{4} \delta \Gamma \left( \frac{1}{q} - q \right) c_1 \right]^2 + \left[ c_3 + (1 - \alpha) \frac{\delta q}{4q} c_1 \right]^2$$

We minimize this $\chi^2$ (i.e. we calculate $\frac{\partial \chi^2}{\partial \delta q} = 0; \frac{\partial \chi^2}{\partial \delta \Gamma} = 0; \frac{\partial \chi^2}{\partial \delta x} = 0; \frac{\partial \chi^2}{\partial \delta y} = 0$). The best fitting values of $\delta \Gamma$ and $\delta q$ are then given by

$$\delta q = -\frac{4qc_3}{(c_1 + c_3)(1 - \alpha)}, \quad \delta \Gamma = 4q \sqrt{\frac{\left[ (3q^2 + 1 - \alpha(1 - q^2) \right] s_1 + (1-q^2)(1-\alpha)s_3}{\left[ (3q^2 + 1 - (1-q^2)\alpha) + (1-\alpha)^2(1-q^2) \right] (c_1 + c_3)}}.$$
Equation (73) is minimized with respect to $\delta x$ and $\delta y$ when
\[
\delta x = 2Rc_2/(1 - \alpha), \quad \delta y = 2qRs_2/(1 - \alpha),
\] (76)
Substituting these terms into equation (72), we find that the resulting expressions for $c_i, s_i$ under a circular fit are:
\[
\begin{align*}
c_0 &= c_0 + \frac{1 + \alpha}{1 - \alpha}c_2, \\
c_2 &= 0, \\
c_1 &= c_1[1 - \frac{\delta q}{4q}(1 - \alpha)] = c_1 + c_3, \\
s_1 &= s_1 - c_1[(3q + 1/q) - (1/q - q)\alpha] \frac{\delta \Gamma}{4}, \\
c_3 &= c_3 + (1 - \alpha)\frac{\delta q}{4q}c_1 = 0, \\
s_3 &= s_3 - c_1(1/q - q)(1 - \alpha)\delta \Gamma/4.
\end{align*}
\] (77)
Now we can calculate which terms will be measured for any potential perturbation. In particular we will investigate two interesting cases: an $m = 1$ perturbation (which can be used to investigate kinematic lopsidedness, see Chapter 4 and 6 of this thesis) and an $m = 2$ perturbation, useful for measuring the elongations of orbits.

### A.7.1 Effect of $m = 1$ distortion

For an $m = 1$ distortion equation (51) gives the correct coefficients if the correct centre is chosen:
\[
\begin{align*}
c_0 &= v_0\frac{1}{3}[(1 - 2\omega_1 + \alpha)a_{11} + (1 - \omega_1)a_{31}] \cos \varphi_1 + \frac{1}{2}[(1 - \omega_1)a_{41} + (\omega_1 - 2\omega_1)a_{21}] \sin \varphi_1, \\
c_2 &= 0, \\
s_2 &= v_0\frac{1}{3}[(1 - \alpha)a_{11} - (1 - \omega_1)a_{31}] \sin \varphi_1 + \frac{1}{2}[(1 - \omega_1)a_{41} - (2 - \omega_1 - \alpha)a_{21}] \cos \varphi_1.
\end{align*}
\] (78)
If the velocity field itself is used to derive the centre, these equations reduce according to equation (77) to:
\[
\begin{align*}
c_0 &= v_0\frac{1}{3(\alpha - 1)}\{[(1 - \alpha)\omega_1a_{11} - (1 - \omega_1)a_{31}] \cos \varphi_1 - 2[(1 - \omega_1)a_{41} + (\alpha - 1)a_{21}] \sin \varphi_1\}, \\
c_2 &= 0, \\
s_2 &= 0.
\end{align*}
\] (79)
These terms are not affected by an error in $q$ and $\Gamma$.

### A.7.2 Effect of $m = 2$ distortion

With $m = 2$, equation (51) gives the coefficients if a correct inclination and position angle are chosen:
\[
\begin{align*}
c_1 &= \frac{1}{4}v_0\left\{1 - \frac{1}{4}[(2 - 3\omega_2 + \alpha)a_{12} + (1 - \omega_2)a_{32}] \cos 2\varphi_2 + \frac{1}{2}[(2(1 - \omega_2)a_{42} + (1 - 2\omega_2 + \alpha)a_{22}] \sin 2\varphi_2\right\},
\end{align*}
\]
152APPENDIX A  HARMONIC ANALYSIS OF THE VELOCITY FIELD OF A GAS DISK

