1 A behavioral approach to model reduction

1.1 Introduction and outline of the thesis

The aim of this thesis is to study a number of problems in the theory of model reduction and approximation for dynamical systems within the behavioral approach.

The problem of model reduction and approximation plays an important role in systems and control theory. The problem is to replace a given complex model by a model of lower complexity, while the new model is a reasonable approximation of the original one. This means that the error between the original model and its approximation needs to be small. For some model reduction and approximation methods, error bounds are known, for example in the methods of balanced truncation [42] or Hankel optimal norm approximation [18]. Often, in addition to reduction of complexity of the original model, preservation of certain properties of the model is required. An example of this is preservation of stability and passivity. Moreover, a very important requirement is that the reduction procedure is computationally efficient. All requirements together make the problem of model reduction one of the most difficult problems in system and control theory.

In [1] it is argued that currently a ‘best’ method, i.e. a reduction method that meets all requirements, does not yet exist. Some methods are theoretically beautiful, provide global error bounds and preserve stability, passivity, but are not computationally efficient, for example SVD based methods (see [42, 18, 6, 48]). On the other hand, some methods are efficient from a computational point of view but do not have error bounds and do not preserve stability, for example Padé approximation (see [4]) or moment-matching methods (see [84, 36]).

In this thesis, we study a class of theoretical methods focusing especially on the preservation of passivity (or dissipativity in general) of the systems. Passivity is an important characteristic of linear systems. It appears in many engineering applications, such as electrical systems, mechanical systems, thermodynamics, etc. Several methods for model reduction with stability and passivity preservation have been introduced in the past (see for example [34, 35, 70, 6, 45, 48]) and the new methods presented recently by Antoulas (see [2]) and Sorensen (see [60]).
The approach of this thesis is inspired by the axiomatic foundation for the general theory of dynamical systems developed by J.C. Willems in [76, 61], called the behavioral approach. In this theory, a system is not considered as a mapping that transforms inputs to outputs. Instead, the system is identified with the set of all time functions compatible with its laws. This set, called the behavior of the system, plays an important role in this theory. One of the main advantages of the behavioral approach is that, since there is a distinction between the behavior of the system and its representation, it allows to treat a wide variety of model classes, more directly emanating from modeling. In particular, the behavioral approach allows to deal with state space models and transfer functions as special cases of more general model specifications.

The behavioral approach to systems and control is a natural and powerful setting for the problem of model reduction and approximation. Due to the fact that the approach deals with models on a more abstract and general level it is able to capture much better the essential concepts and structural intricacies of model reduction and approximation. An example of this is Theorem 4.3.1 in Chapter 4 where an intrinsic, behavioral, description is given of the system invariants appearing in passivity and bounded realness preserving model reduction by balanced truncation.

We now give a description of the outline of this thesis.

**Chapter 1: A behavioral approach to model reduction.** This chapter contains the mathematical background which will be used in later chapters of this thesis. The chapter focuses on definitions and basic results from the behavioral theory. The notions of linear differential systems, dissipative systems, quadratic differential forms, input-state-output, driving-variable and output-nulling representations are explained in the chapter. References for the contents of the chapter are [61, 80, 41, 71].

**Chapter 2: Review of balancing methods for linear systems.** In this chapter we give a review of the existing balanced truncation methods. We start with the classical method of Lyapunov balancing, which is introduced in [42]. This method is applicable only to stable linear systems. Based on this method, several extensions exist, applicable to unstable linear systems. These methods are the LQG balancing method and methods based on balancing of the normalized coprime factor. These methods are also discussed in this chapter. We also summarize the work of Weiland [71] in order to understand the LQG balancing method in a behavioral framework. Finally, we review the positive real and bounded real balancing methods applied for positive real and bounded real linear systems.
Chapter 3: Dissipativity preserving model reduction by retention of trajectories of minimal dissipation. The aim of this chapter is to extend the technique of Antoulas (see [2]) and Sorensen (see [60]) from the classical state space framework to the general framework of the behavioral theory of dissipative systems. The main results of the chapter are algorithmic procedures to compute, for a given behavior represented in driving variable representation or output nulling representation, a reduced-order behavior that preserves dissipativity, while a given subbehavior of the antistable part of the stationary trajectories of the original behavior is retained. It will turn out that the transfer matrix of the reduced order behavior interpolates the transfer matrix of the original behavior in certain directions, with interpolation points at some of the antistable spectral zeroes of the original behavior, and their mirror images, see also [11].

Chapter 4: Dissipativity preserving model reduction by balanced truncation with error bounds. The aim of this chapter is to study the method of balanced truncation for strictly half line dissipative systems whose input cardinality equals the positive signature of the supply rate. The main results of this chapter are Theorem 4.3.1 and the balanced truncation algorithm in section 4.6. Theorem 4.3.1 can be seen as an extension of the work of Weiland [71] in the context of strictly half line dissipative systems, in which only the past behavior is an inner product space, while on the future behavior we only have an indefinite inner product. The reduction algorithm in section 4.6 can be seen as a generalization of the positive real and bounded real balancing methods introduced in Chapter 2.

Chapter 5: Balanced state-space representations: a polynomial algebraic approach. In this chapter we illustrate several results of independent interest. We show how to iteratively compute from the supply rate and an image representation of the system, the two-variable polynomial matrix representing any storage function. Moreover, we also provide an algorithm to compute the factorization of any storage function as a quadratic function of a minimal state map (see [61]). This factorization can be used in order to ascertain whether a storage function is positive along a behavior, a property which has important consequences in the behavioral theory of dissipative systems (see [80]). Based on this factorization, the problem of obtaining a balanced state map from the an image representation of the system is solved.
Let \( w \) denote a vector-valued variable whose components consists of the system variables. We define the signal space \( W \) as the space where the variable \( w \) takes its values. Usually \( w \) itself is a function of an independent variable called time, which takes its values in a set called the time axis. \( T \) denotes the time axis. Let \( W^T \) denote the set of functions from \( T \) to \( W \). Thus \( w \) is an element of \( W^T \). Not every element in \( W^T \) is allowed by the laws governing the behavior/dynamics of the system. The set of functions that are allowed by the system is precisely the object of our study, and this set is called the behavior. The laws that govern a system bring about this restriction of \( W^T \) to its behavior. Thus a system is viewed as an exclusion law indicating which trajectories are admissible for the system. This leads to the following definition of a dynamical system (Willems [76]).

**Definition 1.2.1.** A dynamical system \( \Sigma \) is a triple \( \Sigma = (T,W,B) \) with \( T \) a set, called the time axis; \( W \) a set, called the signal space, and \( B \subseteq W^T \) the behavior of the system.

We now come to properties of dynamical systems. We call the set of functions \( W^T \) the universum \( \mathcal{U} \). The behavior of a system is a subset of the universum \( \mathcal{U} \). An element of the behavior is a function with domain \( T \) and co-domain \( W \). In this thesis we will study dynamical systems which have three properties: linear, time-invariant and described by ordinary differential equations.

**Definition 1.2.2.** A dynamical system \( \Sigma = (T,W,B) \) is called linear if

1. \( W \) is a vector space over \( \mathbb{R} \), and,
2. the behavior \( B \) is a subspace of \( W^T \), i.e.

\[
w_1, w_2 \in B \quad \text{and} \quad \alpha_1, \alpha_2 \in \mathbb{R} \Rightarrow \alpha_1 w_1 + \alpha_2 w_2 \in B.
\]

The latter is called the superposition principle.

If the time axis \( T \) is a semi-group, and \( \sigma^t w \) is defined by \( (\sigma^t w)(\tau) = w(t+\tau) \) for all \( t, \tau \in T \), then we can also define time invariance of a behavior.

**Definition 1.2.3.** A dynamical system \( \Sigma = (T,W,B) \) is called time-invariant if for each trajectory \( w \in B \) the shifted trajectory \( \sigma^t w \) is again an element of \( B \), for all \( t \in T \).
1.2 Linear differential systems

In this thesis we will restrict ourselves to a special class of linear time-
invariant dynamical systems, called linear differential systems. Suppose we
have \( g \) scalar linear constant coefficient differential equations written in the
following form:

\[
R_0 w + R_1 \frac{dw}{dt} + \cdots + R_L \frac{d^L}{dt^L} w = 0 \tag{1.1}
\]

where \( R_i \in \mathbb{R}^{g \times w} \) for each \( i = 0 \ldots L \). By introducing the polynomial matrix
\( R(\xi) = R_0 + R_1 \xi + \cdots + R_L \xi^L \), the \( g \) equations in equation (1.1) can be
written in the form \( R\left(\frac{d}{dt}\right)w = 0 \). Consider the set of \( C^\infty(\mathbb{R}, \mathbb{R}^w) \) solutions of
the equations (1.1), where \( C^\infty(\mathbb{R}, \mathbb{R}^w) \) denotes the space of all infinitely often
differentiable functions from \( \mathbb{R} \) to \( \mathbb{R}^w \). This set of solutions defines the system
\( \Sigma = (T, R_w, B) \) with

\[
B = \left\{ w \in C^\infty(\mathbb{R}, \mathbb{R}^w) \mid R\left(\frac{d}{dt}\right)w = 0 \right\}. \tag{1.2}
\]

Obviously, this system is linear and time-invariant. Such a system \( \Sigma \) is called
a linear differential system. The set of all linear differential systems with \( w \)
variables will be denoted by \( L^w \). We simply write \( B \in L^w \) instead of writing
\( \Sigma \in L^w \). The representation (1.2) of \( B \) is called a kernel representation
of \( B \), and we often write \( B = \text{ker} R\left(\frac{d}{dt}\right) \).

Another type of representation by which linear differential systems can
be given is the latent variable representation, defined through polynomial
matrices \( R \) and \( M \) by

\[
R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell,
\]

where \( B = \left\{ w \in C^\infty(\mathbb{R}, \mathbb{R}^w) \mid \exists \ell \in C^\infty(\mathbb{R}, \mathbb{R}^\ell) \ s.t. \ R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell \right\}. \tag{1.3}
\]

The behavior \( \mathcal{B} \) is then called the external or manifest behavior, and \( \mathcal{B}_{\text{full}} := \{(w, \ell) \in C^\infty(\mathbb{R}, \mathbb{R}^{w+\ell}) \mid R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell \} \), the full behavior. Here, \( w \) is
called manifest variable and \( \ell \) is called latent variable. If \( \mathcal{B} \) is the external
behavior of \( \mathcal{B}_{\text{full}} \), then we often write \( \mathcal{B} = (\mathcal{B}_{\text{full}})_{\text{ext}} \).

We also need the notion of state for a behavior. A latent variable representa-
tion of \( \mathcal{B} \in L^w \) is called a state representation if the latent variable
(denoted here by \( x \) ) has the property of state, i.e.: if \( (w_1, x_1), (w_2, x_2) \in \mathcal{B}_{\text{full}} \)
are such that \( x_1(0) = x_2(0) \) then \( (w_1, x_1) \wedge (w_2, x_2) \), the concatenation (at
t = 0, here) defined as

\[
(w_1, x_1) \wedge (w_2, x_2)(t) := \begin{cases} (w_1(t), x_1(t)), & t < 0 \\ (w_2(t), x_2(t)), & t \geq 0, \end{cases}
\]

belongs to (the \( C^\infty \)-closure) of \( \mathcal{B}_{\text{full}} \). We call such an \( x \) a state for \( \mathcal{B} \).
A latent variable representation is a state representation of its manifest behavior if and only if its full behavior \( B_{\text{full}} \) can be represented by a differential equation that is zero-th order in \( w \) and first order in \( x \), i.e., by \( R_0w = M_0x + M_1 \frac{d}{dt}x \), with \( R_0, M_0, M_1 \) constant matrices (see [53] Theorem 4.3.5 and the remark follows). A state representation is said to be \textit{minimal} if the state has minimal dimension among all state representations that have the same manifest behavior. The dimension of this state is the same for all state representation of a given behavior. This number denoted by \( n(B) \) and is called the McMillan degree of \( B \). There are many, more structured, types of state representations as, for instance, the \textit{driving variable representation} \( \frac{d}{dt}x = Ax + Bu, \; w = Cx + Du \), with \( v \) an, obviously free, additional latent variable; the \textit{output nulling representation} \( \frac{d}{dt}x = Ax + Bw, \; 0 = Cx + Dw \); or the \textit{input-state-output representations} \( \frac{d}{dt}x = Ax + Bu, \; y = Cx + Du, \; w = (u,y) \), the most popular of them all.

