Bifurcations of attractors in 3D diffeomorphisms
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Chapter 1

Introduction

The present work fits within the study of nonlinear deterministic dynamical systems depending on parameters. These systems are used for modelling in many disciplines, such as biology, mechanics, meteorology, physics and economy. The general goal is to understand the qualitative behaviour of these systems, and in particular:

1. The asymptotic, long term dynamics, where the system has settled down on an attractor.

2. The transitions between attractors due to variations of certain parameters (bifurcations).

We focus on discrete dynamical systems, generated by iteration of smooth invertible maps (diffeomorphisms). The orderly asymptotic dynamics takes place on attractors that have a simple geometry, like fixed points, periodic orbits, and invariant circles or tori. Usually the chaotic dynamics lives on strange attractors with a fractal structure. The purpose of this work is to understand the geometry of certain types of strange attractors, as well as the bifurcations leading to their formation.

1.1 The structure of three-dimensional strange attractors

The research presented in this thesis has been motivated by the following question:

What is the structure of strange attractors for three-dimensional diffeomorphisms?

The term ‘strange attractor’ first appeared in the work of Ruelle and Takens [NRT, RT]. Evolutions starting near a strange attractor rapidly converge towards it, and are characterised by chaotic behaviour: two evolutions starting from nearby points typically separate exponentially fast [BC, ER, GH, MV]. Strange attractors carrying chaotic dynamics have been described in many experimental and numerical investigations, in models coming from various fields of research: biology [Kuz], economics [Hom], mathematics [ASFK, BST, Tat], mechanics [GH], meteorology [Lor, SNN, Vee, VOV], laser physics [WKL].
Figure 1.1: Structure of Hénon-like strange attractors. (A) Two-dimensional projection of an attractor occurring in a three-dimensional diffeomorphism $P$. The small box pointed by an arrow is magnified in the lower left corner. (B) A saddle fixed point $p$ of $P$, plotted with its one-dimensional unstable manifold. The same windows are used in the two plots; the same holds for the magnifications. Notice the striking similarity.

The structure of strange attractors of two-dimensional diffeomorphisms is fairly well understood [BC, DRV, MV, Sim, Via1, LSY]. In particular, Hénon-like strange attractors [MV, Via1] are characterised by the property that there exists a periodic point $p$ of saddle type such that the strange attractor coincides with the closure of the unstable manifold $\text{clos} W^u(\text{Orb}(p))$, where $\text{Orb}(p)$ denotes the orbit of $p$. For an illustration of this, see Figure 1.1. In Figure 1.1 (A) an attractor of a three-dimensional diffeomorphism $P$ is plotted, and in (B) a saddle fixed point $p$ of $P$, with its one-dimensional unstable manifold. The occurrence of this type of strange attractors has been proved for a class of diffeomorphisms called Hénon-like [MV, Via1]. The first numerical example of this type of attractors was given in [Hen].

A periodic point is the simplest invariant set for a diffeomorphism $P$. The ‘next’ object in terms of geometrical and dynamical complexity is an invariant circle, i.e., a closed smooth curve which is mapped onto itself by $P$. The diffeomorphism $P$ generically has two types of dynamics on an invariant circle $\mathcal{C}$: quasi-periodic or Morse-Smale [Arn, BHS, GH, Kuz]. In the first case, repeated application of $P$ on a point of $\mathcal{C}$ is equivalent to rotating the point along $\mathcal{C}$ in a rigid fashion. In the second case, the circle $\mathcal{C}$ contains a finite number of attracting periodic orbits, and the same number of repelling periodic orbits. Under iteration of $P$, any point of $\mathcal{C}$ not belonging to the repelling periodic orbits converges to one of the attracting periodic orbits.

This leads us to the central question:

Given a diffeomorphism $P$ and an invariant circle $\mathcal{C}$ of saddle type, are there strange attractors contained in the closure $\text{clos} W^u(\mathcal{C})$?

This problem forms the basis of the third Chapter. There we consider a diffeomorphism $T$ of the solid torus $\mathbb{R}^2 \times S^1$ into itself. The diffeomorphism is obtained by the weak coupling of a planar Hénon-like map [MV] with a diffeomorphism of the circle $S^1$ which is a perturbation of a rigid rotation. When the dynamics on $S^1$ is
of Morse-Smale type, by using a theorem of Díaz-Rocha-Viana [DRV], we prove that Hénon-like strange attractor occur for the diffeomorphism $T$. In Figure 1.2 (A) we display one of the attractors of the diffeomorphism $T$ in the period three case, where the picture is obtained by numerical simulation.

For a subset of parameter values, a different result is obtained for $T$. In this case, $T$ is a perturbation of the product map given by a planar diffeomorphism $K$ with a rigid rotation on $S^1$, where $K$ satisfies the properties:

1. $K$ is dissipative, i.e., it contracts area;
2. $K$ has a saddle fixed point with a transversal homoclinic point [PT].

