We study the influence of the cosmological constant on global properties of dark matter halos. In particular, we study the shape and evolution of the mass functions. To this end, we perform thirteen high resolution $N$-body simulations, which include open, flat and closed Universes, with or without a cosmological constant. We find that the mass function of models with the same value of the matter density are indistinguishable at low redshift, independent of the value of the cosmological constant. We compare our simulated mass functions with the Press & Schechter formalism, and found that it shows a rough agreement at low redshift, but it differs substantially at higher redshifts.
2.1 Introduction

Cosmological observations strongly indicate that we are living in a flat, accelerating Universe with a low matter density. Observations of distant supernovae (Riess et al. 1998; Perlmutter et al. 1999) and the precise measurements of cosmic microwave background fluctuations (Spergel et al. 2003) have established a new cosmological paradigm. Most of the matter in the Universe is in the form of an unknown species of dark matter, probably cold dark matter (CDM). While this is responsible for the formation of structure and the corresponding clustering of matter in the Universe, most of its energy is in the form of a mysterious dark energy. This dark energy behaves like Einstein’s cosmological constant, $\Lambda$, and it is responsible for the acceleration of the cosmic expansion. The estimated amount of dark energy appears to be precisely sufficient to yield a flat geometry of our Universe. In all, it also solves the apparent conflict suggested by the old age of globular cluster stars.

Structure in the Universe arose out of the gravitational growth of tiny primordial density and velocity perturbations. In the current standard view this process is hierarchical, with small clumps being the first objects to form and gradually merging and accreting while assembling into ever larger structures. The history of this process is highly dependent on the amount of (dark) matter in the Universe: structure formation in low $\Omega_m$ cosmologies comes to a halt at much earlier times than that in cosmologies with high density values.

An issue that remains to be clarified is the role of the cosmological constant in structure formation. In this we may identify various influences of the cosmological constant. Here we discuss three effects.

- Dark energy strongly influences the dynamical time scales involved with the structure formation process. Possibly this is its main effect.

- Dark energy implies a modified spectrum of primordial density fluctuations. Its main effect concerns the amplitude of the perturbations.

- The dynamical accelerating influence of dark energy may also play a role in the dynamics of the emerging and evolving structures. In the linear regime this is a minor effect, e.g., Lahav et al. (1991) showed that it only has around $\sim 1/70$ of the influence of matter perturbations.

There has not yet been a lot of attention to situations of an open or closed Universe with a cosmological constant. Nor, for that matter, on structure formation in closed pure matter-dominated Universes. Of the few studies that addressed such cosmologies we may mention Bjornsson & Gudmundsson (1995), who discussed how a closed Universe would appear to astronomers living at different cosmic epochs. White & Scott (1996) considered structure formation and CMB anisotropies in a closed Universe, both with and without cosmological constant. They found that there are a range of closed models models that are consistent with observational constraints while being older than flat models with a cosmological constant. While perhaps less feasible than the currently popular flat Lambda-dominated Universes, or low-density matter-dominated Universes, such cosmologies are possible products of an early inflationary phase. For example, Linde (1995) showed that it is possible to produce inflationary models that result in generic closed Universes.

One of the most important representatives in the cosmic hierarchy of structures, and therefore important probes for the study of cosmic structure and evolution, are clusters of galaxies. They are the most massive and most recently collapsed objects in the Universe. Their density is in the order of several hundred times the critical density of the Universe, with collapse times comparable to the age of the Universe. The substructure observed in many galaxy clusters reaffirms this idea.

Observationally, it is almost impossible to study the evolution of galaxy clusters, especially if one wants to investigate differences between different cosmological models. This makes $N$-body simulations a necessary tool. They represent a realistic description of the formation and evolution of galaxy clusters.

To assess the influence of a positive cosmological constant on the formation and evolution of dark matter halos and galaxy clusters we study this in a set of dissipationless $N$-body simulations. All
simulations involve variants of the Cold Dark Matter scenarios, embedded with a range of cosmological parameters. By investigating structure formation for models with different values of $\Omega_m < 1$ and $\Omega_\Lambda \neq 0$, we do seek to learn more about the influence of $\Omega_\Lambda$.

One important aspect is the mass function of emerging objects. Several authors have used mass functions as a diagnostic for $\Omega_m$ (e.g., Eke et al. 1996; Lee & Shandarin 1999; Governato et al. 1999; Gardini et al. 1999; Pierpaoli et al. 2001; Sánchez et al. 2002; Reed et al. 2003; Younger et al. 2005). Our approach will be similar to that of these authors, but will include a wider spectrum of cosmologies. By carefully choosing the range of our cosmologies we hope to find more information on the various influences of the cosmological constant on the structure formation process.