We will collect the basic material on these representations in section 1.6.

In [55] it was shown that a state variable (and in particular, a minimal one) for \( B \) can be obtained from the external- or full trajectories by applying to them a state map, defined as follows. Let \( X \in \mathbb{R}^{n\times\ell}[\xi] \) be such that the subspace of \( \mathcal{C}_\infty(\mathbb{R},\mathbb{R}^{w+n}) \) defined by

\[
\{(w,X\left(\frac{d}{dt}\right)w) \mid w \in B\}
\]

is a state system; then \( X\left(\frac{d}{dt}\right) \) is called a state map for \( B \), and \( X\left(\frac{d}{dt}\right)w \) is a state variable for \( B \). In the following we will consider state maps for systems in image form; in this case it can be shown (see [55]) that a state map can be chosen acting on the latent variable \( \ell \) alone, and we will consider state systems

\[
w = M\left(\frac{d}{dt}\right)\ell
\]

\[
x = X\left(\frac{d}{dt}\right)\ell,
\]

with \( x \) a state variable.

The definition of minimal state map follows in a straightforward manner. In [55] algorithms are stated to construct a state map from the equations describing the system.

We call \( B \in \mathcal{L}^w \text{ controllable} \) if for all \( w_1,w_2 \in B \), there exists a \( T \geq 0 \) and a \( w \in B \) such that \( w(t) = w_1(t) \) for \( t < 0 \) and \( w(t+T) = w_2(t) \) for \( t \geq 0 \). Denote the subset of all controllable elements of \( \mathcal{L}^w \) by \( \mathcal{L}^w_{\text{cont}} \). For controllable systems, it can be shown in [53] Theorem 6.6.1 that \( B \in \mathcal{L}^w_{\text{cont}} \) if and only if it admits an \textit{image representation}, i.e. there exists an \( M \in \mathbb{R}^{w\times\ell}[\xi] \) such that
\[ \mathcal{B} = \{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \mid \text{there exists } \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^1) \ w = M \frac{d}{dt} \ell \}. \]

If \( \mathcal{B} \) is represented in this way, we also write \( \mathcal{B} = \text{im} M \left( \frac{d}{dt} \right) \).

The **controllable part** of a behavior is defined as follows. Let \( \mathcal{B} \in \mathcal{L}^w \). There exists \( \mathcal{B}' \in \mathcal{L}^w_{\text{cont}}, \mathcal{B}' \subset \mathcal{B} \) such that \( \mathcal{B}'' \in \mathcal{L}^w_{\text{cont}}, \mathcal{B}'' \subset \mathcal{B} \) implies \( \mathcal{B}'' \subset \mathcal{B}' \), i.e., \( \mathcal{B}' \) is the largest controllable sub-behavior contained in \( \mathcal{B} \). Denote this system as \( \mathcal{B}_{\text{cont}} \). It can be shown that \( \mathcal{B}_{\text{cont}} \) is the closure in the \( \mathcal{C}^\infty \)-topology of \( \mathcal{B} \cap \mathcal{D} \).

A behavior \( \mathcal{B} \in \mathcal{L}^w \) is called **autonomous** if it is a finite-dimensional subspace of \( \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \). It can be shown that an autonomous behavior admits a kernel representation \( \mathcal{B} = \ker(\mathcal{R}(\frac{d}{dt})) \) in which the matrix \( \mathcal{R} \) is \( w \times w \) and nonsingular. The polynomial \( \det \mathcal{R}(\xi) \), denoted by \( \chi_{\mathcal{B}} \), and is called the **characteristic polynomial** of \( \mathcal{B} \); their roots are called the **characteristic frequencies** of \( \mathcal{B} \). It can be proved (see Theorem 3.2.16 of [53]) that if \( \mathcal{B} \) is autonomous and \( \lambda_i \in \mathbb{C}, i = 1, \ldots, r \) are the distinct roots of the characteristic polynomial \( \chi_{\mathcal{B}} \), each with multiplicity \( n_i \), then \( w \in \mathcal{B} \) if and only if

\[
\begin{align*}
w(t) &= \sum_{i=1}^{r} \sum_{j=0}^{n_i-1} v_{ij} t^j e^{\lambda_i t} \quad (1.4)
\end{align*}
\]

where the vectors \( v_{ij} \in \mathbb{C}^w \) satisfy \( \sum_{j=0}^{n_i-1} (j) R^{(j-k)}(\lambda_i)v_{ij} = 0 \), with \( R^{(j-k)} \) denoting the \((j-k)\)-th derivative of the matrix polynomial \( R \). Consequently, in the autonomous case every trajectory \( w \in \mathcal{B} \) is a linear combination of polynomial-exponential trajectories associated with the characteristic frequencies \( \lambda_i \).

Given a behavior \( \mathcal{B} \in \mathcal{L}^w \), it is in general possible to choose some components of \( w \) as any function in \( \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \). The maximal number of such components that can be chosen arbitrarily is called the **input cardinality** of \( \mathcal{B} \) and is denoted as \( m(\mathcal{B}) \). The number \( m(\mathcal{B}) \) is exactly equal to the dimension of the input \( u \) in any input/state/output representation of \( \mathcal{B} \). The complementary number \( w - m(\mathcal{B}) \) is called the **output cardinality** of \( \mathcal{B} \) and denoted by \( p(\mathcal{B}) \). If none of the components of \( w \) can be chosen arbitrarily, then \( \mathcal{B} \) is autonomous, and we write \( m(\mathcal{B}) = 0 \).

Let \( \mathcal{B} \in \mathcal{L}^w_{\text{cont}} \) and \( \Sigma \in \mathbb{R}^{w \times w} \) be symmetric, we define the \( \Sigma \)-**orthogonal complement** \( \mathcal{B}^{\perp\Sigma} \) of \( \mathcal{B} \) as

\[
\mathcal{B}^{\perp\Sigma} := \{ w \in \mathcal{C}^\infty \mid \int_{-\infty}^{+\infty} w^T \Sigma \delta \ dt = 0 \text{ for all } \delta \in \mathcal{B} \cap \mathcal{D} \}.
\]

It can be proven that \( \mathcal{B}^{\perp\Sigma} \) is also a controllable behavior, see section 10 of [80]. If \( \Sigma = I \), we simply write \( \mathcal{B}^{\perp} \), called the **orthogonal complement** of \( \mathcal{B} \).
1.3 Some notations

We denote by \( \mathbb{R}_- \) the set of negative real numbers, and by \( \mathbb{R}_+ \) the complementary set of nonnegative real numbers. \( \mathbb{C}_- (\mathbb{C}_+) \) is the subset of \( \mathbb{C} \) of all \( \lambda \) such that \( \text{Re}(\lambda) < 0 \) (\( \text{Re}(\lambda) > 0 \)). The Euclidean linear space of \( n \) dimensional real, respectively complex, vectors is denoted by \( \mathbb{R}^n \), respectively \( \mathbb{C}^n \), and the space of \( m \times n \) real, respectively complex, matrices, by \( \mathbb{R}^{m \times n} \), respectively \( \mathbb{C}^{m \times n} \). Given two column vectors \( x \) and \( y \), we denote with \( \text{col}(x,y) \) the vector obtained by stacking \( x \) over \( y \); a similar convention holds for the stacking of matrices with the same number of columns. If \( A \in \mathbb{R}^{k \times n} \), then \( A^T \in \mathbb{R}^{n \times p} \) denotes its transpose. If \( A \in \mathbb{C}^{k \times n} \), then \( A^* \in \mathbb{C}^{n \times p} \) denotes its complex conjugate transpose.

In this thesis we use the following function spaces:

- \( \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^n) \): the space of all infinitely differentiable functions from \( \mathbb{R} \) to \( \mathbb{R}^n \).
- \( \mathcal{D}(\mathbb{R},\mathbb{R}^n) \): the subspace of \( \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^n) \) consisting of all compactly supported functions.
- \( \mathcal{L}^{1,\text{loc}}(\mathbb{R},\mathbb{R}^n) \): the space of all locally integrable functions \( w \) from \( \mathbb{R} \) to \( \mathbb{R}^n \), i.e. all \( w : \mathbb{R} \rightarrow \mathbb{R}^n \) such that \( \int_a^b \|w(t)\| \, dt < \infty \) for all \( a,b \in \mathbb{R} \).
- \( \mathcal{L}^{2,\text{loc}}(\mathbb{R},\mathbb{R}^n) \): the space of all locally square integrable functions \( w \) from \( \mathbb{R} \) to \( \mathbb{R}^n \), i.e. all \( w : \mathbb{R} \rightarrow \mathbb{R}^n \) such that \( \int_a^b \|w(t)\|^2 \, dt < \infty \) for all \( a,b \in \mathbb{R} \).
- \( \mathcal{L}_2(\mathbb{R},\mathbb{R}^n) \): the space of all square integrable functions on \( \mathbb{R} \); i.e. all \( w : \mathbb{R} \rightarrow \mathbb{R}^n \) such that \( \int_\mathbb{R} \|w(t)\|^2 \, dt < \infty \). The \( L_2 \)-norm of \( w \) is \( \|w\|_2 := (\int_\mathbb{R} \|w(t)\|^2 \, dt)^{1/2} \).
- \( \mathcal{L}_2(\mathbb{R}_-\mathbb{R}^n) \): the space of all square integrable functions on \( \mathbb{R}_- \); i.e. all \( w : \mathbb{R}_- \rightarrow \mathbb{R}^n \) such that \( \int_{\mathbb{R}_-} \|w(t)\|^2 \, dt < \infty \).
- \( \mathcal{L}_2(\mathbb{R}_+\mathbb{R}^n) \): the space of all square integrable functions on \( \mathbb{R}_+ \); i.e. all \( w : \mathbb{R}_+ \rightarrow \mathbb{R}^n \) such that \( \int_{\mathbb{R}_+} \|w(t)\|^2 \, dt < \infty \).

Sometimes, when the domain and co-domain are obvious from the context, we simply write \( \mathcal{C}, \mathcal{D}, \mathcal{L}^{1,\text{loc}}, \mathcal{L}^{2,\text{loc}}, \mathcal{L}_2, \mathcal{L}_2(\mathbb{R}_-) \) and \( \mathcal{L}_2(\mathbb{R}_+) \). If \( F(t) \) is a real \( p \times m \) matrix valued function, then the space of all functions formed as linear combinations of the columns of \( F(t) \) is denoted by \( \text{span}\{F(t)\} := \{F(t)x_0 \mid x_0 \in \mathbb{R}^m\} \). For a given function \( w \) on \( \mathbb{R} \), we denote by \( w|_{\mathbb{R}_-} \) and \( w|_{\mathbb{R}_+} \) the restrictions of \( w \) to \( \mathbb{R}_- \) and \( \mathbb{R}_+ \), respectively. The exponential function whose value at \( t \) is \( e^{\lambda t} \) will be denoted with \( \exp\lambda \).
1.4 Quadratic differential forms

The ring of polynomials with real coefficients in the indeterminate \( \xi \) is denoted by \( \mathbb{R}[\xi] \); the ring of two-variable polynomials with real coefficients in the indeterminates \( \zeta \) and \( \eta \) is denoted by \( \mathbb{R}[\zeta, \eta] \). The space of all \( n \times m \) polynomial matrices in the indeterminate \( \xi \) is denoted by \( \mathbb{R}^{n \times m}[\xi] \), and that consisting of all \( n \times m \) polynomial matrices in the indeterminates \( \zeta \) and \( \eta \) by \( \mathbb{R}^{n \times m}[\zeta, \eta] \). Given a matrix \( R \in \mathbb{R}^{n \times m}[\xi] \), we define \( R(\xi) \sim = [P_0 \quad P_1 \ldots \quad P_N \ldots] \).