Under these hypotheses, we prove that the coupled diffeomorphism $T$ has an invariant circle $C$ of saddle type, such that the following property holds: the stable and unstable manifolds of $C$ bound an open region $U$ such that the orbits of all points in $U$ are attracted to the closure of the unstable manifold $W^u(C)$. This shows the occurrence of an attractor inside $\text{clos} W^u(C)$. When the dynamics on $C$ is quasi-periodic, numerical evidence suggests that this attractor is strange; compare Figure 1.2 (B). The latter property remains conjectural and is topic of future research.

**Remark 1.1.** When speaking about numerical evidence for strange attractors in a concrete system, one should be aware of the following:

1. Most theoretical results concerning Hénon-like strange attractors are of a perturbative nature, meaning that these only hold in settings which are sufficiently close to a one-dimensional situation [BC, MV, LSY]. Furthermore, no explicit bounds are given for the size of the allowed perturbation.
2. The above theoretical results yield the occurrence of strange attractors for a nowhere dense parameter set $\mathcal{S}$ of positive Lebesgue measure. Given a specific, numerical parameter value, there is no algorithm to check whether this parameter value belongs to the set $\mathcal{S}$ or not.

This implies that, in general, the above theory does not guarantee that a numerically observed strange attractor actually exists in the examined system. For example, despite many efforts it still remains unproven whether the attractor numerically studied by Hénon a quarter of a century ago [Hen] is a Hénon-like strange attractor. Therefore, the structure and the properties expressed in [BC, MV, Via1, LSY] are considered as a general paradigm for the interpretation of numerical results which suggest the existence of strange attractors. In other words, Hénon-like strange attractors are conjectured to occur in a large class of dynamical systems. Throughout the thesis, when describing our numerical results, we will tacitly assume this conjecture.

The above discussion forms the mathematical core of the thesis, and is mainly reported in Chapter three. We now describe the remaining Chapters (two and four).

## 1.2 The Lorenz-84 climate model with seasonal forcing

Chapter two contains the basic examples which motivated the research presented in Chapters three and four. These examples are found in a model which is a perturbation
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of a three-dimensional autonomous system of ordinary differential equations coming from meteorology, the Lorenz-84 model [Lor]. The Lorenz-84 model is the simplest approximation of the general atmospheric circulation at mid-latitude, on a scale of a few thousand kilometres in space and about a week in time.

In meteorology, low-dimensional models such as the Lorenz-84 system have drawn the attention of investigators over the last decades [Vee, VOV]. The reason is that both the equation of Fluid Dynamics for the atmospheric flow and the computer models used for weather prediction are hard to analyse mathematically. Low-dimensional models have the advantage that they can be studied by the qualitative methods of Dynamical Systems theory, such as bifurcation theory [Arn, GH, Kuz]. On the one hand, low-dimensional systems are not suitable to produce quantitatively correct predictions about the climate. Indeed, strong simplifications imposed to obtain such models. See [Vee] for a discussion about this aspect. On the other hand, the mathematical understanding of low-dimensional models helps clarify the qualitative, large scale dynamics of more complicated systems. Moreover, low-dimensional models have always played a central role as motivating examples for the development of the theory of Dynamical Systems [GH].

The model we study in the second Chapter is a non-autonomous perturbation of the Lorenz-84 model, constructed by introducing a periodic forcing, which simulates seasonal changes in north-south and ocean-land temperature contrasts. The non-autonomous model is analysed in terms of the Poincaré map [Arn, GH, Kuz], which is a diffeomorphism $P_{F,G,\varepsilon} : \mathbb{R}^3 \to \mathbb{R}^3$ depending on three control parameters $(F, G, \varepsilon)$. The parameter $\varepsilon$ is the relative amplitude of the periodic perturbation. The study focuses on the Dynamical Systems aspects of the model, and in particular on the question, posed at the beginning of this introduction, concerning the structure of strange attractors for three-dimensional diffeomorphisms. Various types of strange attractors are detected in the model; see Figure 1.3 and compare Remark 1.1. We conjecture that the attractors in Figure 1.3 are Hénon-like (A) and quasi-periodic Hénon-like (B), see the previous section. To be precise:

(A) There exists a saddle periodic point $p$ of the map $P_{F,G,\varepsilon}$ such that the attractor in Figure 1.3 (A) coincides with the closure $\text{clos} W^u(\text{Orb}(p))$. 

Figure 1.2: (A) Two-dimensional projection of a Hénon-like strange attractor of the diffeomorphism $T$ studied in the third Chapter (see text for details). (B) Two-dimensional projection of a quasi-periodic Hénon-like strange attractor of $T$. 