\[ s_1 = v_n \left\{ -\frac{1}{4} \left[ (2 - 3\omega_2 + \alpha) a_{12} + (1 - \omega_2) a_{32} \right] \sin 2\varphi_2 + \frac{1}{2} \left[ 2(1 - \omega_2) a_{42} + (1 - 2\omega_2 + \alpha) a_{22} \right] \cos 2\varphi_2 \right\}, \]
\[ c_3 = v_n \left\{ -\frac{1}{4} \left[ (2 - \omega_2 - \alpha) a_{12} - (1 - \omega_2) a_{32} \right] \cos 2\varphi_2 + \frac{1}{2} \left[ 2(1 - \omega_2) a_{42} - (3 - 2\omega_2 - \alpha) a_{22} \right] \sin 2\varphi_2 \right\}, \]
\[ s_3 = v_n \left\{ -\frac{1}{4} \left[ (-2 + \omega_2 + \alpha) a_{12} + (1 - \omega_2) a_{32} \right] \sin 2\varphi_2 + \frac{1}{2} \left[ 2(1 - \omega_2) a_{42} - (3 - 2\omega_2 - \alpha) a_{22} \right] \cos 2\varphi_2 \right\}. \]

When the velocity field itself is used to derive the position angle and inclination, we find using equation (77):

\[ \dot{c}_1 = v_n \left\{ 1 + \frac{1}{4} \left[ (\alpha - \omega_2) a_{12} + (1 - \omega_2) a_{32} \right] \cos 2\varphi_2 + \frac{1}{2} (1 - \omega_2) a_{42} + (\alpha - 1) a_{22} \right\} \sin 2\varphi_2, \]
\[ \dot{s}_1 = v_n \left\{ -\frac{1}{4} \left[ (2 - 3\omega_2 + \alpha) a_{12} + (1 - \omega_2) a_{32} \right] \sin 2\varphi_2 + \frac{1}{2} \left[ 2(1 - \omega_2) a_{42} + (1 - 2\omega_2 + \alpha) a_{22} \right] \cos 2\varphi_2 \right\}, \]
\[ \frac{1}{4} \left[ 3q^2 + 1 + \alpha(q^2 - 1) \right] \left\{ (2 - \alpha) (\omega_2 - \alpha) + 2[2 - \alpha^2 - \omega_2 + \alpha(2 + \omega_2)]q^2 \right\} a_{12} - \left\{ 1 + q^2 - \alpha(1 - q^2) \right\} (1 - \omega_2) a_{32} \sin 2\varphi_2 - \left\{ 4[1 + q^2 - \alpha(1 - q^2)](1 - \omega_2) a_{42} - 2[1 - (1 - \omega_2) a_{32}]\right\} \cos 2\varphi_2, \]
\[ \dot{c}_3 = 0, \]
\[ \dot{s}_3 = v_n \left\{ -\frac{1}{4} \left[ (\omega_2 + \alpha - 2) a_{12} + (1 - \omega_2) a_{32} \right] \sin 2\varphi_2 + \frac{1}{2} \left[ 2(1 - \omega_2) a_{42} - (3 - 2\omega_2 - \alpha) a_{22} \right] \cos 2\varphi_2 \right\}, \]
\[ + \frac{1}{4} \left[ 1 + 2q^2 + 5q^4 + \alpha^2(1 - q^2)^2 + 2\alpha(q^4 - 1) \right] \left\{ 2(1 - \alpha) \omega_2 \right\} a_{12} - \left\{ 1 + q^2 - \alpha(1 - q^2) \right\} (1 - \omega_2) a_{32} \sin 2\varphi_2 + \left\{ 4[1 + q^2 - \alpha(1 - q^2)](1 - \omega_2) a_{42} - 2[1 - (3 - 4\omega_2) q^2 - \alpha(1 + q^2)] a_{22} \right\} \cos 2\varphi_2 \right\} \right\}. \]

These equations are independent of a small error in the position of the centre. For a triaxial non-rotating halo model, in which the potential is scale free, we have \( \omega_2 = 0, \) \( \alpha = 0 \) (flat rotation curve), the phase of the perturbation is constant with radius \( \varphi_2(R) \sim \) const (therefore \( a_{42} = a_{22} = 0 \), \( a_{32} = 2a_{12} = 2\epsilon_{polar} \) (PvGdZ) and the above equations simplify to the equations they found.

A.7.3  Solid body rotation

Let us finally consider the effect of solid body rotation, \( \alpha = 1 \). Equations (75) and (76) show that in this case it is impossible to measure the inclination. Indeed, in solid body rotation, all iso-velocity contours are parallel and each inclination fits equally well. The position angle on the other hand can be determined, \( \delta \Gamma = s_1 / q \). Centre and systemic velocity cannot be determined uniquely. Therefore, assumptions for all these parameters have to be made in order to get a rotation velocity. It is difficult to interpret harmonic terms under these conditions in this way we did in this Appendix.