Observe that \( \text{mat}(P) \) has only a finite number of nonzero entries; moreover,

\[
P(\xi) = \text{mat}(P) = \begin{bmatrix}
I_w \\
I_w \xi \\
\vdots \\
I_w \xi^k \\
\vdots 
\end{bmatrix}.
\]

Finally, if \( \Sigma = \Sigma^T \), then we denote with \( \sigma_+ (\Sigma) \) the number of positive eigenvalues, with \( \sigma_- (\Sigma) \) the number of negative eigenvalues, and with \( \sigma_0 (\Sigma) \) the multiplicity of zero as an eigenvalue, of \( \Sigma \).

1.4 Quadratic differential forms

Let \( \Phi \in \mathbb{R}^{w_1 \times w_2}[\zeta, \eta] \), written out in terms of its coefficient matrices \( \Phi_{k,\ell} \) as the (finite) sum \( \Phi(\zeta, \eta) = \sum_{k,\ell \in \mathbb{Z}_+} \Phi_{k,\ell} \zeta^k \eta^\ell \). It induces the map \( L_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w_2}) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \), defined by

\[
L_\Phi(w_1, w_2) = \sum_{k,\ell \in \mathbb{Z}_+} \left( \frac{d^k}{dt^k} w_1 \right)^T \Phi_{k,\ell} \left( \frac{d^\ell}{dt^\ell} w_2 \right).
\]

This map is called the bilinear differential form (BDF) induced by \( \Phi \). When \( w_1 = w_2 = w \), \( L_\Phi \) induces the map \( Q_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^u) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \), defined by

\[
Q_\Phi(w) = L_\Phi(w, w), \text{ i.e.,}
\]

\[
Q_\Phi(w) = \sum_{k,\ell \in \mathbb{Z}_+} \left( \frac{d^k}{dt^k} w \right)^T \Phi_{k,\ell} \left( \frac{d^\ell}{dt^\ell} w \right).
\]
This map is called the quadratic differential form (QDF) induced by $\Phi$. When considering QDF’s, we can without loss of generality assume that $\Phi$ is symmetric, i.e. $\Phi(\zeta,\eta) = \Phi(\eta,\zeta)^\top$. We denote the set of real symmetric $w$-dimensional two-variable polynomial matrices with $\mathbb{R}_w^{w\times w}[\zeta,\eta]$. We associate with $\Phi(\zeta,\eta) = \sum_{k,\ell \in \mathbb{Z}_+} \Phi_{k,\ell} \zeta^k \eta^\ell \in \mathbb{R}_w^{w\times w}[\zeta,\eta]$, its coefficient matrix, defined as the infinite block-matrix:

$$
\text{mat}(\Phi) := \begin{bmatrix}
\Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,N} & \cdots \\
\Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,N} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\Phi_{N,0} & \Phi_{N,1} & \cdots & \Phi_{N,N} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots 
\end{bmatrix}
$$

Observe that $\text{mat}(\Phi)$ has only a finite number of nonzero entries, and that

$$
\Phi(\zeta,\eta) = \begin{bmatrix} I_w & I_w \zeta & \cdots & I_w \zeta^k & \cdots \end{bmatrix} \text{mat}(\Phi) \begin{bmatrix} I_w \\
I_w \eta \\
\vdots \\
I_w \eta^k \\
\vdots 
\end{bmatrix}
$$

When $\Phi$ is symmetric, it holds that $\text{mat}(\Phi) = (\text{mat}(\Phi))^\top$; in this case, we can factor $\text{mat}(\Phi) = \tilde{M}^\top \Sigma_{\Phi} \tilde{M}$ with $\tilde{M}$ a matrix having a finite number of rows, full row rank, and an infinite number of columns; and $\Sigma_{\Phi}$ a signature matrix. This decomposition leads to the following decomposition of $\Phi$

$$
\Phi(\zeta,\eta) = M(\zeta)^\top \Sigma_{\Phi} M(\eta)
$$

and is called a canonical symmetric factorization of $\Phi$. A canonical symmetric factorization is not unique; they can all be obtained from one by replacing $M(\zeta)$ with $UM(\zeta)$, with $U$ a square polynomial matrix such that $U^\top \Sigma_{\Phi} U = \Sigma_{\Phi}$.

The features of the calculus of QDFs which will be used in this thesis are the following. The first one is the delta operator, defined as

$$
\partial : \mathbb{R}_w^{w\times w}[\zeta,\eta] \rightarrow \mathbb{R}_w^{w\times w}[\xi]
$$

$$
\partial \Phi(\zeta) := \Phi(-\xi,\zeta).
$$

Observe that $\partial \Phi$ is a para-Hermitian matrix, i.e. $\partial \Phi(\xi) = \partial \Phi(-\xi)^\top$. 

1.5 Dissipative systems

Another concept we use is that of derivative of a BDF. The functional \( \frac{d}{dt} L \Phi \) defined by \( (\frac{d}{dt} L \Phi)(w_1,w_2) := \frac{d}{dt}(L \Phi(w_1,w_2)) \), is again a BDF. It is easy to see that the two-variable polynomial matrix inducing it is \( (\zeta + \eta) \Phi(\zeta,\eta) \).

Next, we introduce the notion of integral of a QDF. In order to make sure that the integral exists, we assume that the QDF acts on \( \mathcal{D}(\mathbb{R},\mathbb{R}^\omega) \), the space of infinitely-differentiable compact support functions. The integral of \( Q \Phi \) is defined as

\[
\int Q \Phi(w) := \int_{-\infty}^{\infty} Q \Phi(w) dt.
\]

We now introduce the notion of observability of a QDF. Fix \( w_1 \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^\omega) \), and consider the map \( L \Phi(w_1,\bullet) \) acting on infinitely differentiable trajectories, defined by \( L \Phi(w_1,\bullet)(w_2) := L \Phi(w_1,w_2) \), and the map \( L \Phi(\bullet,w_2) \) defined in an analogous way. Then \( \Phi \) is observable if

\[
[ L \Phi(w_1,\bullet) = 0 ] \implies [ w_1 = 0 ],
\]

\[
[ L \Phi(\bullet,w_2) = 0 ] \implies [ w_2 = 0 ].
\]

It can be shown that if \( \Phi \) is symmetric, and \( M^T(\zeta) \Sigma \Phi M(\eta) \) is a symmetric canonical factorization of \( \Phi \), then \( \Phi \) is observable if and only if \( M(\lambda) \) has full column rank for all \( \lambda \in \mathbb{C} \).

Finally, we show how to associate a QDF acting on \( \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^\omega) \) to a behavior \( \mathfrak{B} \in \mathfrak{L}_x^\text{cont} \). Let \( \mathfrak{B} = \text{im} \left( \frac{d}{dt} \right) \), and let \( \Sigma \) be a nonsingular matrix. Define \( \Phi \in \mathbb{R}^{\ell \times \ell(\zeta,\eta)} \) as

\[
\Phi(\zeta,\eta) := M^T(\zeta) \Sigma M(\eta).
\]

It is easily verified that if \( w \) and \( \ell \) satisfy \( w = M \left( \frac{d}{dt} \right) \ell \), then \( Q \Sigma(w) = Q \Phi(\ell) \), where \( Q \Sigma(w) := w^T \Sigma w \). The introduction of the two-variable matrix \( \Phi \) allows to study \( Q \Sigma \) along \( \mathfrak{B} \) in terms of properties of the QDF \( Q \Phi \) acting on free trajectories of \( \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^\ell) \).

1.5 Dissipative systems

For an extensive treatment of dissipative systems in a behavioral context we refer to [79, 80, 81, 61]. Here we review the basic material.

**Definition 1.5.1.** Let \( \Sigma \in \mathbb{R}^{w \times w} \) be symmetric, \( \mathfrak{B} \in \mathfrak{L}_x^\text{cont} \) and \( Q \Sigma(w) := w^T \Sigma w \).
1. $\mathcal{B}$ is said to be \textit{dissipative} with respect to $Q_\Sigma$ (or briefly, $\Sigma$-dissipative) if $\int_{-\infty}^{\infty} Q_\Sigma(w) \, dw \geq 0$ for all $w \in \mathcal{B} \cap \mathcal{D}$. Further, it is said to be dissipative on $\mathbb{R}_-$ with respect to $Q_\Sigma$ (or briefly, $\Sigma$-dissipative on $\mathbb{R}_-$) if $\int_{-\infty}^{0} Q_\Sigma(w) \, dw \geq 0$ for all $w \in \mathcal{B} \cap \mathcal{D}$. We also use the analogous definition of dissipativity on $\mathbb{R}_+$.

2. $\mathcal{B}$ is said to be \textit{strictly dissipative} with respect to $Q_\Sigma$ (or briefly, strictly $\Sigma$-dissipative) if there exists an $\epsilon > 0$ such that $\mathcal{B}$ is dissipative with respect to $Q_{\Sigma-\epsilon I}$. Further, it is said to be strictly dissipative on $\mathbb{R}_-$ with respect to $Q_\Sigma$ (or briefly, strictly $\Sigma$-dissipative on $\mathbb{R}_-$) if there exists an $\epsilon > 0$ such that $\mathcal{B}$ is dissipative on $\mathbb{R}_-$ with respect to $Q_{\Sigma-\epsilon I}$. We have the obvious definitions for strict dissipativity on $\mathbb{R}_+$.

\textbf{Remark 1.5.2.} It is easily seen that if $\mathcal{B}$ is $\Sigma$-dissipative on $\mathbb{R}_-$ or $\mathbb{R}_+$, then it is $\Sigma$-dissipative. Similarly, if $\mathcal{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}_-$ or $\mathbb{R}_+$, then it is strictly $\Sigma$-dissipative.

\textbf{Definition 1.5.3.} The QDF $Q_\Psi$ induced by $\Psi \in \mathbb{R}_s^{w \times w}[\zeta, \eta]$ is called a \textit{storage function} for $(\mathcal{B}, Q_\Sigma)$ if

$$\frac{d}{dt} Q_\Psi(w) \leq Q_\Sigma(w) \text{ for all } w \in \mathcal{B} \cap \mathcal{C}_\infty. \quad (1.5)$$

The QDF $Q_\Delta$ induced by $\Delta \in \mathbb{R}_s^{w \times w}[\zeta, \eta]$ is called a \textit{dissipation function} for $(\mathcal{B}, Q_\Sigma)$ if $Q_\Delta(w) \geq 0$ for all $w \in \mathcal{B} \cap \mathcal{C}_\infty$ and

$$\int_{-\infty}^{\infty} Q_\Sigma(w) \, dw = \int_{-\infty}^{\infty} Q_\Delta(w) \, dt \text{ for all } w \in \mathcal{B} \cap \mathcal{D}.$$ 

If the supply rate $Q_\Sigma$, the dissipation function $Q_\Delta$, and the storage function $Q_\Psi$ satisfy

$$\frac{d}{dt} Q_\Psi(w) = Q_\Sigma(w) - Q_\Delta(w) \text{ for all } w \in \mathcal{B} \cap \mathcal{C}_\infty \quad (1.6)$$

then we call the triple $(Q_\Sigma, Q_\Psi, Q_\Delta)$ \textit{matched on $\mathcal{B}$}.

Equation (1.6) expresses that, along $w \in \mathcal{B}$, the increase in internal storage is equal to the rate at which supply is delivered minus the rate at which supply is dissipated. The following is well-known, see e.g. [61]-Theorem 5.4.

\textbf{Proposition 1.5.4.} The following conditions are equivalent

1. $(\mathcal{B}, Q_\Sigma)$ \textit{is dissipative},

\textit{[Notes: The rest of the text is omitted for brevity.]}

\textit{[End of notes: The rest of the text is omitted for brevity.]}
2. \((\mathcal{B}, Q_{\Sigma})\) admits a storage function,

3. \((\mathcal{B}, Q_{\Sigma})\) admits a dissipation function.