1.2 The Lorenz-84 climate model with seasonal forcing

Figure 1.3: (A) Two-dimensional projection of a Hénon-like strange attractor of the diffeomorphism $P_{F,G,\varepsilon}$. The four components are mapped onto each other by $P_{F,G,\varepsilon}$ in the order given by the labels 1 to 4. The component in the box is magnified in Figure 1.1 (A). This attractor coexists with the period four saddle point $p$ in Figure 1.1 (B). (B) Two-dimensional projection of a quasi-periodic Hénon-like strange attractor of $P_{F,G,\varepsilon}$.

(B) There exists a quasi-periodic invariant circle of saddle type such that this attractor coincides with the closure $\text{clos} W^u(\mathcal{C})$.

One of the four components of the attractor in Figure 1.3 (A) is magnified in Figure 1.1 (A). In Figure 1.1 (B) we display a period four saddle point $p$ together with its one-dimensional unstable manifold. This empirical observation suggests that the attractor in Figure 1.3 (A) coincides with $\text{clos} W^u(\text{Orb}(p))$.

The formation of strange attractors is often understood in terms of bifurcation scenarios, also called routes to chaos upon variation of parameters. In this respect, the numerical results are interpreted in the light of bifurcation theory, i.e., the theory of quasi-periodic bifurcations [BHS, BHTB] and of hetero- and homoclinic tangency bifurcations [PT]. The attractor in Figure 1.3 (B) is created after two consecutive quasi-periodic period doublings of an invariant circle, followed by a homoclinic tangency of a saddle periodic point on the circle, occurring inside a resonance tongue [Arn, BST, GH]. The attractor in Figure 1.3 (B) is created after a period doubling cascade of periodic points.

Several papers on the autonomous Lorenz-84 system (see [BSV1] and references therein) are the starting point for the analysis of the map $P_{F,G,\varepsilon}$. The bifurcation diagram of the autonomous system is organised by a codimension two Hopf-saddle-node bifurcation of equilibria [SNN]. This dynamically rich bifurcation has been intensively studied in the general vector field case [BV, GH, Kuz]. In particular, under appropriate hypotheses, near a Hopf-saddle-node of equilibria the following bifurcations occur:

1. Shil’nikov bifurcations of equilibria of saddle-focus type.

2. Hopf–Neimark–Sacker bifurcations of a limit cycle, where a two-dimensional invariant torus is created.

3. Heteroclinic bifurcations, by which the torus is destroyed.
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Figure 1.4: (A) Two-dimensional projection of a quasi-periodic Hénon-like strange attractor of the Poincaré map $P_{F,G,\varepsilon}$ of the forced Lorenz-84 system. (B) Two-dimensional projection of a quasi-periodic Hénon-like strange attractor of the model $G$ studied in Chapter four.

See [BV, GH, Kuz] for the terminology. On the basis of perturbation theory, the Poincaré map $P_{F,G,\varepsilon}$ is expected to display many related phenomena. Indeed, for small $\varepsilon$ we find that the map $P_{F,G,\varepsilon}$ has a Hopf-saddle-node bifurcation of a fixed point. Near this bifurcation point, the map $P_{F,G,\varepsilon}$ exhibits quasi-periodic Hopf bifurcations of an invariant circle [BHTB], where an invariant torus is created. Near these Hopf bifurcations, quasi-periodic Hénon-like attractors are numerically detected, see Figure 1.4 (A).

As far as we know, no systematic investigations of the Hopf-saddle-node bifurcation of fixed points of diffeomorphisms are known. This led to the research in the fourth Chapter.

1.3 Hopf-saddle-node bifurcation for fixed points of diffeomorphisms

In the fourth Chapter we analyse the Hopf-saddle-node bifurcation for diffeomorphisms in two model maps $G$ and $Q$, constructed ‘as generic as possible’ when restricting to the unfolding class they belong to.

Quasi-periodic Hopf bifurcations [BHS, BHTB] of an invariant circle $\mathcal{C}$ occur for $G$ and $Q$, at which a two-dimensional torus $\mathcal{T}$ branches off from $\mathcal{C}$. Bifurcations of an invariant circle $\mathcal{C}$ are well understood only in the case that the dynamics on $\mathcal{C}$ is quasi-periodic. To be more precise, at least two parameters are required for a generic, theoretical analysis of bifurcations of an invariant circle. In two-parameter families of maps having an invariant circle $\mathcal{C}$, Hopf bifurcations of $\mathcal{C}$ do not occur along continuous curves inside the parameter plane. They can only be described in a Cantor-like boundary in the parameter plane, formed by the intersection of a Cantor set of lines with a continuous curve. The resonance gaps in this Cantor-like boundary are called bubbles [BHS, Che]. Not much is known about the behaviour of the bifurcating circle $\mathcal{C}$ for parameter values inside the bubbles.