This chapter is organized as follows: in section 2.2 we give a description of the Friedmann-Robertson-Walker equation and the various cosmologies. In section 2.3 we describe the different $N$-body simulations. In section 2.4 we describe the techniques to construct the halo catalogues that we use in order to calculate the different mass functions. We use these catalogues to extract the mass function of each cosmology. We study the evolution of the mass function in the various cosmologies and compare with the Press-Schechter mass function. Conclusions are presented in section 2.5.

### 2.2 Cosmological Background

We are going to investigate structure formation in Friedmann-Robertson-Walker Universes containing matter and a cosmological constant, with a negligible radiation contribution, and with a generic, not necessarily flat, geometry. Structure forms as a result of gravitational instability and we assume that the dark matter is some cold dark matter particle.

#### 2.2.1 FRW Universes

The general Friedmann-Robertson-Walker equation for the expansion of the Universe is (neglecting the contribution by radiation)

$$\frac{\dot{a}}{a} = H_0 \sqrt{\Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda},$$

where $a$ is the expansion factor is related to the redshift via $1 + z = a^{-1}$. At present time, $a_0 = 1$, $\Omega_m$ is the matter density parameter, $\Omega_\Lambda$ is the vacuum energy density parameter and $\Omega_k$ is the curvature density parameter,

$$\Omega_m = \frac{\rho_m}{\rho_c}, \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}, \quad \Omega_k = -\frac{k}{a_0^2 H_0^2},$$

where $\rho_c = 3H_0^2/8\pi G$ is the critical density (the energy density needed to get a flat $k = 0$ Universe). If $k > 0$, then the Universe is closed, if $k = 0$, it is flat and if $k < 0$, it is open. Dividing Eqn. 2.1 by $H_0$ and evaluating at present time we get

$$\Omega_K = 1 - \Omega_m - \Omega_\Lambda.$$  

This equations tells us that the sum of the matter density parameter and the cosmological constant density parameter describes the geometry of the Universe. It is convenient to define $\Omega_{\text{total}} = \Omega_m + \Omega_\Lambda$. Then, Eqn. 2.3 becomes $\Omega_K = 1 - \Omega_{\text{total}}$.

In our study we assess and compare all three possible geometries. Table 2.1 shows a $4 \times 4$ matrix with a combination of $\Omega_m$ and $\Omega_\Lambda$ values. The sums in light gray refer to those Universes modelled for the present work. We chose the values in such a way that we could investigate systematically the influence of $\Omega_\Lambda$, and reproduce earlier works (open models with $\Omega_m = 0.3$ and no cosmological constant) and the accepted flat model ($\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$).

We study thirteen cosmologies in total. Six of them are open models, four are flat models and the remaining three are closed Universes. Of the six open models, three of them are pure matter dominated
while the other three have a cosmological constant. Of the flat models, one is an EdS Universe (for the CDM structure formation scenarios which we investigate here this is known as SCDM). The other three flat models involve a cosmological constant. The remaining three are closed Universes with a cosmological constant.

### 2.2.2 Cosmic Structure Formation

In the homogeneous and isotropic FRW Universes, initially the tiny density perturbations $\delta(x, t)$ grow linearly with a universal rate independent of their comoving spatial scale, the linear density growth factor $D(a)$,

$$\delta(x, t) = D(a) \cdot \delta_0(x),$$

in which $\delta_0(x)$ is the initial density fluctuation at comoving position $x$ (linearly extrapolated to the present time). The density growth factor $D(a)$ is sensitively dependent on the cosmology at hand. An explicit expression for $D(a)$ is (Heath 1977)

$$D(a) = \frac{5}{2} \Omega_m H_0^2 H(a) \int_0^a \frac{da'}{a'^3 H(a')^3} = a g(a),$$

where $g(a)$ is the linear growth factor. An accurate approximation for $g(a)$ in the case of Universes with a cosmological constant is (Carroll et al. 1992):

$$g(a) \approx \frac{5}{2} \Omega_m(a) \left[ \frac{\Omega_m(a)^{4/7} \Omega_\Lambda(a) + 1 + \Omega_m(a)}{2} \right] \left(1 + \frac{\Omega_\Lambda(a)}{70}\right)^{-1}. \quad (2.6)$$