Furthermore, for any dissipation function \(Q_{\Delta}\) there exists a unique storage function \(Q_{\Psi}\), and for any storage function \(Q_{\Psi}\) there exists a unique dissipation function \(Q_{\Delta}\) such that \((Q_{\Sigma}, Q_{\Psi}, Q_{\Delta})\) is matched on \(\mathcal{B}\).

In general there exist an infinite number of storage functions of \(\mathcal{B}\), but it turns out that all of them lie between two extremal storage functions.

**Proposition 1.5.5.** Let \(\mathcal{B}\) be \(\Sigma\)-dissipative; then there exist storage functions \(\Psi_-\) and \(\Psi_+\) such that any other storage function \(\Psi\) for \((\mathcal{B}, \Sigma)\) satisfies

\[
Q_{\Psi_-} \leq Q_{\Psi} \leq Q_{\Psi_+}
\]

**Proof.** See [80]-Theorem 5.7. \(\square\)

Every storage function is a quadratic function of the state, in the following sense.

**Proposition 1.5.6.** Let \(\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}\) and \(\mathcal{B} \in \mathcal{L}_\text{cont}^w\) be \(\Sigma\)-dissipative. Let \(Q_{\Psi}\) be a storage function. Then \(Q_{\Psi}\) is a state function, i.e. for every \(X\) inducing a minimal state map for \(\mathcal{B}\), there exists a real symmetric matrix \(K\) such that

\[
Q_{\Psi}(w) = \left( X \left( \frac{d}{dt} \right) w \right)^\top K \left( X \left( \frac{d}{dt} \right) w \right).
\]

**Proof.** See Theorem 5.5 of [80]. \(\square\)

The storage function \(Q_{\Psi}\) is called a nonnegative storage function if the matrix \(K\) in the above proposition is positive semidefinite. It is called a positive storage function if the matrix \(K\) in the above proposition is positive definite.

If \(m(\mathcal{B}) = \sigma_+(\Sigma)\), then the positivity of all storage functions is equivalent with half-line \(\Sigma\)-dissipativity of \(\mathcal{B}\), as the following result shows.

**Proposition 1.5.7.** Let \(\mathcal{B} \in \mathcal{L}_\text{cont}^w\) and \(\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}\) be nonsingular. Assume that \(m(\mathcal{B}) = \sigma_+(\Sigma)\). Then the following statements are equivalent.

1. \(\mathcal{B}\) is \(\Sigma\)-dissipative on \(\mathbb{R}_-\);

2. there exists a positive storage function of \(\mathcal{B}\);

3. all storage functions of \(\mathcal{B}\) are positive;
4. there exists a real symmetric matrix $K > 0$ such that

$$Q_K(w) := \left( X \left( \frac{d}{dt} \right) w \right)^\top \ K \left( X \left( \frac{d}{dt} \right) w \right)$$

is a storage function of $\mathcal{B}$;

5. there exists a storage function of $\mathcal{B}$, and every real symmetric matrix

$K$ such that $Q_K(w) := \left( X \left( \frac{d}{dt} \right) w \right)^\top \ K \left( X \left( \frac{d}{dt} \right) w \right)$ is a storage function

of $\mathcal{B}$ satisfies $K > 0$.

Proof. See [80]-Proposition 6.4. \hfill $\Box$

1.6 Basics of input-state-output, driving-variable and output-nulling representations

As already noted in section 1.2, linear differential behaviors often result as external behavior of systems with latent variables. Three particular instances of such latent variable representations are input-state-output, driving-variable and output-nulling representations. In these latent variable systems, the latent variable in fact satisfies the axiom of state. In this section we have collected the basic material on input-state-output, driving variable and output nulling representations.

1.6.1 Input-state-output representations

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, and consider the equations

$$\frac{d}{dt} x = Ax + Bu,$$

$$y = Cx + Du. \quad (1.7)$$

These equations represent the full behavior

$$\mathcal{B}_{iso}(A,B,C,D) := \{(u,x,y) \in C^\infty(\mathbb{R},\mathbb{R}^m) \times C^\infty(\mathbb{R},\mathbb{R}^n) \times C^\infty(\mathbb{R},\mathbb{R}^p) \mid (1.7) \text{ hold}\}.$$

If we interpret $w = (u,y)$ as manifest variable and $x$ as latent variable, then $\mathcal{B}_{iso}(A,B,C,D)$ is a latent variable representation of its external behavior

$$\mathcal{B}_{iso}(A,B,C,D)_{ext} = \{(y) \in C^\infty(\mathbb{R},\mathbb{R}^{m+p}) \mid \exists \ x \in C^\infty(\mathbb{R},\mathbb{R}^n) \text{ such that } (u,x,y) \in \mathcal{B}_{iso}(A,B,C,D)\}.$$
The variable $x$ is in fact a state variable according to [53]-Theorem 4.3.5.

If $B = B_{iso}(A,B,C,D)_{ext}$ then we call $B_{iso}(A,B,C,D)$ a input-state-output representation of $B$. A input-state-output representation $B_{iso}(A,B,C,D)$ of $B$ is called minimal if the state dimension $n$ is minimal over all state representations of $B$, i.e. if $n = n(B)$, the McMillan degree of $B$. In the following, let $m(B)$ and $p(B)$ denote the input cardinality and the output cardinality of $B$, respectively. $(A,B)$ is called a controllable pair if the rank of the matrix $[B AB \ldots A^{n-1}B]$ is equal to $n$ and $(C,A)$ is called an observable pair if the rank of the matrix $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ is equal to $n$. The following result is well known:

**Proposition 1.6.1.** Let $B \in L^w_{cont}$ be given. Denote $n = n(B)$, $m = m(B)$ and $p = p(B)$. Then

1. there exists matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$ such that $B_{iso}(A,B,C,D)$ is a minimal input-state-output representation of $B$,

2. if $B_{iso}(A,B,C,D)$ represents $B$, then it is a minimal representation if and only if $(C,A)$ is an observable pair,

3. if $B_{iso}(A,B,C,D)$ is a minimal representation of $B$, then $B_{iso}(A',B',C',D')$ is a minimal representation of $B$ if and only if there exists an invertible matrix $S$ such that

$$(A',B',C',D') = (S^{-1}AS,S^{-1}B,CS,D).$$

**Proof.** See [59].

The next proposition explains how to compute a minimal input-state-output representation from a given one.

**Proposition 1.6.2** (Kalman decomposition). Let the input-output behavior $B \in L^{m+p}_{out}$ be represented in input-state-output representation as $B = B_{iso}(A,B,C,D)_{ext}$. Then there is a nonsingular matrix $S$ such that

$$S^{-1}AS = \begin{bmatrix} A_{11}' & 0 \\ A_{21}' & A_{22}' \end{bmatrix}, \quad CS = \begin{bmatrix} C_1' \\ 0 \end{bmatrix},$$

and such that

1. the pair $(C_1',A_{11}')$ is observable, and

2. $B_{iso}(A_{11}',B_1',C_1',D)_{ext} = B_{iso}(A,B,C,D)_{ext}$.

Consequently, $B_{iso}(A_{11}',B_1',C_1',D)$ is a minimal input-state-output representation of $B$. 

\[ \square \]
1.6.2 Driving-variable representations

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times v}$, $C \in \mathbb{R}^{w \times n}$, $D \in \mathbb{R}^{w \times v}$, and consider the equations

$$\frac{dx}{dt} = Ax + Bv,$$

$$w = Cx + Dv. \quad (1.8)$$

These equations represent the full behavior $\mathfrak{B}_{DV}(A,B,C,D) := \{(w,x,v) \in C^\infty(\mathbb{R},\mathbb{R}^w) \times C^\infty(\mathbb{R},\mathbb{R}^n) \times C^\infty(\mathbb{R},\mathbb{R}^v) \mid (1.8) \text{ hold}\}$.

In we interpret $w$ as manifest variable and $(x,v)$ as latent variable, then $\mathfrak{B}_{DV}(A,B,C,D)$ is a latent variable representation of its external behavior $\mathfrak{B}_{DV}(A,B,C,D)_{\text{ext}} = \{w \in C^\infty(\mathbb{R},\mathbb{R}^w) \mid \exists x \in C^\infty(\mathbb{R},\mathbb{R}^n) \text{ and } v \in C^\infty(\mathbb{R},\mathbb{R}^v) \text{ such that } (w,x,v) \in \mathfrak{B}_{DV}(A,B,C,D)\}$.

The variable $x$ is in fact a state variable, the variable $v$ is undefined, and is called the driving variable.

If $\mathfrak{B} = \mathfrak{B}_{DV}(A,B,C,D)_{\text{ext}}$ then we call $\mathfrak{B}_{DV}(A,B,C,D)$ a driving variable representation of $\mathfrak{B}$. A driving variable representation $\mathfrak{B}_{DV}(A,B,C,D)$ of $\mathfrak{B}$ is called minimal if the state dimension $n$ and the driving variable dimension $v$ are minimal over all such driving variable representations. The following result holds:

**Proposition 1.6.3.** Let $\mathfrak{B} \in \mathcal{L}^\omega$ be given. Denote $n = n(\mathfrak{B})$ and $m = m(\mathfrak{B})$. Then

1. there exists matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{w \times n}$, $D \in \mathbb{R}^{w \times m}$ such that $\mathfrak{B}_{DV}(A,B,C,D)$ is a minimal driving variable representation of $\mathfrak{B}$,
2. if $\mathfrak{B}_{DV}(A,B,C,D)$ represents $\mathfrak{B}$, then it is a minimal representation if and only if $D$ is injective and the pair $(C + DF, A + BF)$ is observable for all $F$,
3. if $\mathfrak{B}_{DV}(A,B,C,D)$ is a minimal representation of $\mathfrak{B}$, then $\mathfrak{B}_{DV}(A',B',C',D')$ is a minimal representation of $\mathfrak{B}$ if and only if there exist invertible matrices $S$ and $R$ and a matrix $F$ such that

$$(A',B',C',D') = (S^{-1}(A + BF)S, S^{-1}BR, (C + DF)S, DR).$$

**Proof.** A proof of 1.) is given in [59]-Section 3. For a proof of 2.), see [59]-Corollary 4.2. For 3.) we refer to [77]-Theorem 7.2, (see also Remark 8.3 in this thesis).
The next proposition states that in order to compute a minimal driving variable representation from a given one, we can use state feedback.

**Proposition 1.6.4.** Let \( \mathfrak{B} \in \mathcal{L}^n \) and let \( \mathfrak{B}_{DV}(A,B,C,D) \) be a driving variable representation of \( \mathfrak{B} \), with \( D \) injective. Define \( F := -(D^\top D)^{-1}D^\top C \). Then there is a nonsingular matrix \( S \) such that \( S^{-1}(A + BF)S = \begin{bmatrix} A_{11}' & 0 \\ A_{21}' & A_{22}' \end{bmatrix} \), \( S^{-1}B = \begin{bmatrix} B_1' \\ B_2' \end{bmatrix} \), \( (C + DF)S = \begin{bmatrix} C_1' & 0 \end{bmatrix} \) such that

1. the pair \( (C_1' + DF', A_{11}' + B_1'F') \) is observable for all \( F' \),

2. \( B_{DV}(A_{11}', B_1', C_1', D)_{ext} = \mathfrak{B}_{DV}(A,B,C,D)_{ext} \).

Consequently, \( \mathfrak{B}_{DV}(A_{11}', B_1', C_1', D) \) is a minimal driving variable representation of \( \mathfrak{B} \).

**Proof.** Let \( \mathcal{V}^* \) be the weakly unobservable subspace of \((A,B,C,D)\) (see [64], section 7.3). By [64]-Exercise 7.5, \( \mathcal{V}^* \) is equal to the unobservable subspace of the pair \((C + DF,A + BF)\), with \( F = -(D^\top D)^{-1}D^\top C \). With respect to a basis adapted to \( \mathcal{V}^* \), \( A + BF \), \( C + DF \) and \( B \) have matrices partitioned as claimed above. By construction, the weakly unobservable subspace of \((A_{11}', B_1', C_1', D)\) is zero and therefore, by [64]-Theorem 7.16, statement (1) of the proposition holds.

