Chapter 7

The Monty Hall dilemma

7.1 Introduction

The Monty Hall Dilemma is a puzzle that often leads to furious discussions. The puzzle received worldwide attention when it was discussed in Marilyn vos Savant’s column in *Parade Magazine* (Vos Savant (1990)). In her column ‘Ask Marilyn’ she answers questions sent in by the readers.

> Suppose you’re on a game show, and you’re given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say number 1, and the host, who knows what’s behind the doors, opens another door, say number 3, which has a goat. He says to you, “Do you want to pick door number 2?” Is it to your advantage to switch your choice of doors?

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The Monty Hall Dilemma got its name from the American game show host Monty Hall, who figured in the problem in an article by Selvin (1975). Other versions of this puzzle circulated at least as early as the sixties, see Mosteller (1965, p.4, for example). Answering this question, Vos Savant argued that it is to your advantage to switch. If you switch you lose in one third of the cases and win in two thirds of the cases.

The claim made by Vos Savant, who is listed in the Guinness Book of World Records for the highest IQ, can be argued as follows. If you switch you get a goat in one third of the cases and win the car in two thirds of the cases. This could be argued as follows. Suppose you initially pick the door with the car, then you should not switch. This happens in one third of the cases. Suppose on the other hand you initially pick a door that contains a goat, which happens in two third of the cases. Monty Hall cannot open the door with the car and he cannot open
the door you picked. He has to open the other door with a goat. So, if you pick a door with a goat, Monty Hall only has one option. After he opens that door, the remaining unopened door you did not pick must contain the car. Therefore, if you initially pick a door with a goat, switching will guarantee that you win the car. You pick such a door in two third of the cases. Hence by switching you lose in one third of the cases and you win in two third of the cases.

Many people did not agree with this solution. They argued the chances of winning do not increase when you switch. Among them are some considered to be experts in the field of probability. Three Ph.D.’s wrote letters explaining that Vos Savant was wrong. The mathematician Paul Erdős also did not want to believe switching is to your advantage. The discussion made its way to the Netherlands after Rob van den Berg reported the discussion in the newspaper NRC-Handelsblad (May 18th, 1995). The response to his article was overwhelming. People called to say they could not sleep because they were thinking about the puzzle and demanded an explanation, many e-mails were sent and the newspaper received over eighty letters. Some of these letters were published (the 1st, 8th, and 15th of June). I found the letter by H. von Saher to be the most surprising:

I will show with an analogous example that her thesis is incorrect. Marilyn vos Savant wins a quiz (naturally), at the back of the stage a wall is placed with 100 doors. She takes her stand in front of door 1; in that case she has a 1% chance of standing in front of the right door and there is a 99% chance that the prize is behind one of all the other doors. Then the quiz master comes along; he opens all doors from 2 to 99, so 98 doors in total. Behind none of these doors is the prize. It must be behind door 1 or 100. According to Marilyn vos Savant the whole 99% chance passes to door 100. It is evident that this is absurd. Here too another situation has arisen with only two alternatives and therefore equal chances for each of the remaining doors. (NRC June 1st, 1995, my translation)

Although it seems that some of the crucial information eludes Von Saher, namely that the quiz master only opens a door if he knows it does not contain a prize, he apparently presents a good argument for Vos Savant’s thesis, instead of against it. Suppose you pick door number one and all the other doors except door number 53 are opened; to me it seems even more obvious that in that case you should switch than in the three door case. These arguments seem to be difficult to understand and can even make people quite angry. Marilyn vos Savant was ridiculed in many of the letters written to her.

There were also letters by people who reported that they had actually played the game or had made a computer simulation of the game confirming Vos Savant’s claim. Several good simulations of the Monty Hall Dilemma can be found on the Internet by searching for the Monty Hall Dilemma. Theo Kuipers wrote that
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playing the game with his wife not only convinced his wife, but also made them find an elegant solution, see Wooters (1991). Yet some who reported that their findings concurred with Vos Savant were still not convinced. Some even offered their computer programs to be distributed hoping that someone could find a bug. Experimental data can give a correct answer to the question whether you should switch, but experimental data alone do not yield an intuitively appealing analysis of the problem.