The density growth factor in a Einstein-De Sitter Universe is simply proportional to the cosmic expansion factor $a(t)$,

$$D(a) = a(t) \propto t^{2/3}, \quad (2.7)$$

while in a freely expanding empty Universe ($\Omega_m = 0$), $D(a)$ becomes a constant and therefore structure formation comes to a halt. This is the asymptotic situation for a low density matter dominated universe ($\Omega_m < 1$). Such a Universe starts of as a near EdS Universe and attains free expansion at

$$1 + z_{mf} = \frac{1}{\Omega_m} - 1. \quad (2.8)$$

As a result in matter dominated $\Omega_m < 1$ Universes at early times we see structure growing with a rate $D(a)$ proportional to $a(t)$, while it freezes out after $z_{mf}$.
2.2. COSMOLOGICAL BACKGROUND

In the case of \( \Lambda \) dominated Universes structure formation comes to a halt when the Universes sets in its accelerated expansion at redshifts \( z_{m\Lambda} \),

\[
1 + z_{m\Lambda} = \left( \frac{2\Omega_{\Lambda}}{\Omega_m} \right)^{1/3}.
\]

For models with \( \Omega_m = 0.5 \) and \( \Omega_{\Lambda} = 0.7 \) and 0.9, this happens at \( z \sim 0.40 \) and \( z \sim 0.53 \), respectively.

For the cosmological models treated in this study, we have illustrated the density growth function \( D(a) \) in the right hand frame of Fig. 2.1. In the figure we distinguish the various cosmologies by means of grey scale values and linestyle. The linestyle represents the different values of \( \Omega_{\Lambda} \) while the grey scale represents the various values of \( \Omega_m \) so that their combination forms a complete representation of the cosmology at hand. For example, solid lines correspond to models without a cosmological constant, \( \Omega_{\Lambda} = 0 \). Black lines are \( \Omega_m = 1 \) models and as we go from dark to light grey the value of \( \Omega_m \) decreases. Throughout this chapter we will use these color scheme.

2.2.3 Cosmological Scenarios

The cosmological models which we study are variants of the cold dark matter (CDM) scenario. The primordial density field is fully characterized by the power spectrum, for which we use the functional form of the matter power spectrum Bardeen et al. (1986),

\[
P(k) = A T^2(q) k^n = A \frac{k^n}{\left[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4\right]^{1/2}} \times \frac{[\ln(1 + 2.34q)]^2}{(2.34q)^2},
\]

where \( T(q) \) is the transfer function of fluctuations, \( A \) is the amplitude and \( q = k/\Gamma \). \( \Gamma \) is the shape parameter and \( k = 2\pi/\lambda \) is the wavenumber in units of \( h^{-1}\text{Mpc} \). The index \( n \) is the slope of the pri-
Table 2.2 — Cosmological parameters for the runs. The columns give the identifications of the runs, the present matter density parameter, the density parameter associated with the cosmological constant, the age of the Universe in Gyr since the Big Bang, the mass per particle in units of $10^{10}h^{-1}M_\odot$ and the shape parameter of the power spectrum. $\sigma_8$ and the Hubble parameter is the same for every model, $\sigma_8 = 0.8$ and $h = 0.7$.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\Omega_m$</th>
<th>$\Omega_\Lambda$</th>
<th>$\Omega_k$</th>
<th>Age</th>
<th>$m_{dm}$</th>
<th>$\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCDM</td>
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<td>0</td>
<td>9.31</td>
<td>13.23</td>
<td>0.7</td>
</tr>
<tr>
<td>OCDM01</td>
<td>0.1</td>
<td>0.9</td>
<td>0.0</td>
<td>12.55</td>
<td>1.32</td>
<td>0.07</td>
</tr>
<tr>
<td>OCDM03</td>
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<td>0.0</td>
<td>11.30</td>
<td>3.97</td>
<td>0.21</td>
</tr>
<tr>
<td>OCDM05</td>
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<td>0.5</td>
<td>0.0</td>
<td>10.53</td>
<td>6.62</td>
<td>0.35</td>
</tr>
<tr>
<td>OCDM01</td>
<td>0.1</td>
<td>0.5</td>
<td>0.4</td>
<td>14.65</td>
<td>1.32</td>
<td>0.07</td>
</tr>
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<td>0.07</td>
</tr>
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<td>17.85</td>
<td>3.97</td>
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</tr>
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<td>12.70</td>
<td>3.97</td>
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<td>3.97</td>
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<td>OCDMC1</td>
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<td>3.97</td>
<td>0.21</td>
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<tr>
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<td>0.0</td>
<td>11.61</td>
<td>6.62</td>
<td>0.35</td>
</tr>
<tr>
<td>OCDMC2</td>
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<td>0.9</td>
<td>-0.4</td>
<td>12.84</td>
<td>6.62</td>
<td>0.35</td>
</tr>
<tr>
<td>OCDMC3</td>
<td>0.5</td>
<td>0.9</td>
<td>-0.4</td>
<td>12.84</td>
<td>6.62</td>
<td>0.35</td>
</tr>
</tbody>
</table>