In order to prove that \( B_{DV}(A_{11}', B_1', C_1', D)_{ext} = \mathfrak{B}_{DV}(A,B,C,D)_{ext} \), observe that since coordinate transformations and state feedback do not change the external behavior, we have \( \mathfrak{B}_{DV}(S^{-1}(A + BF)S,S^{-1}B,(C + DF)S,D)_{ext} = \mathfrak{B}_{DV}(A,B,C,D)_{ext} \). We now prove that \( \mathfrak{B}_{DV}(S^{-1}(A + BF)S,S^{-1}B,(C + DF)S,D)_{ext} = B_{DV}(A_{11}', B_1', C_1', D)_{ext} \). The inclusion \( \subseteq \) follows immediately. In order to prove the converse inclusion, let \( w \in \mathfrak{B}_{DV}(A_{11}', B_1', C_1', D)_{ext} \). Then there exist \( x_1, v \) such that

\[
\frac{d}{dt} x_1 = A_{11}' x_1 + B_1' v \\
w = C_1' x_1 + D v.
\]

Then, let \( x_2 \) be any solution of \( \frac{d}{dt} x_2 = A_{21}' x_1 + A_{22}' x_2 + B_2' v \). This proves that \( w \in \mathfrak{B}_{DV}(S^{-1}(A + BF)S,S^{-1}B,(C + DF)S,D)_{ext} \), so statement (2) of the proposition holds. Finally, the minimality of \((A_{11}', B_1', C_1', D)\) as a representation of \( \mathfrak{B} \) follows from the fact that \( D \) is injective and from statement (1).

\( \square \)

In this thesis, in the context of dissipative systems, we mostly work with controllable behaviors, and with the controllable part of a behavior. We now examine under what conditions a behavior represented in driving variable form is controllable.
Proposition 1.6.5. Let $\mathcal{B} \in \mathcal{L}^w$ be given. Then the following statements are equivalent

1. $\mathcal{B}$ is controllable,

2. there exist matrices $A,B,C$ and $D$ such that $\mathcal{B} = \mathcal{B}_{DV}(A,B,C,D)_{\text{ext}}$ with $(A,B)$ controllable,

3. for every minimal representation $\mathcal{B} = B_{DV}(A,B,C,D)_{\text{ext}}$, the pair $(A,B)$ is controllable.

Proof. See [71]-Theorem 3.11. □

Now let $\mathcal{B} \in \mathcal{L}^w$, possibly non-controllable, and let $\mathcal{B}_{DV}(A,B,C,D)$ be a driving variable representation. The following result shows how to compute a driving variable representation of the controllable part of $\mathcal{B}$.

Proposition 1.6.6. Let $\mathcal{B} \in \mathcal{L}^w$ and let $\mathcal{B}_{DV}(A,B,C,D)$ be a driving variable representation of $\mathcal{B}$. Then there exists a nonsingular matrix $S$ such that

1. $S^{-1}AS = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}$, $S^{-1}B = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$, $CS = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}$,

2. $(\bar{A}_{11},\bar{B}_1)$ is controllable.

Then $\mathcal{B}_{DV}(\bar{A}_{11},\bar{B}_1,\bar{C}_1,D)$ is a driving variable representation of the controllable part $\mathcal{B}_{\text{cont}}$ of $\mathcal{B}$.

Proof. First, because of Proposition 1.6.5 the full behavior $\mathcal{B}_{DV}(\bar{A}_{11},\bar{B}_1,\bar{C}_1,D)$ is controllable. Define $\mathcal{B}_0 := \{(w,(x_1,0),v) \mid (w,x_1,v) \in \mathcal{B}_{DV}(\bar{A}_{11},\bar{B}_1,\bar{C}_1,D)\}$. Then $\mathcal{B}_0$ is also controllable. Also we have $\mathcal{B}_0 \subseteq \mathcal{B}_{DV}(S^{-1}AS,S^{-1}B,CS,D)$, and the input cardinalities of these two behaviors coincide. By [3]-Lemma 2.10.3, their controllable parts then coincide, so we have $\mathcal{B}_0 = \mathcal{B}_{DV}(S^{-1}AS,S^{-1}B,CS,D)_{\text{cont}}$. Finally, the two operations of taking the controllable part and taking external behavior commute (see [3]-Lemma 2.10.4). Thus we obtain $\mathcal{B}_{DV}(\bar{A}_{11},\bar{B}_1,\bar{C}_1,D)_{\text{cont}} = (\mathcal{B}_0)_{\text{ext}} = (\mathcal{B}_{DV}(S^{-1}AS,S^{-1}B,CS,D)_{\text{cont}})_{\text{ext}} = (\mathcal{B}_{DV}(S^{-1}AS,S^{-1}B,CS,D))_{\text{cont}}$. □

1.6.3 Output-nulling representations

Output-nulling representations are defined as follows. Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times w}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times w}$, and consider the equations
\[
\frac{d}{dt}x = Ax + Bw,
\]
\[
0 = Cx + Dw.
\]  
(1.9)

These equations represent the \textit{full behavior}

\[
\mathcal{B}_{ON}(A,B,C,D) := \{(w,x) \in \mathcal{C}^{\infty}(\mathbb{R},\mathbb{R}^{w}) \times \mathcal{C}^{\infty}(\mathbb{R},\mathbb{R}^{n}) \mid (1.9) \text{ hold}\}.
\]

Again, if we interpret \(w\) as manifest variable and \(x\) as latent variable, then \(\mathcal{B}_{ON}(A,B,C,D)\) is a latent variable representation of its external behavior

\[
\mathcal{B}_{ON}(A,B,C,D)_{\text{ext}} = \{w \in \mathcal{C}^{\infty}(\mathbb{R},\mathbb{R}^{w}) \mid \exists x \in \mathcal{C}^{\infty}(\mathbb{R},\mathbb{R}^{n}) \text{ such that } (w,x) \in \mathcal{B}_{ON}(A,B,C,D)\}.
\]

Also here, the variable \(x\) is a state variable. If \(\mathcal{B} = \mathcal{B}_{ON}(A,B,C,D)_{\text{ext}}\) then we call \(\mathcal{B}_{ON}(A,B,C,D)\) an output nulling representation of \(\mathcal{B}\). \(\mathcal{B}_{ON}(A,B,C,D)\) is called a \textit{minimal output nulling representation} if \(n\) and \(p\) are minimal over all output nulling representations of \(\mathcal{B}\). Again, the following is well-known:

\textbf{Proposition 1.6.7.} Let \(\mathcal{B} \in \mathcal{L}^{w}\) be given. Denote \(n = n(\mathcal{B})\) and \(p = p(\mathcal{B})\). Then

1. there exist matrices \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times w}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times w}\) such that \(\mathcal{B}_{ON}(A,B,C,D)\) is a minimal output nulling representation of \(\mathcal{B}\),

2. if \(\mathcal{B}_{ON}(A,B,C,D)\) represents \(\mathcal{B}\), then it is a minimal representation if and only if \(D\) is surjective and \((C,A)\) is observable,

3. if \(\mathcal{B}_{ON}(A,B,C,D)\) is a minimal representation of \(\mathcal{B}\), then \(\mathcal{B}_{ON}(A',B',C',D')\) is a minimal representation of \(\mathcal{B}\) if and only if there exist invertible matrices \(S\) and \(R\) and a matrix \(J\) such that

\[
(A',B',C',D') = (S^{-1}(A + JC)S, S^{-1}(B + JD), RCS, RD).
\]

\textit{Proof.} See [71]-Theorem 3.20. \hfill \Box

The next proposition shows how to compute a minimal output nulling representation of \(\mathcal{B}\) from a given one.

\textbf{Proposition 1.6.8.} Let \(\mathcal{B} \in \mathcal{L}^{w}\) and let \(\mathcal{B}_{ON}(A,B,C,D)\) be an output nulling representation of \(\mathcal{B}\) with \(D\) surjective. Then there exist a nonsingular matrix \(S\) such that
1. \( S^{-1}AS = \begin{bmatrix} A'_{11} & 0 \\ A'_{21} & A'_{22} \end{bmatrix} \), \( S^{-1}B = \begin{bmatrix} B'_1 \\ B'_2 \end{bmatrix} \), \( CS = \begin{bmatrix} C'_1 & 0 \end{bmatrix} \),

2. the pair \((C'_1, A'_{11})\) is observable.

3. \( \text{BON}(A'_{11}, B'_1, C'_1, D)_{\text{ext}} = \text{BON}(A, B, C, D)_{\text{ext}} \).

Consequently, \( \text{BON}(A'_{11}, B'_1, C'_1, D) \) is a minimal output nulling representation of \( \mathcal{B} \).

**Proof.** The existence of a nonsingular transformation matrix \( S \) such that the conditions (1) and (2) hold follows from a standard argument. In order to prove that \( \text{BON}(A'_{11}, B'_1, C'_1, D)_{\text{ext}} = \text{BON}(A, B, C, D)_{\text{ext}} \), observe first that the transformation \( S \) does not change the external behavior. We now prove that \( \text{BON}(S^{-1}AS, S^{-1}B, CS, D)_{\text{ext}} = \text{BON}(A'_{11}, B'_1, C'_1, D)_{\text{ext}} \). The inclusion \( \subseteq \) follows immediately from the equations. In order to prove the converse inclusion, let \( w \in \text{BON}(A'_{11}, B'_1, C'_1, D)_{\text{ext}} \). Then there exist \( x_1 \) such that

\[
\frac{d}{dt} x_1 = A'_{11}x_1 + B'_1w \\
0 = C'_1x_1 + D'w.
\]

With these \( x_1 \) and \( v \), let \( x_2 \) be any solution of \( \frac{d}{dt} x_2 = A'_{21}x_1 + A'_{22}x_2 + B'_2w \). The claim follows. \( \square \)

As noted before, in the context of dissipative systems we work with controllable behaviors, and with the controllable part of a behavior. We now examine under what conditions a behavior represented in output-nulling form is controllable.

**Proposition 1.6.9.** Let \( \mathcal{B} \in \mathcal{L}^w \) be given. Then the following statements are equivalent

1. \( \mathcal{B} \) is controllable,

2. there exist matrices \( A, B, C \) and \( D \) such that \( \mathcal{B} = \text{BON}(A, B, C, D)_{\text{ext}} \) with \((A + JC, B + JD)\) controllable for all \( J \),

3. for every minimal representation \( \mathcal{B} = \text{BON}(A, B, C, D)_{\text{ext}} \) we have: the pair \((A + JC, B + JD)\) is controllable for all \( J \).

**Proof.** See [71]-Theorem 3.11. \( \square \)

Next, we show how to compute an output nulling representation of the controllable part of \( \mathcal{B} \) from a given output nulling representation.
Proposition 1.6.10. Let $\mathcal{B} = \mathcal{B}_{\text{ON}}(A,B,C,D)_{\text{ext}}$, with $D$ surjective. Define an output injection by $G := -BD^\top (DD^\top)^{-1}$. Then there exists a nonsingular matrix $S$ such that

1. $S^{-1}(A + GC)S = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}$, 
   $S^{-1}(B + GD) = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$, 
   $CS = \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix}$,

2. $(\bar{A}_{11} + G' \bar{C}_1, \bar{B}_1 + G'D)$ is controllable for all real matrices $G'$.

Furthermore, $\mathcal{B}_{\text{ON}}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, D)$ is an output nulling representation of the controllable part $\mathcal{B}_{\text{cont}}$ of $\mathcal{B}$.

Proof. A proof of this can be given using the the notion of strongly reachable subspace (see [64]-section 8.3), combined with similar ideas as in the proof of Proposition 1.6.6.

1.6.4 Relations between DV and ON representations

Driving variable and output nulling representations of the same behavior enjoy certain duality properties. These will be examined in this section. The first result we prove explains how an output nulling representation can be obtained from a driving-variable representation of the same behavior, and the other way around. In order to state it, we need to introduce the notion of “annihilator” of a matrix, defined as follows. Let $D$ be a $p \times m$ matrix of full column rank; then $D^\perp$ denotes any full row rank $(p - m) \times p$ matrix such that $D^\perp D = 0$. If $D$ is a $p \times m$ matrix of full row rank, then $D^\perp$ denotes any $m \times (m - p)$ full column rank matrix such that $DD^\perp = 0$.