The Monty Hall Dilemma is a puzzle for which intuitions fail many people. It is surprising that these wrong intuitions are very strong. But there are many puzzles and paradoxes where one can have strong intuitions that are wrong. One might think for example that there are more natural numbers than there are prime numbers, but this is not true. The best way to show that such counterintuitive results are in fact correct is to use some formal method such as logical analysis. In this chapter I use the logic that was introduced in chapter 6 to analyze this puzzle.

7.2 A semantical analysis

In section 6.4.1 a method was introduced to build a model for specific situations. In this section this method is applied to the Monty Hall Dilemma. The set of propositional variables $\mathcal{P}$ is the union of the three sets $A = \{A_1, A_2, A_3\}$ (where $A_i$ means that the car is behind door number $i$), $C = \{C_1, C_2, C_3\}$ (where $C_i$ means that the contestant initially chooses door number $i$), and $O = \{O_1, O_2, O_3\}$, (where $O_i$ means that door number $i$ is opened by Monty Hall).

There are twelve possible outcomes, which can be seen as follows. There are three possible locations for the car. For each of these the contestant can choose three doors. For each of these choices Monty Hall can either open one or two doors. If the same door is chosen as where the car is, both the other doors can be opened. If another door is chosen, the remaining door that does not contain the car must be opened. Initially the contestant cannot distinguish any of these possibilities, and Monty Hall just knows where the car is. An epistemic model for this situation is given in figure 7.1.

The question is what would be considered an appropriate prior probability distribution over these worlds. In the face of ignorance, the usual approach is to consider a uniform probability distribution. But that is inappropriate. There are twelve possible worlds and all would get a probability of $\frac{1}{12}$. If we were to calculate the probabilities assigned to worlds according to Monty Hall we get something odd. Suppose the car is behind door number one. In this case, the domain of the probability function assigned to Monty Hall consists of those worlds where the car is behind door number one only. These are four worlds and each would get probability $\frac{1}{4}$. Consequently Monty Hall would assign probability $\frac{1}{2}$ that the contestant initially chooses door number one. But it seems more appropriate that
Monty Hall would assign probability $\frac{1}{2}$ to that. What the contestant is concerned a uniform distribution would also be inappropriate. Suppose that in that case the player chooses door number one. After updating with his own choice there are only four worlds remaining with probability $\frac{1}{4}$ each. But that would mean that although the contestant initially assigned probability $\frac{1}{4}$ to the car being behind door number one, after choosing door number one it would be $\frac{1}{4}$. It seems that afterwards it should still be $\frac{1}{3}$. So a uniform distribution is not appropriate.

So the requirements seem to be that Monty Hall should assign probability $\frac{1}{3}$ to all choices the contestant can make and the contestant should assign probability $\frac{1}{3}$ to all the possible locations of the car, before and after choosing a door. There are many ways to satisfy these constraints, but these requirements do fix the probability assigned to worlds where the door chosen by the contestant is not the door with the car; this probability must be $\frac{1}{3}$. The rest depends on what Monty Hall does if the contestant chooses the door with the car. In that case he can open either of the two remaining doors. Let us assume that in that case he chooses according to a uniform distribution over those two doors. This seems reasonable. The model is shown in figure 7.2.