mordial power spectrum, for which we assume a Harrison-Zeldovich spectrum: $n = 1$ (Harrison 1970; Zeldovich 1972). The shape parameter $\Gamma$ of the power spectrum completely describe the scale dependence of the density fluctuations. For the shape parameter we use the form given by Sugiyama (1995)

$$\Gamma = \Omega_m h \exp \left[ -\Omega_b \left( 1 + \frac{\sqrt{2h}}{\Omega_m} \right) \right],$$

(2.11)

where $\Omega_m$ is the matter density and $\Omega_b$ is the baryonic density. For our models we assume a baryon density parameter of $\Omega_b = 0.047$ and a Hubble parameter of $h = 0.7$. The shape of the power spectrum, via $\Gamma$, will be mainly determined by the value of $\Omega_m$ (see Table 2.2).

The amplitude of the power spectrum is determined on the basis of $\sigma_8$, the rms fluctuation (in linear theory) of the mass contained in spheres of $8h^{-1}$Mpc, and for all our scenarios we use

$$\sigma_8 = 0.8.$$

(2.12)

Note that we did not normalize according to the number of clusters present in each simulation.

To compare the linear power spectra $P(k, z)$ of the models at a particular redshift $z$, we simply compute the expression

$$P(k, z) = AT^2(k)kD^2(z).$$

(2.13)

In this way, we can find the linear power spectrum at $z = 49$ in Fig. 2.1. To distinguish the power spectra of the different scenarios we use the same color and linestyle scheme as described above for the linear density growth factor $D(a)$ (2.2.2). We may observe the following facts:

- Models without a cosmological constant have more power on small scales than models with $\Omega_\Lambda$, independent of the values of $\Omega_m$.
- The higher the value of $\Omega_m$, the less power on small scales, independent of the values of $\Omega_\Lambda$.
- Models with $\Omega_m = 0.5$ with a cosmological constant are the ones that have less power on both scales, followed by the models with $\Omega_m = 0.3$
- Given a value of $\Omega_m$, the amplitude of the linear power spectrum increases as $\Omega_\Lambda$ decreases.
2.3 N-Body Simulations

In order to study formation and evolution of structures in the thirteen different cosmologies, we perform an N-body simulation in each of these cosmological backgrounds. Each simulation consists of 256$^3$ dark matter particles in a box of size 200$h^{-1}$Mpc with periodic boundary conditions. Every simulation has the same Hubble parameter, $h = 0.7$ (where the Hubble constant is given by $H_0 = 100h$ km s$^{-1}$ Mpc$^{-1}$) and the same normalization of the power spectrum, $\sigma_8 = 0.8$.

The simulations started from initial conditions at $z = 49$. In order to facilitate comparison between the outcome of the simulations in the various cosmologies, the primordial Gaussian density fields were assumed to have the same phases $\phi(k)$ for each of its Fourier components,

$$\delta(x) = \int \frac{d^3k}{(2\pi)^3} \hat{\delta}(k) e^{ikx},$$  \hspace{1cm} (2.14)

with

$$\hat{\delta}(k) = |\hat{\delta}(k)| e^{i\phi(k)}.$$  \hspace{1cm} (2.15)

While the spectral dependence figures in via $|\hat{\delta}(k)|$, the choice for equal phases $\phi(k)$ in each of the simulations means that we recognize the same morphological pattern in the cosmic mass distribution. Each difference between the structure that form in each of the cosmologies can therefore be related to the difference in cosmology.

We followed the gravitational evolution of the structures from the initial density fields at $z = 49$ until the present, $z = 0$, using the massive parallel tree N-Body/SPH code GADGET-2 (Springel 2005). We restricted ourselves to the dark matter particles, the gaseous SPH component was turned off. The Plummer-equivalent softening was set at $\epsilon_{Pl} = 15 h^{-1}$kpc in physical units from $z = 2$ to $z = 0$. At higher redshifts the softening length was fixed in comoving units. Of each simulation we saved 100 snapshots from $z = 4$ till the present time. These time steps were equally spaced in log($a$) (where $a$ is the expansion factor).