Quasi-periodic Hénon-like attractors are observed in $G$ and $Q$ (Figure 1.4 left),
near the breakdown of the invariant torus $\mathcal{T}$. The same kind of attractor is found in the Poincaré map $P_{FG,\varepsilon}$ of the forced Lorenz-84 model, compare Figure 1.4 right. In both cases, homoclinic and heteroclinic tangency bifurcations [PT] are involved in the destruction of the torus.

The model $G$ is explicitly constructed for the study of a 1:5 bubble on the Hopf boundary $H$, where $\mathcal{C}$ has a weak resonance of order 1:5 [Arn, Tak]. It turns out that the 1:5 bubble on $H$ is bounded by two secondary Hopf-saddle-node bifurcations of period five points of the map $G$. Both bifurcations take place on the invariant circle $\mathcal{C}$. The bifurcation diagram of $G$ in the neighbourhood of the two secondary Hopf-saddle-node points is quite rich. It involves at least two families of period five invariant circles, two families of invariant tori, and one family of period five invariant tori. For example, in certain regions of the parameter plane a torus $\mathcal{T}$ coexists with a period five circle $\mathcal{C}_5$ (Figure 1.5 left), where both are attractors. Their basins of attraction are separated by a period five repelling invariant torus $\mathcal{T}_5$ (Figure 1.5 right). By varying the parameters, the torus $\mathcal{T}_5$ collides with the circle $\mathcal{C}_5$ through a quasi-periodic Hopf bifurcation. Again, this occurs on a ‘Cantor-like’ bifurcation boundary, interspersed by resonance bubbles. Therefore it seems that near a Hopf-saddle-node bifurcation for diffeomorphisms infinitely many subordinate resonant Hopf-saddle-node bifurcations take place, of various orders and at arbitrarily small scales.

1.4 Overview

Many mathematical problems arise in Dynamical Systems theory from the study of concrete models, such as the Lorenz-84 system with seasonal forcing. Chapter two is an inventory of the dynamics observed in the periodically forced Lorenz-84 model. This material provides the basic examples for the investigations pursued in Chapters three and four.

In Chapter three, two simplified settings are considered for the occurrence of a strange attractor inside $\text{clos}\, W^u(\mathcal{C})$, the closure of the unstable manifold of an invariant circle $\mathcal{C}$ of saddle type. In the first scenario, Hénon-like strange attractors are proved to occur for a diffeomorphism $T$ of the solid torus $\mathbb{R}^2 \times S^1$. For certain parameter values, there exists an invariant circle $\mathcal{C}$ with rational rotation number such that the above strange attractors are contained in $\text{clos}\, W^u(\mathcal{C})$. In a slightly dif-
ferent setting it is proved that \( \text{clos} W^u(\mathcal{C}) \) contains an attractor, where the dynamics on \( \mathcal{C} \) is quasi-periodic. The precise characterisation of the quasi-periodic Hénon-like attractors remains conjectural.

In Chapter four the Hopf-saddle-node bifurcation for fixed points of diffeomorphisms is studied in two model maps. These maps are constructed in such a way that they are likely to be representative of a large class of Hopf-saddle-node diffeomorphisms. The main point of interest is the rich bifurcation diagram near a 1:5 resonance bubble on a Hopf bifurcation boundary.

Throughout the thesis, particular emphasis is put on the study of strange attractors that are typical for three-dimensional diffeomorphisms. The theory of strange attractors for three-dimensional vector fields and for two-dimensional maps is quite developed. See e.g. the results on the Lorenz attractor [GH, Tuc] and on Hénon-like attractors [BC, DRV, MV, Via2, LSY]. On the other hand, attractors of diffeomorphisms having a genuine three-dimensional nature lie at the verge of the theory developed so far [Tat, Via1, Via2]. In this respect, the phenomena discussed in Chapters three and four display many similarities with those encountered in the forced Lorenz-84 model. In particular, quasi-periodic Hénon-like strange attractors seem to occur in a persistent and abundant way.

Many questions remain open for future research. We just mention a few:

1. The study of the Shil’nikov homoclinic tangency bifurcation for fixed points of diffeomorphisms of saddle-focus type; in particular, the analysis of the related attractors.

2. A rigorous theoretical analysis concerning the existence and structure of quasi-periodic Hénon-like strange attractors, in the spirit of [BC, MV, Via2, LSY].

3. A discussion of the Hopf-saddle-node bifurcation for fixed points of diffeomorphisms in a more general setting. Analysis of the global bifurcations occurring near the secondary Hopf-saddle-node points.

The three Chapters of this thesis are based on the papers [BSV1, BSV2, BSV3] respectively.
Bibliography for Chapter one