Now we can use the semantics of probabilistic dynamic epistemic logic to see what happens to the information of both the contestant and Monty Hall during the quiz. The contestant chooses a door, let us assume it is door number one and Monty Hall opens a door containing a goat, let us assume it is door number three. Let us also assume that in fact the car is behind door number two. So now we want to know whether the sentence $[C_1]_{O_2} [P_c(A_1) = \frac{1}{3} \land P_c(A_2) = \frac{2}{3}]$ is true in the world where the car is behind door number two, door number one is chosen and door number three is opened. Therefore we have to calculate the results of the updates. This is shown in figure 7.3, and indeed in the resulting model the probability that the car is behind door number one is one third according to the

![Figure 7.1: An epistemic model of the initial situation of the Monty Hall Dilemma.](image)
7.3 A syntactic analysis

In a syntactic approach the idea is that we try to represent the inference in the language of probabilistic epistemic logic, and show that it is valid. So, what are the premises of the inference. One of the rules is that there is only one car behind the doors, the contestant may only choose one door, and Monty Hall may only open one door. We use the same sets of propositional variable as in the previous
section.

\[
\text{onecar} = \bigoplus A \\
\text{onechoice} = \bigoplus C \\
\text{oneopen} = \bigoplus O
\]

Where \(\bigoplus\) means exclusive or. I assume that the contestant should assign a probability of \(\frac{1}{3}\) to the car being behind a particular door. This is an assumption that has to be made to get vos Savant’s answer. Moreover I assume the contestant does not learn anything about the location of the car by picking a door. Therefore the contestant should still assign a probability of \(\frac{1}{3}\) after picking a door: the contestant’s choice is independent of where the car is.

\[
\text{equal} = \bigwedge_{i \in \{1,2,3\}} P_c(A_i) = \frac{1}{3}
\]

\[
\text{independentAC} = \bigwedge_{j \in \{1,2,3\}} [C_j]\text{equal}
\]

This is a nice way of expressing independence. This assumption remains implicit in most other analyses I found of the Monty Hall Dilemma. The crucial part of the analysis of the Monty Hall Dilemma is to see under what conditions Monty Hall opens a door. He opens exactly one door such that the contestant did not pick it and the car is not behind it.

\[
\text{conditions} = \bigwedge_{i,j \in \{1,2,3\}} [C_i](O_j \leftrightarrow (\neg A_j \land \neg C_j \land \bigwedge_{k \in \{1,2,3\}} \neg O_k))
\]

Let us use initial as an abbreviation for the conjunction of onecar, onechoice, oneopen, equal, independentAC, and conditions.

The question is whether the contestant should switch or not:

\[
\text{switch} = [C_i][O_3]P_c(A_i) \leq P_c(A_2)
\]

If this sentence is true, then the chances that the contestant wins the car do not decrease by switching. It turns out that initial is not enough to deduce this result. What is needed is that the contestant is informed about the game: \(P_c(\text{initial}) = 1\). We also need two other very natural assumptions, namely that \(P_c(C_i) > 0\) and \([C_i]P_c(O_3) > 0\). This suffices to deduce switch.

The independentAC assumption implies that \([C_i]P_c(A_i) = \frac{1}{3}\), and therefore:

\[
[C_i]P_c(O_3 \land A_1) \leq \frac{1}{3}
\]

By conditions, onechoice, and oneopen we get \([C_i]P_c(A_2 \rightarrow O_3) = 1\). Some probabilistic reasoning gives us that \([C_i]P_c(O_3 \land A_2) = P_c(A_2)\). This, together with \([C_i]P_c(A_2) = \frac{1}{3}\) (from independentAC), allows us to infer that \([C_i]P_c(O_3 \land A_2) = \frac{1}{3}\), which yields

\[
[C_i]P_c(O_3 \land A_1) \leq P_c(O_3 \land A_2)
\]
7.2. A syntactic analysis

By atomic permanence we get

\[ [C_1] \mathbf{P}_c(O_3 \land [O_3] A_1) \leq \mathbf{P}_c(O_3 \land [O_3] A_2) \]

From the Probability-Update 1 axiom it follows (using some rewriting and the 0-terms axiom) that

\[ \mathbf{P}_c(O_3) > 0 \rightarrow (\{[O_3] \mathbf{P}_c(A_1) \leq \mathbf{P}_c(A_2)\} \leftrightarrow (\mathbf{P}_c(O_3 \land [O_3] A_1) \leq \mathbf{P}_c(O_3 \land [O_3] A_2))) \]