2.3.1 Simulation results

Fig. 2.2 show slices of 2$h^{-1}$Mpc thick through the center of the simulation box of the $\Lambda$CDM$F_2$ (top panel), $\Lambda$CDM$O_2$ (middle) and $\Lambda$CDM$C_2$ model at $z = 0$. Visual inspection of these figure directly reveals a few outstanding observations:

- The effect of choosing identical random phases is clearly visible: the patterns of the large scale structure are similar. The differences between the cosmologies manifests itself in the different levels and behavior of clustering.

- The $\Lambda$CDM$O_2$ model contains less structure than the other ones. To a large extent this is the result of the extremely low value of $\Omega_m$ in this scenario, possible in combination with the presence of a cosmological constant. As a result, structure growth came to a halt at a significantly earlier epoch.

- The detailed view in the zoom-ins strengthen these conclusions: higher $\Omega_m$ produces considerably more pronounced and evolved patterns, characterized by higher level of clustering.

Fig. 2.3 shows the evolution of the zoomed region in five different redshifts. We also show the time (in Gyr) since the Big Bang for comparison. In different cosmologies the same redshift corresponds to a different cosmic time, even for the same Hubble parameter. This depends sensitively on the values of $\Omega_m$ and $\Omega_\Lambda$. The low-$\Omega_m$ model shows a faster growth and evolution of structure. In fact, we see that at $z = 4$, it already shows some defined features, which are loosely present in the other models. The formation and evolution of structures in the closed model is slower. But if we compare by cosmic time, we see that the structure looks more or less the same. If we take $t = 4.76$ Gyr ($z = 1.29$) in the
Figure 2.2 — Slices of $2 h^{-1}$Mpc thick through the center of the box of three different cosmological models: $\Lambda$CDM$^2$ (top), $\Lambda$CDMO$^2$ (middle) and $\Lambda$CDMC$^2$ (bottom). On the right, a zoom into the region selected in the slice.
Figure 2.3 — Evolution of zoomed regions of Fig. 2.2 in five different redshift. In the lower region, the time (in Gyr) since the Big Bang is depicted. Time is different although redshift is the same.
ΛCDMF2 model and compare with the ΛCDMO2 model at \( t = 4.01 \) Gyr \( (z = 2.38) \), we see that the structures are similar.

Perhaps here we see one of the main consequences of having a cosmological constant. It strongly affects the available dynamical time scales for the formation of structure.

2.4 The Mass Function

The mass function is the number density of objects of a given mass. The abundance of the most massive halos is sensitive to the overall amplitude of mass fluctuations, \( \sigma_8 \). The evolution of this abundance can give us clue on the cosmological density parameter, \( \Omega_m \).

In order to calculate the mass function, we first need to construct the halo catalogues of each cosmology.

<table>
<thead>
<tr>
<th>Model</th>
<th>Number of Clusters</th>
<th>Model</th>
<th>Number of Clusters</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCDM</td>
<td>1835</td>
<td>ΛCDMO3</td>
<td>192</td>
</tr>
<tr>
<td>OCDM01</td>
<td>31</td>
<td>ΛCDMF2</td>
<td>189</td>
</tr>
<tr>
<td>OCDM03</td>
<td>197</td>
<td>ΛCDMC1</td>
<td>189</td>
</tr>
<tr>
<td>OCDM05</td>
<td>559</td>
<td>ΛCDMF3</td>
<td>539</td>
</tr>
<tr>
<td>ΛCDMO1</td>
<td>28</td>
<td>ΛCDMC2</td>
<td>538</td>
</tr>
<tr>
<td>ΛCDMO2</td>
<td>27</td>
<td>ΛCDMC3</td>
<td>527</td>
</tr>
<tr>
<td>ΛCDMF3</td>
<td>28</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.3 — Number of cluster detected in each of the simulations of this study. For the parameters of each of the models we refer to Table 2.2.

2.4.1 Halo Catalogues

To extract the groups present in the simulation, we use HOP (Eisenstein & Hut 1998). HOP associates a density to every particle by smoothing the density field with a spline cubic kernel using the \( n \) nearest neighbors of a given particle. Particles are then linked by associating each particle to the densest particle from the list of its closest neighbor. This process is repeated until it reaches a particle that is its own highest density neighbor. All particles linked to a local density maximum are identified as a group.