Proposition 1.6.11. Let $\mathcal{B} \in \mathcal{L}_{\text{cont}}^w$ and let $\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}$ be nonsingular.

1. Let $\mathcal{B}_{\text{DV}}(A,B,C,D)$ be a minimal driving variable representation of $\mathcal{B}$ such that
   a. $D^\top \Sigma D = I$,
   b. $D^\top \Sigma C = 0$.

Define $\hat{A} := A$, $\hat{B} := BD^\top \Sigma$, $\hat{C} := -D^\perp C$, $\hat{D} := D^\perp$. Then $\mathcal{B}_{\text{ON}}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is a minimal output nulling representation of $\mathcal{B}$ and

2. $\hat{D}\Sigma^{-1}\hat{D}^\top = J$, where $J := \text{block diag}(I_{\text{row}(\hat{D}) - q}, -I_q)$, $q := \sigma_-(\Sigma)$, and $\text{row}(\hat{D})$ is row number of $\hat{D}$.
3. $\hat{B}\Sigma^{-1}\hat{D}^\top = 0$. 


2. Assume that $\mathcal{B}_{ON}(\hat{A},\hat{B},\hat{C},\hat{D})$ is a minimal output nulling representation of $\mathcal{B}$ such that the conditions c and d of statement 1 hold. Define $A := \hat{A}$, $B := \hat{B}D_{\perp}$, $C := -\Sigma^{-1}\hat{D}^\top J\hat{C}$, $D := \hat{D}_{\perp}$. Then $\mathcal{B}_{DV}(A,B,C,D)$ is a minimal driving variable representation of $\mathcal{B}$ satisfying the conditions a and b of statement 1.

Before giving a proof of Proposition 1.6.11 we need two lemmas. In the following, $D_{-R}$ denotes a right inverse of a full row rank matrix $D$, i.e. $DD_{-R} = I$, and $D_{-L}$ denotes a left inverse of a full column rank matrix $D$, i.e. $D_{-L}D = I$. The following result appears as Lemma 5.1.5 in [41].

**Lemma 1.6.12.** Let $\mathcal{B}_{DV}(A,B,C,D)$ define a minimal driving variable representation of $\mathcal{B}$. Then $\mathcal{B}_{ON}(\hat{A},\hat{B},\hat{C},\hat{D})$ given by $\hat{A} := A - BD_{-L}C$, $\hat{B} := BD_{-L}$, $\hat{C} := -D_{\perp}C$, $\hat{D} := D_{\perp}$ defines a minimal output nulling representation of $\mathcal{B}$.

Let $\mathcal{B}_{ON}(\hat{A},\hat{B},\hat{C},\hat{D})$ define a minimal output nulling representation of $\mathcal{B}$. Then $\mathcal{B}_{DV}(A,B,C,D)$ given by $A := \hat{A} - \hat{B}D_{-R}\hat{C}$, $B := \hat{B}D_{\perp}$, $C := -\hat{D}_{-R}\hat{C}$, $D := \hat{D}_{\perp}$ defines a minimal driving variable representation of $\mathcal{B}$.

We now prove the following “inertia Lemma”.

**Lemma 1.6.13.** Let $\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}$ be nonsingular.

1. Let $D$ be such that $D^\top \Sigma D = I$. Then there exists a full row rank matrix $\hat{D}$ such that $\hat{D} = D_{\perp}$ and $\hat{D}\Sigma^{-1}\hat{D}^\top = \text{block diag}(I_{\text{row}(\hat{D}) - q}, -I_q) =: J$, where $q$ is the number of negative eigenvalues of $\Sigma$.

2. Let $\hat{D}$ be such that $\hat{D}\Sigma^{-1}\hat{D}^\top = \text{block diag}(I_{\text{row}(\hat{D}) - q}, -I_q) =: J$, where $q$ is the number of negative eigenvalues of $\Sigma$. Then there exists a full column rank matrix $D$ such that $D = \hat{D}_{\perp}$ and $D^\top \Sigma D = I$.

**Proof.** (1). Let $\hat{D}$ be a full row rank matrix such that $\hat{D} = D_{\perp}$. It follows from
\[
\begin{bmatrix}
\hat{D} \\
D^\top
\end{bmatrix}
\begin{bmatrix}
D^\top & \Sigma D
\end{bmatrix}
= \begin{bmatrix}
\hat{D}D^\top & \hat{D}\Sigma D \\
0 & D^\top \Sigma D
\end{bmatrix}
= \begin{bmatrix}
\hat{D}D^\top & \hat{D}\Sigma D \\
0 & D^\top \Sigma D
\end{bmatrix}
\]
that $[\hat{D}^\top & \Sigma D]$ is nonsingular. Moreover,
\[
\begin{bmatrix}
\hat{D}^\top & \Sigma^{-1} & D^\top \\
D^\top & \Sigma
\end{bmatrix}
\begin{bmatrix}
\Sigma^{-1} & 0 \\
0 & D^\top \Sigma D
\end{bmatrix}
= \begin{bmatrix}
\hat{D}\Sigma^{-1}\hat{D}^\top & 0 \\
0 & D^\top \Sigma D
\end{bmatrix}
= \begin{bmatrix}
\hat{D}\Sigma^{-1}\hat{D}^\top & 0 \\
0 & I
\end{bmatrix}
\]
(1.10)
is then also nonsingular, and this implies that \( \tilde{D}\Sigma^{-1}\tilde{D}^\top \) is nonsingular as well. By Sylvester’s inertia law, the identity (1.10) implies that \( \sigma_- (\tilde{D}\Sigma^{-1}\tilde{D}^\top) = \sigma_- (\Sigma^{-1}) = q \). Consequently, there exists a nonsingular matrix \( W \) such that \( \tilde{D}\Sigma^{-1}\tilde{D}^\top = WJW^\top \). Set \( \hat{D} := W^{-1}\tilde{D} \). It follows that \( \hat{D} = D_\perp \) and \( \tilde{D}\Sigma^{-1}\tilde{D}^\top = J \).

(2). Let \( \hat{D} \) be a full column rank matrix such that \( \hat{D} = D_\perp \). It follows from

\[
\begin{bmatrix}
\hat{D} \\
\hat{D}^\top
\end{bmatrix}
\begin{bmatrix}
\Sigma^{-1}\hat{D}^\top \\
\Sigma^{-1}\hat{D}_\perp
\end{bmatrix}
= \begin{bmatrix}
\hat{D}\Sigma^{-1}\hat{D}^\top \\
\Sigma^{-1}\hat{D}_\perp
\end{bmatrix}
= \begin{bmatrix}
J \\
0
\end{bmatrix}
\]

that \( \begin{bmatrix}
\Sigma^{-1}\hat{D}^\top \\
\Sigma^{-1}\hat{D}_\perp
\end{bmatrix} \) is nonsingular. This implies that

\[
\begin{bmatrix}
\hat{D}\Sigma^{-1}\hat{D}^\top \\
0
\end{bmatrix}
\begin{bmatrix}
\Sigma^{-1}\hat{D}^\top \\
\Sigma^{-1}\hat{D}_\perp
\end{bmatrix}
= \begin{bmatrix}
J \\
0
\end{bmatrix}
\]

(1.11)
is nonsingular, and consequently also \( \hat{D}^\top\Sigma\hat{D} \). Observe that \( q = \sigma_- (\Sigma) \) equals the number of negative eigenvalues of the right hand side of (1.11). It follows that \( \hat{D}^\top\Sigma\hat{D} > 0 \), and that there exists a nonsingular matrix \( W \) such that \( \tilde{D}\Sigma^{-1}\tilde{D}^\top = WW^\top \). Set \( D := W^{-1}\tilde{D} \). Then \( D = D_\perp \) and \( D^\top\Sigma D = I \). □

We now give a proof of Proposition 1.6.11.

Proof. (1). Use Lemma 1.6.13 to conclude that there exists \( \hat{D} \) such that \( \hat{D} = D_\perp \) and \( \tilde{D}\Sigma^{-1}\tilde{D}^\top = J \). Since \( D^\top\Sigma D = I \), we can choose \( D^{-L} := D^\top\Sigma \).

It follows from Lemma 1.6.12 that \( \mathfrak{B}_{ON}(\hat{A},\hat{B},\hat{C},\hat{D}) \) given by \( \hat{A} := A - BD^{-L}\Sigma C = A - BD^\top\Sigma C = A, \hat{B} := BD^{-L} = BD^\top\Sigma, \hat{C} := -D_\perp C, \hat{D} := D_\perp \) is a minimal driving variable representation of \( \mathfrak{B} \). It is straightforward to see that \( \mathfrak{B}_{ON}(\hat{A},\hat{B},\hat{C},\hat{D}) \) satisfies conditions (c) and (d).

(2). Lemma 1.6.13 implies that there exists \( D \) such that \( D = \hat{D}_\perp \) and \( \tilde{D}\Sigma D^\top = I \). The fact that \( \tilde{D}\Sigma^{-1}\tilde{D}^\top = J \) implies that we can choose \( D^{-R} := \Sigma^{-1}\tilde{D}^\top J \). It follows from Lemma 1.6.12 that \( \mathfrak{B}_{DV}(A,B,C,D) \) given by \( A := \hat{A} - BD^{-R}\Sigma \hat{C} = \hat{A} - \hat{B}\Sigma^{-1}\hat{D}^\top \hat{J} \hat{C} = A, B := BD^{-L}_\perp, C := -D^{-R}\hat{C} = -\Sigma^{-1}\hat{D}^\top J \hat{C}, D := D_\perp \) is a minimal driving variable representation of \( \mathfrak{B} \). It is straightforward to check that \( \mathfrak{B}_{DV}(A,B,C,D) \) satisfies conditions (a) and (b). □

To conclude this section, we recall how driving variable and output nulling representations of a behavior can be used in order to obtain representations for the orthogonal behavior.
Proposition 1.6.14. Let $B \in C_{\text{contr}}$ and let $\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}$ be nonsingular. Then

1. If $B_{DV}(A,B,C,D)$ is a minimal driving variable representation of $B$, then $B_{ON}(-A^\top, C^\top \Sigma, B^\top, -D^\top \Sigma)$ is a minimal output nulling representation of $B_{+w}$.

2. If $B_{ON}(A,B,C,D)$ is a minimal output nulling representation of $B$, then $B_{DV}(-A^\top, C^\top \Sigma^{-1} B^\top, -\Sigma^{-1} D^\top)$ is a minimal driving variable representation of $B_{-w}$.

Proof. See section VI.A of [81].

1.6.5 Dissipativity in terms of DV representations

We first examine dissipativity for the case that a system is represented by a DV-representation.

Proposition 1.6.15. Let $B \in C_{\text{contr}}$ with minimal DV-representation $B_{DV}(A,B,C,D)$ and let $\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}$ be nonsingular. Assume $D^\top \Sigma D > 0$. Then

1. $B$ is $\Sigma$-dissipative if and only if there exists a real symmetric solution $K = K^\top \in \mathbb{R}^{n \times n}$ of the algebraic Riccati equation (ARE)

$$A^\top K + KA - C^\top \Sigma C + (KB - C^\top \Sigma D)(D^\top \Sigma D)^{-1}(B^\top K - D^\top \Sigma C) = 0.$$  (1.12)

If this is the case, then there exist real symmetric solutions $K_-$ and $K_+$ of the ARE (1.12) such that every real symmetric solution $K$ satisfies $K_- \leq K \leq K_+$.

2. $B$ is $\Sigma$-dissipative on $\mathbb{R}_-$ if and only if there exists a positive semidefinite solution $K = K^\top \in \mathbb{R}^{n \times n}$ of the ARE (1.12).