By applying necessitation with \([C_1]\), distribution, the assumption that \([C_1] \mathbf{P}_c(O_3) > 0\), and propositional reasoning we get:

\[ [C_1][O_3] \mathbf{P}_c(A_1) \leq \mathbf{P}_c(A_2) \]

Thus far we have made no assumptions about the strategy used by the contestant or Monty Hall. We do not need this to deduce switch, but we do need to assume something about the strategy of Monty Hall if we want to deduce that the probability that the contestant wins the car by switching equals two thirds. Then we need to assume that if Monty Hall can choose between opening two doors (if the door the contestant picked is the same as where the car is), then the probability he opens one door is the same as the probability he opens the other door. This boils down to:

\[
\text{equalopen} = \bigwedge_{\{i, j, k\} = \{1, 2, 3\} \atop i \neq j} [C_1] \mathbf{P}_c(O_j) = \mathbf{P}_c(O_k)
\]

With this we can deduce:

\[
\text{Savant} = [C_1][O_3] \mathbf{P}_c(A_1) = \frac{1}{3} \land \mathbf{P}_c(A_2) = \frac{2}{3}
\]

In order to deduce this, first of all we have to see that using conditions and one car we can get

\[
[C_1] \mathbf{P}_c(O_3) = \mathbf{P}_c(O_3 \land A_1) + \mathbf{P}_c(O_3 \land A_2)
\]

Moreover given similar reasoning as before we get \([C_1] \mathbf{P}_c(O_3 \land A_1) = \mathbf{P}_c(A_1)\). Therefore using independent\(\Box\) we get

\[
[C_1] \mathbf{P}_c(O_3) = \mathbf{P}_c(O_3 \land A_1) + \frac{1}{3}
\]

Moreover from one open and equalopen and conditions we get that \(\mathbf{P}_c(O_3) = \frac{1}{3}\). Therefore \(\mathbf{P}_c(O_3 \land A_1) = \frac{1}{3}\). In order to apply the Probability-Update 1 axiom we need to get the right form. But given what we have we can deduce

\[
[C_1] \mathbf{P}_c(O_3 \land A_1) = \frac{1}{3} \times \mathbf{P}_c(O_3)
\]
with atomic permanence we get

\[ [C_1]p(O_3 \land [O_3]A_1) = \frac{1}{3} \times p(O_3) \]

As before, using Probability-Update 1, we get

\[ [C_1][O_3]p(A_1) = \frac{1}{3} \]

Given conditions we have that

\[ [C_1][O_3]p(A_1) + p(A_2) = 1 \]

From this we can deduce Savant.

7.4 Conclusion

There are other formal methods one might use to analyze the Monty Hall Dilemma. One might use a Bayesian approach. One can also think of decision theory or game theory. I do not at all reject these methods as successful means of analyzing the Monty Hall Dilemma. There is a vast literature about the Monty Hall Dilemma. In my view a logical approach to the problem is the best.

Why does the solution of the Monty Hall Dilemma seem so counterintuitive? The question whether the contestant should switch door in order to increase his chance of winning the car is usually answered by stating that it does not matter whether you should switch or not. There is one very natural way by which to arrive at this answer. The information Monty Hall provides by opening door number three is simply seen as an update with the sentence \( \neg A_3 \). If one updates with this sentence, then the probability that the car is behind any of the remaining doors equals \( \frac{1}{2} \). But this is not the only information Monty Hall provides. The conditions that must hold in order for him to be able to provide the information that the car is not behind door number three are also provided. The question is whether the formalism employed to analyze the dilemma is rich enough to be able to express these conditions. In other words, one should be able to write them down. As was seen in the previous section the logic presented in chapter 6 is such a system. Because logic deals with the question whether an inference is valid, I think logic is the best way to analyze problems such as the Monty Hall Dilemma.