So far, no distinction between a high density region and its surrounding has been made. To identify halos above a density threshold, a regrouping merging procedure is performed. This procedure is based on three parameters. The code first includes only particles that are above some density threshold \( \delta_{\text{outer}} \). It then merges all groups for which the boundary density between them exceeds \( \delta_{\text{saddle}} \). Finally, all groups identified must have one particle that exceeds \( \delta_{\text{peak}} \) to be accepted as an independent group.

We associate the value of \( \delta_{\text{peak}} \) with the virial density, which is given by the solution of the spherical collapse model. For the other two density parameters, we follow the suggestion of Eisenstein & Hut (1998), who claim that the values are in the ratio \( \delta_{\text{outer}}:\delta_{\text{saddle}}:\delta_{\text{peak}} = 1:2.5:3 \).

We apply HOP to every output of every run giving the corresponding value of \( \Delta_{\text{vir}} \). We only consider groups with more than 100 particles in each run, this means that the minimum mass will vary for each cosmological model (see Table 2.2 for the masses of each particle). From each output of each simulation we construct halo catalogues which will allow us to study the mass function. At \( z = 0 \), the number of groups in each run ranges from \( \sim 5300 \) for \( \Lambda\text{CDMF1} \) to \( \sim 14300 \) for SCDM. This indicates that low \( \Omega_m \) Universe have less structure. The number of groups in cosmologies with the same \( \Omega_m \) but different \( \Omega_{\Lambda} \) is similar (with the exception of SCDM), showing that the dynamics on this scale does not tell much about the cosmological constant.

Table 2.3 shows the number of galaxy clusters in each cosmology at present time. We define a galaxy cluster as an object with \( M > 10^{14}h^{-1}M_\odot \). We see that low matter universes have few clusters at
$z = 0$, independently of the value of $\Omega_\Lambda$. As the value of $\Omega_m$ increases, the number of clusters increases, which is to be expected, since there is more matter in the Universe to form structures. In models where $\Omega_m = 0.1$, the amount of matter is too low to suppress the action of $\Omega_\Lambda$.

![Figure 2.4](image_url) — Evolution of the mass function for the cosmological models discussed in 2.3. The panels depict the mass function in four different redshifts. Colors represent models with the same value of $\Omega_m$, different linestyles represent models with the same value of $\Omega_\Lambda$. At early times, $z \sim 3$, $\Omega_m$ dominates the evolution.
2.4.2 Results

Fig. 2.4 shows the evolution of the cumulative number of dark halos with mass above \( M \) per comoving volume, \( N(>M) \), for all the cosmoologies described. Each panel shows the mass function at a specific redshift, \( z = 2.98, z = 1.49, z = 1.01 \) and \( z = 0 \). Colors and linestyle are as described in section 2.2.

Hierarchical clustering predicts that the number of objects of a certain mass increases as a function of time, while lower mass structures start to form and get incorporated into more massive ones. This is confirmed in Fig. 2.4, where we see that, for every model, the mass and the number of objects increases as we go from \( z = 2.98 \) to \( z = 0 \). In the figure we observe the following:

- For the open models, the redshift at which structure growth stops is in the range of \( z \approx 10 \) (\( \Omega_m = 0.1 \)) to \( z \approx 0 \) (\( \Omega_m = 0.5 \)). For the other cases, the range is from \( z \approx 1.62 \) for the ΛCDMF1 model to \( z \approx 0.25 \) for the ΛCDMF3 model.
- At low redshifts, open models (OCDM01, OCDM03 and OCDM05) show a higher number of objects than the other models.
- As time evolves, structure formation in low \( \Omega_m \) models is suppressed, while in high \( \Omega_m \) objects become more massive and more numerous.
- From \( z \approx 1 \) to the present, it is possible to distinguish models with the same \( \Omega_m \). The higher the value of \( \Omega_m \), the more massive the objects.
- The role of \( \Omega_\Lambda \) is less clear. This is specially obvious at \( z = 0 \), where we see that it is not possible to distinguish models with the same \( \Omega_m \) and different \( \Omega_\Lambda \). Their mass functions tend to largely overlap.

2.4.3 Mass Functions: Press-Schechter formalism

The previous analysis showed us that the influence of \( \Omega_\Lambda \) is negligible, to the point where at present time is almost impossible to distinguish between models with same \( \Omega_m \) but different \( \Omega_\Lambda \). In this section we will study this effect together with the Press-Schechter formalism.