3. If $m(B) = \sigma_+ (\Sigma)$ then $B$ is $\Sigma$-dissipative on $\mathbb{R}_-$ if and only if all solutions of ARE (1.12) are positive definite, equivalently $K_- > 0$.

Proof. 1) is proved in [79]-Theorem 8.4.5 and 2), 3) are proved in [80]-Theorem 6.4. □
1.6.6 Strict dissipativity in terms of DV representations

In this subsection we examine strict dissipativity for the case that our system is represented by a DV-representation.

The connection between dissipativity, the algebraic Riccati equation (ARE), and the Hamiltonian matrix of the system is well-known, see [79, 74, 75, 85]. In the following, we will review this connection for the case of half-line dissipativity. First note the following:

**Lemma 1.6.16.** Let \( \Sigma = \Sigma^\top \in \mathbb{R}^{w \times w} \). Let \( B \in \mathcal{L}_{\text{contr}}^w \) be strictly \( \Sigma \)-dissipative. Then there exists a minimal driving variable representation \( B_{DV}(A,B,C,D) \) of \( B \) with \( D^\top \Sigma D = I \) and \( D^\top \Sigma C = 0 \).

**Proof.** Let \( \hat{A}, \hat{B}, \hat{C}, \hat{D} \) be such that \( B_{DV}(\hat{A},\hat{B},\hat{C},\hat{D}) \) is a minimal DV representation of \( B \). Then \( \hat{D} \) has full column rank (see Proposition 1.6.3). We prove now that \( \hat{D}^\top \Sigma \hat{D} > 0 \). Take the driving variable \( v(t) = \sqrt{\delta(t)} v_0 \), with \( \delta(t) \) representing the Dirac pulse. Then, with state trajectory \( x(t) = 0 \) for all \( t \in \mathbb{R} \), \( \dot{x} = Ax + Bv \) holds, so \( w(t) = \hat{D} v(t) \). There exists \( \epsilon > 0 \) such that

\[
v_0^\top \hat{D}^\top \Sigma \hat{D} v_0 = \int_{-\infty}^{\infty} v(t)^\top \hat{D}^\top \Sigma \hat{D} v(t) dt \\
= \int_{-\infty}^{\infty} w(t)^\top \Sigma w(t) dt \\
\geq \epsilon \int_{-\infty}^{\infty} w(t)^\top w(t) dt \\
= \epsilon v_0^\top \hat{D}^\top \hat{D} v_0.
\]

Of course, a rigorous proof can be given using smooth approximations of \( \delta \). This proves the claim. Let \( W \) be a nonsingular matrix such that \( D^\top \Sigma D = W^\top W \). By applying the state feedback transformation \( \dot{\hat{v}} = -(\hat{D}^\top \Sigma \hat{D})^{-1} \hat{D}^\top \Sigma \hat{C} x + W^{-1} v \) to \( B_{DV}(\hat{A},\hat{B},\hat{C},\hat{D}) \) we obtain a new driving variable representation \( B_{DV}(A,B,C,D) \) of \( B \), with

\[
A = \hat{A} - \hat{B}(\hat{D}^\top \Sigma \hat{D})^{-1} \hat{D}^\top \Sigma \hat{C} \\
B = \hat{B} W^{-1} \\
C = \hat{C} - \hat{D}(\hat{D}^\top \Sigma \hat{D})^{-1} \hat{D}^\top \Sigma \hat{C} \\
D = \hat{D} W^{-1}.
\]

Observe that \( D \) is injective, and that from the minimality of \( B_{DV}(\hat{A},\hat{B},\hat{C},\hat{D}) \) and statement (2) of Proposition 1.6.3 it follows that \( B_{DV}(A,B,C,D) \) is also a minimal representation of \( B \). It is easy to see that \( D^\top \Sigma D = I \), and moreover
\[ D^\top \Sigma C = W^{-\top} \hat{D}^\top \Sigma (\hat{C} - \hat{D} (\hat{D}^\top \Sigma \hat{D})^{-1} \hat{D}^\top \Sigma \hat{C}) = 0. \]

This concludes the proof. \(\square\)

We then have the following:

**Proposition 1.6.17.** Let \( \mathcal{B} \in \mathcal{B}_{\text{constr}}^w \), and let \( \Sigma = \Sigma^\top \in \mathbb{R}^{w \times w} \). Let \( \mathcal{B}_{DV}(A,B,C,D) \) be a minimal driving variable representation of \( \mathcal{B} \) such that \( D^\top \Sigma D = I \) and \( D^\top \Sigma C = 0 \). If \( \mathcal{B} \) is strictly \( \Sigma \)-dissipative then the Hamiltonian matrix

\[ H = \begin{bmatrix} A & BB^\top \\ C^\top \Sigma C & -A^\top \end{bmatrix} \tag{1.13} \]

has no eigenvalues on the imaginary axis. Furthermore, the following statements are equivalent:

1. \( \mathcal{B} \) is strictly \( \Sigma \)-dissipative on \( \mathbb{R}_- (\mathbb{R}_+) \),

2. the ARE

\[ A^\top K + KA - C^\top \Sigma C + KBB^\top K = 0 \tag{1.14} \]

has a real symmetric solution \( K \) with \( K > 0 \) (\( K < 0 \)) and \( A + BB^\top K \) is antistable (stable),

3. The Hamiltonian matrix (1.13) has no eigenvalues on the imaginary axis, and there exists \( X_1,Y_1 \in \mathbb{R}^{n \times n} \), with \( X_1 \) nonsingular, and \( M \in \mathbb{R}^{n \times n} \) antistable (stable) such that

\[ H \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} M, \]

with \( X_1^\top Y_1 > 0 \) (\( X_1^\top Y_1 < 0 \)).

If \( K \) satisfies the conditions in (2.) above then it is unique, and it is the largest (smallest) real symmetric solution of (1.14). We denote it by \( K^+ \) (\( K^- \)). If \( X_1,Y_1 \) satisfy the conditions in (3.) above, then \( Y_1 X_1^{-1} \) is equal to this largest (smallest) real symmetric solution \( K^+ \) (\( K^- \)) of the ARE (1.14).

**Proof.** Assume that \( H \) has an eigenvalue \( i\omega \), with eigenvector \( (x_1^*, x_2^*)^* \). Then

\[ Ax_1 + BB^\top x_2 = i\omega x_1 \] and \( C^\top \Sigma C x_1 - A^\top x_2 = 0 \). We will first prove that the vector

\[ w_0 := Cx_1 + DB^\top x_2 \tag{1.15} \]
is unequal to 0. Indeed, assume \( w_0 = 0 \). Then premultiplying with \( D^T \Sigma \) yields \( B^T x_2 = 0 \). This yields \( Ax_1 = i \omega x_1 \) and \( Cx_1 = 0 \). By observability of the pair \((C,A)\) we then obtain \( x_1 = 0 \). Thus \((i \omega I + A^T) x_2 = 0\), whence \( x_2^* (A - i \omega I) = 0 \). Together with \( x_2^* B = 0 \), by controllability of the pair \((A,B)\) this yields also \( x_2 = 0 \). Consequently, \( w_0 \neq 0 \). It is also easily verified that \( w_0^T \Sigma w_0 = 0 \).

Let \( \Delta > 0 \). Consider the differential equation \( \dot{x} = Ax + Bv \). Using controllability of the pair \((A,B)\), let \( \tilde{v}_1 : (-\infty,0] \rightarrow \mathbb{R}^y \) be a driving variable trajectory that drives state 0 at \( t = -\Delta \) to state \( x_1 \) at time \( t = 0 \). Choose \( \tilde{v}_1 \) such that \( \tilde{v}_1(t) = 0 \) for \( t < -\Delta \). Let \( \tilde{x}_1(t) (t \leq 0) \) be the corresponding state trajectory, and \( \tilde{w}_1(t) := C\tilde{x}_1 + D\tilde{v}_1(t) \). Likewise, let \( \tilde{v}_2 : [0,\infty) \rightarrow \mathbb{R}^y \) be a driving variable trajectory that drives state \( x_1 \) at \( t = 0 \) to state 0 at time \( t = \Delta \). Choose \( \tilde{v}_2 \) such that \( \tilde{v}_2(t) = 0 \) for \( t > \Delta \). Let \( \tilde{x}_2(t) (t \geq 0) \) be the corresponding state trajectory, and \( \tilde{w}_2(t) := C\tilde{x}_2 + D\tilde{v}_2(t) \).

Denote \( T = \frac{2\pi}{\omega} \) and define a sequence of driving variable trajectories \( v_n \) by

\[
 v_n(t) = \begin{cases} 
  \tilde{v}_1(t + nT) & t < -nT, \\
  e^{i\omega t} B^T x_2 & nT \leq t < nT, \\
  \tilde{v}_2(t - nT) & t \geq nT.
\end{cases}
\]

Define

\[
 x_n(t) = \begin{cases} 
  \tilde{x}_1(t + nT) & t < -nT, \\
  e^{i\omega t} x_1 & nT \leq t < nT, \\
  \tilde{x}_2(t - nT) & t \geq nT.
\end{cases}
\]

Then \( \dot{x}_n(t) = Ax_n(t) + Bv_n(t) \) for all \( t \in \mathbb{R} \). Also, for \( t \in (-\infty, -nT - \Delta] \cup [nT + \Delta, \infty) \) we have \( x_n(t) = 0 \), so \( w_n(t) = 0 \). For \( t \in (-nT - \Delta, -nT) \) we have \( w_n(t) = \tilde{w}_1(t + nT) \), for \( t \in (-nT,nT) \) we have \( w_n(t) = w_0 e^{i\omega t} \) (with \( w_0 \) given by (1.15)), and for \( t \in [nT,nT + \Delta) \) we have \( w_n(t) = \tilde{w}_2(t - nT) \).

Now, clearly, \( \int_{-\infty}^{\infty} |w_n(t)|^2 dt \rightarrow \infty \) as \( n \rightarrow \infty \). On the other hand however, \( w_n(t)^* \Sigma w_n(t) = w_0^* \Sigma w_0 = 0 \) for \( t \in (-nT,nT) \), so \( \int_{-\infty}^{\infty} w_n(t)^* \Sigma w_n(t) dt = \int_0^\Delta \tilde{w}_1(t) dt + \int_0^\Delta \tilde{w}_2(t) dt \), independent of \( n \). Thus, for \( n \) sufficiently large the inequality \( \int_{-\infty}^{\infty} w_n(t)^* \Sigma w_n(t) dt \geq \varepsilon \int_{-\infty}^{\infty} |w_n(t)|^2 dt \) fails to hold, contradicting the assumption that our system is strictly \( \Sigma \)-dissipative. This proves that \( H \) has no eigenvalues on the imaginary axis.

(1) \( \implies \) (2). In both cases, \( B \) is strictly \( \Sigma \)-dissipative, so the Hamiltonian \( H \) has no eigenvalues on the imaginary axis. It then follows from standard results on the Hamiltonian matrix, using controllability of \((A,B)\), (see e.g. [85]) that the ARE (1.14) has a real symmetric solution \( K \) such that \( A + BB^T K \) is antistable, and also a real symmetric solution \( K \) such
that $A + BB^T K$ is stable. It was proven in [41]-Theorem 5.3.4 that if $\mathcal{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}_-$ then the antistabilizing solution $K$ is positive definite. In a similar way it can be proven that if $\mathcal{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}_+$ then the stabilizing solution $K$ is negative definite.

(2) $\Rightarrow$ (1). This was also proven in [41]-Theorem 5.3.4.

(2) $\Leftrightarrow$ (3). This equivalence follows from standard results on the relation between the algebraic Riccati equation and Hamiltonian matrices, see e.g. [85].

Proposition 1.6.17 examined strict dissipativity for the case that our system is represented by a DV-representation with the properties $D^\top \Sigma D = I$ and $D^\top \Sigma C = 0$. In the remainder of this subsection we examine strict dissipativity for the case that the system is represented by general a DV-representation. The proof of the following statement is omitted since the computation in the proof is similar to the one in the proof of Proposition 1.6.17.