The Press-Schechter formalism (Press & Schechter 1974, hereafter PS, see also Bond et al. (1991)) states that the comoving number density of objects of mass \( M \) is

\[
\frac{dn}{dM} = \sqrt{\frac{2}{\pi}} \frac{\delta_c}{M^2 \sigma(M,z)} \left| \frac{d\ln \sigma(M,z)}{d\ln M} \right| \frac{1}{\left| \frac{d\ln \sigma(M,z)}{d\ln M} \right|} e^{-\frac{\delta_c^2}{2}},
\]

(2.16)

where \( \rho_b \) is the mean density of the Universe and \( \delta_c \) is the effective linear overdensity required for the collapse.

2.4.3.1 Critical collapse density and Cosmology

The critical value \( \delta_c \) is only weakly dependent on \( \Omega_m \) and virtually independent of \( \Omega_\Lambda \).

A good numerical approximation of \( \delta_c \) is given by Navarro et al. (1997):

\[
\delta_c(\Omega_m) = \begin{cases} 
0.15(12\pi)^{2/3}\Omega_m^{0.0185} & \text{if } \Omega_m < 1 \text{ and } \Omega_\Lambda = 0, \\
0.15(12\pi)^{2/3}\Omega_m^{0.0055} & \text{if } \Omega_m + \Omega_\Lambda = 1.
\end{cases}
\]

(2.17)

Note that these approximations are not valid for \( \Omega_k > 0 \) or \( \Omega_k < 0 \) in the generic case of a non-zero cosmolological constant. To further investigate possible influences of \( \Omega_\Lambda \) on mass functions, we have explicitly computed \( \delta_c \) for the general spherical collapse model in section 2.A. Fig. 2.5 shows the evolution of the density contrast as a function of the expansion factor in two cases, each showing four different cosmologies: one where \( \Omega_m \) has a constant value but \( \Omega_\Lambda \) is different (left panel) and the other where \( \Omega_\Lambda = 0 \) and \( \Omega_m \) is different. In the first case, we see that the density contrast is weakly dependent
2.4. THE MASS FUNCTION

Figure 2.5 — Evolution of the critical linear collapse density as a function of the expansion factor for two different cases. Left panel: the evolution for cosmologies with the same $\Omega_m$ but different $\Omega_\Lambda$. We see that the influence of the cosmological constant is almost imperceptible. Right panel: the evolution for cosmologies with different values of $\Omega_m$ and $\Omega_\Lambda = 0$. The influence of $\Omega_m$ is more evident, specially between a cosmology with almost no matter ($\Omega_m = 0.1$) and a SCDM one.

The underestimation (overestimation) of the PS formalism at different times are due because the original implementation of PS was based in spherical objects. This is not the case in hierarchical clustering. At early times, mergers of low mass objects dominate the formation and growth of structures, resulting in objects that are triaxial in shape. At present time, objects are more dynamically relaxed, resulting in more spherical objects, which translates into a better fit of the PS formalism.

2.4.3.2 Comparison PS and simulated mass functions

Fig. 2.6 and 2.7 shows the mass functions of four cosmologies together with their respective PS fitted mass function chosen as example. These are: SCDM, $\Lambda$CDMO2, $\Lambda$CDMF2 and $\Lambda$CDMC2. The fitted PS mass functions of the other models with same geometry and/or same $\Omega_m$ are similar to the ones shown here. As observed in Fig. 2.4, the evolution of the mass function is characterized as an increase in the number of objects and in their mass in each simulation as a function of time. The shape is distinctive of each cosmology. At every redshift the SCDM scenario produces a high number of objects spanning a wide range of masses. By contrast, the low density $\Lambda$CDMO2 model only contains a relatively modest population of objects spread over a small mass range. The other two models, $\Lambda$CDMF2 and $\Lambda$CDMC2, are intermediate cases. This can be directly related to their intermediate value of $\Omega_m$.