Proposition 1.6.18. Let $\mathcal{B} \in \mathcal{L}_{\text{con}}^\omega$, and let $\Sigma = \Sigma^\top \in \mathbb{R}^{u \times u}$. Let $\mathcal{B}_{\text{DV}}(A,B,C,D)$ be a minimal driving variable representation of $\mathcal{B}$. Then the following statements are equivalent:

1. $\mathcal{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}_-$ ($\mathbb{R}_+$),

2. the ARE

$$A^\top K + KA - C^\top \Sigma C + (KB - C^\top \Sigma D)(D^\top \Sigma D)^{-1}(B^\top K - D^\top \Sigma C) = 0$$

(1.16)

has a real symmetric solution $K$ with $K > 0$ ($K < 0$) and $A + B(D^\top \Sigma D)^{-1}(B^\top K - D^\top \Sigma C)$ is antistable (stable),

3. The Hamiltonian matrix

$$H = \begin{bmatrix}
A - B(D^\top \Sigma D)^{-1}D^\top \Sigma C & B(D^\top \Sigma D)^{-1}B^\top \\
C^\top \Sigma C - C^\top \Sigma D(D^\top \Sigma D)^{-1}D^\top \Sigma C & -(A - B(D^\top \Sigma D)^{-1}D^\top \Sigma C)^\top
\end{bmatrix}$$

has no eigenvalues on the imaginary axis, and there exists $X_1, Y_1 \in \mathbb{R}^{n \times n}$, with $X_1$ nonsingular, and $M \in \mathbb{R}^{n \times n}$ antistable (stable) such that

$$H \begin{bmatrix}
X_1 \\
Y_1
\end{bmatrix} = \begin{bmatrix}
X_1 \\
Y_1
\end{bmatrix} M,$$

with $X_1^\top Y_1 > 0$ ($X_1^\top Y_1 < 0$).

If $K$ satisfies the conditions in (2.) above then it is unique, and it is the largest (smallest) real symmetric solution of (1.16). We denote it by $K_+$ ($K_-$). If $X_1, Y_1$ satisfy the conditions in (3.) above, then $Y_1 X_1^{-1}$ is equal to this largest (smallest) real symmetric solution $K_+$ ($K_-$) of the ARE (1.16).
1.6.7 Strict dissipativity in terms of ON representations

In this subsection we examine conditions under which a system in output nulling representation is strictly $\Sigma$-dissipative on $\mathbb{R}^-$ or $\mathbb{R}^+$. 

**Proposition 1.6.19.** Let $\mathcal{B} \in \mathcal{L}_\text{contr}^w$. Let $\Sigma = \Sigma^T \in \mathbb{R}^{w \times w}$ be nonsingular. Assume that $\mathcal{B}$ is strictly $\Sigma$-dissipative. Then there exists a minimal output nulling representation $\mathcal{B}_{\text{ON}}(A,B,C,D)$ of $\mathcal{B}$ such that

\[ D\Sigma^{-1}B^T = J, \text{ with } J := \text{blockdiag}(I_{\text{rowdim}(D)} - qI_q) \text{ and } q = \sigma_-(\Sigma), \]

(1.17)

\[ B\Sigma^{-1}D^T = 0. \]

(1.18)

**Proof.** The proof follows easily by combining Lemma 1.6.16 and Proposition 1.6.11. $\square$

The following result is analogous to that of Proposition 1.6.17.

**Proposition 1.6.20.** Let $\mathcal{B} \in \mathcal{L}_\text{contr}^w$. Let $\Sigma = \Sigma^T \in \mathbb{R}^{w \times w}$ be nonsingular, and let $\mathcal{B}_{\text{ON}}(A,B,C,D)$ be a minimal ON representation of $\mathcal{B}$ such that (1.17) and (1.18) hold. If $\mathcal{B}$ is strictly $\Sigma$-dissipative then the Hamiltonian matrix

\[ H = \begin{bmatrix} A & B\Sigma^{-1}B^T \\ C^TJC & -A^T \end{bmatrix} \]

(1.19)

has no eigenvalues on the imaginary axis. Furthermore, the following conditions are equivalent:

1. $\mathcal{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}^-$ ($\mathbb{R}^+$),

2. The ARE

\[ AK + KA^T + B\Sigma^{-1}B^T - KC^TJCK = 0 \]

(1.20)

has a real symmetric solution $K > 0$ such that $A - KC^TJC$ is stable (antistable),

3. the Hamiltonian matrix (1.19) has no eigenvalues on the imaginary axis, and there exist $X_1,Y_1 \in \mathbb{R}^{n \times n}$, with $Y_1$ nonsingular, and $M \in \mathbb{R}^{n \times n}$ antistable (stable) such that

\[ H' \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} M \]

with $Y_1^TX_1 > 0$ ($Y_1^TX_1 < 0$).
If $K$ satisfies the conditions in (2.) above then it is unique, and it is the smallest (largest) real symmetric solution of (1.20). We denote it by $K_-$ ($K_+$). If $X_1Y_1$ satisfy the conditions in (3.) above, then $X_1Y_1^{-1}$ is equal to this smallest (largest) real symmetric solution $K_-$ ($K_+$) of the ARE (1.20).

Proof. The argument of the proof can be given as follows: using Proposition 1.6.11 in this chapter, associate with the given minimal output nulling representation a minimal driving variable representation satisfying the conditions of Proposition 1.6.17. Then apply proposition 1.6.17 to this driving variable representation. Finally, restate the conditions obtained in terms of the original output nulling representation (see also Theorem 5.3.5 in [41]). The details of the proof are given below.

Step 1: Transform the minimal output nulling representation into an equivalent minimal driving variable representation. By Lemma 1.6.11, $\mathcal{B}$ can be represented by a minimal driving variable representation $\mathcal{B}_{DV}(\bar{A},\bar{B},\bar{C},\bar{D})$, with satisfied $\bar{D}^\top \Sigma \bar{D} = I$ and $\bar{D}^\top \Sigma \bar{C} = 0$ and

$$
\bar{A} = A
$$

$$
\bar{B} = BD_{\perp}
$$

$$
\bar{C} = -\Sigma^{-1}D^\top JC
$$

$$
\bar{D} = D_{\perp}.
$$

Step 2: We show that $\bar{B} \bar{B}^\top = B \Sigma^{-1} B^\top$ and $\bar{C}^\top \Sigma \bar{C} = C^\top JC$. Indeed,

$$
\bar{C}^\top \Sigma \bar{C} = (\Sigma^{-1}D^\top JC)^\top \Sigma (\Sigma^{-1}D^\top JC)
$$

$$
= C^\top JD\Sigma^{-1}\Sigma\Sigma^{-1}D^\top JC
$$

$$
= C^\top J(D\Sigma^{-1}D^\top )JC
$$

$$
= C^\top JJJJC
$$

$$
= C^\top JC
$$

To prove that $\bar{B} \bar{B}^\top = B \Sigma^{-1} B^\top$, we start from the fact $\bar{D}^\top \Sigma \bar{D} = I$, which implies that $\bar{D} \bar{D}^\top \Sigma \bar{D} = \bar{D}$, or $(\bar{D} \bar{D}^\top \Sigma - I)\bar{D} = 0$. Since $\bar{D} = D_{\perp}$, there exists a matrix $F$ such that $\bar{D} \bar{D}^\top \Sigma - I = FD$. Post-multiply this equation by $D^{-R}$ to obtain $\bar{D} \bar{D}^\top \Sigma D^{-R} - D^{-R} = F$. Using $D^{-R} = \Sigma^{-1}D^\top J$ (since $D\Sigma^{-1}D^\top = J$) it follows that

$$
F = \bar{D} \bar{D}^\top \Sigma D^{-R} - D^{-R}
$$

$$
= \bar{D} \bar{D}^\top \Sigma \Sigma^{-1}D^\top J - D^{-R}
$$

$$
= \bar{D}(\bar{D} \bar{D})^\top J - D^{-R}
$$

$$
= -D^{-R} \quad \text{(since } \bar{D} \bar{D} = DD_{\perp} = 0).\]
Now substitute $F = -D - R$ into the identity $\bar{D}D^T \Sigma - I = FD$. This implies that $\bar{D}D^T \Sigma - I = -D - R$. By post-multiplying this equation by $\Sigma^{-1}$ we get $\bar{D}D^T - \Sigma^{-1} = -D - R \Sigma^{-1}$. Next, pre-multiply this equation by $B$ and post-multiply by $B^T$ to arrive at

$$
B\bar{D}D^T B^T = B\Sigma^{-1}B^T - BD^{-R}D\Sigma^{-1}B^T
$$

(since $D\Sigma^{-1}B^T = 0$).

Since $BB^T = B\bar{D}D^TB^T$, the proof is complete.

**Step 3:** Apply Proposition 1.6.17 to the driving variable representation $\mathfrak{B}_{DV}(\bar{A}, \bar{B}, \bar{C}, \bar{D})$. $\mathfrak{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}_-$ if and only if

1. The ARE

$$
\bar{A}^T\bar{K} + \bar{K}\bar{A} + \bar{K}BB^T\bar{K} - \bar{C}^T\Sigma\bar{C} = 0; \quad (1.21)
$$

has a unique real symmetric solution $\bar{K}_+$ such that $\bar{A} + \bar{B}\bar{B}^T\bar{K}_+$ is antistable; and

2. this solution $\bar{K}_+$ is positive definite.

**Step 4:** Translate things back in terms of output nulling representations $\mathfrak{B}_{ON}(A,B,C,D)$. Since $BB^T = B\Sigma^{-1}B^T$ and $\bar{C}^T\Sigma\bar{C} = C^TJC$, the ARE (1.21) becomes

$$
A^T\bar{K} + \bar{K}A + \bar{K}B\Sigma^{-1}B^T\bar{K} - C^TJC = 0.
$$

Pre-multiply this equation with $\bar{K}^{-1}$ and post-multiply with $\bar{K}^{-1}$ to get

$$
A\bar{K}^{-1} + \bar{K}^{-1}A^T - \bar{K}^{-1}C^TJC\bar{K}^{-1} + B\Sigma^{-1}B^T = 0. \quad (1.22)
$$

Assume that $\mathfrak{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}_-$. We will show that ARE (1.22) has a unique real symmetric positive definite solution $K_+$ such that $A^T - C^TJC\bar{K}_+$ is stable. Indeed, set $K_+ := (\bar{K}_+)^{-1}$, where $\bar{K}_+$ is the largest solution of ARE (1.21) as shown in Step 3. Obviously, $\bar{K}_+$ is a real symmetric solution of ARE (1.22) and $K_+ > 0$ since $\bar{K}_+ > 0$. It remains to prove that $A^T - C^TJC\bar{K}_+$ is stable. From (1.22) it follows that

$$
K_+(A^T - C^TJC\bar{K}_+) + (A - B\Sigma^{-1}B(K_+)^{-1})K_+ = 0.
$$

This implies that
\[(A^\top - C^\top JCK_+) = -(K_+)^{-1}(A - B\Sigma^{-1}B(K_+)^{-1})K_+\]

or

\[(A^\top - C^\top JCK_+) = -(K_+)^{-1}(\bar{A} + \bar{B}\bar{B}^\top \bar{K}_+)K_+\]

Since \(\bar{A} + \bar{B}\bar{B}^\top \bar{K}_+\) is antistable (as shown in Step 3), this implies that \(A^\top - C^\top JCK_+\) is stable.

Conversely, assume that ARE (1.22) has a unique real symmetric positive definite solution \(K_+\) such that \(A^\top - C^\top JCK_+\) is stable. By setting \(\bar{K}_+ := (K_+)^{-1}\), we conclude that ARE (1.21) has unique real symmetric positive definite solution \(\bar{K}_+\) such that \(\bar{A} + \bar{B}\bar{B}^\top \bar{K}_+\) is antistable. This is equivalent to saying that the system \(\mathfrak{B}\) is strictly \(\Sigma\)-dissipative on \(\mathbb{R}_-\) (as Step 3).

The equivalence of (2) and (3) again follows from standard results on the relation between the algebraic Riccati equation and the Hamiltonian matrix, see e.g. [85]. \(\square\)