The fitted PS mass functions show the same trend in evolution and shape, but do not agree with the theoretical mass function at every redshift for some cosmologies. For the SCDM model, we find that the PS mass function is consistent only at $z = 0$. At lower redshift, it underestimates the number density of objects, with the critical case at $z \sim 2.98$. This is also the case for the $\Lambda$CDMF2 and $\Lambda$CDMC2 cosmologies. For the $\Lambda$CDMO2 model the agreement is roughly consistent at every redshift. We checked the other cosmologies with $\Omega_m$, and found the same behavior, except for the flat model ($\Omega_m = 0.1$, $\Omega_\Lambda = 0.9$), where at low redshift the PS mass function significantly underestimates the true mass distribution.
2.5 Conclusions

We have investigated the evolution of dark matter halos in thirteen cosmological models by means of numerical simulations. The investigated cosmologies include a SCDM Universe, six open Universes with and without a cosmological constant, three flat models and three closed with cosmological constant. The initial conditions were generated with identical phases for the Gaussian random field in order to ensure the presence of identical morphological patterns in each of the simulations. Our simulations are set up in periodic boxes of size $200 h^{-1}$Mpc containing $256^3$ particles. Every simulation has the same Hubble parameter, $h = 0.7$, and the same normalization of the power spectrum amplitude, $\sigma_8 = 0.8$.

We studied global properties of halos, focusing on their mass function. As expected, models with
higher $\Omega_m$ result in more and more massive objects, with SCDM being the most representative case. Both in low density matter dominated Universes as well in Universes dominated by a cosmological constant the formation and development of structures comes to a halt at early epochs.

We find that mass functions do distinguish between the $\Omega_m$ of the different cosmologies. However, at $z = 0$ we do not find any significant influence on the value of $\Omega_\Lambda$. While this relates to some extent on the normalization of the power spectrum on mass fluctuations at the present epoch, we do notice some noticeable effects at other redshifts. This is a result of the different dynamical timescales related to these redshifts as a consequence of the different $\Omega_\Lambda$ (Fig. 2.4).

These conclusions are reaffirmed by comparing our simulated mass functions to the PS prediction. The PS formalism only leads to a rough agreement at low redshift (with the exception of low-$\Omega_m$ models). However, it does differ substantially at higher redshifts. This might be understood if we take into account that the small objects forming at high redshifts are much more susceptible to external dynamical influences while we also should take into account their anisotropic collapse via oblate or prolate shapes. For example, Sheth & Tormen (1999) did demonstrate a substantial change in predicted mass spectrum when taking into account these effects. In this respect it is good to realize that the Press-Schechter formalism makes the implicit – and unrealistic – assumption that proto objects are spherical.
2.A Spherical Collapse Model

The spherical collapse model (Gunn & Gott 1972) describes the evolution of an isolated spherical overdense region in a homogeneous cosmological background of mean density $\rho_b$. Because it is straightforward to see that its equation of motion simplify to that of a one-dimensional equation for the motion of a radial shell, it is the one physical system whose evolution can be analytically followed in its entirety. While in reality isolated spherical systems do not exist, the model provides a necessary basis for understanding and interpreting the considerably more complicated evolution of generic systems. Moreover, it appears to provide a surprisingly accurate description of many physical systems lacking spherical symmetry.

2.A.1 From initial time to turn around

Here we restrict ourselves to a sphere of mass $M$ with a uniform mass distribution within a radius $R(t)$. This region starts to expand at the same rate as the background but, if its density is high enough, its expansion will slow down so much that it will stop at some point, reaching a maximum radius, turn around into contraction and finally collapse and virialize. Generically, we can identify three stages for the evolution of such an overdense region:

- Turn around: the spherical region has stopped expanding and begins to collapse.
- Collapse: the spherical region begins to contract and collapses to a point.
- Virialization: in practice, collapse to a singularity does not occur. Before that happens, shell crossing will occur and it will virialize.

The equation of motion which describes the evolution of the spherical overdense region is given by:

$$\left(\frac{dR}{dt}\right)^2 = \frac{2GM}{R} + \frac{\Lambda R^2}{3} + K.$$  \hspace{1cm} (A-1)

At maximum expansion, the velocity of the perturbation is zero, i.e., $\dot{R} = 0$. By setting the latter equation to zero, we obtain a cubic equation for the radius at maximum expansion or turn around, $R_{ta}$ which has to be solved numerically. The time from $t = 0$ to turn around is then

$$t_{ta} = \int_0^{R_{ta}} \frac{dr}{R}.$$  \hspace{1cm} (A-2)

By symmetry, we note that the time of collapse, $t_{coll}$, is always twice the time of turn around, $t_{ta}$: $t_{coll} = 2t_{ta}$. We can obtain the mass and the radius at turn around, so we can calculate the density at turn around relative to the background density at any expansion factor $a$:

$$\Delta_{ta} = \frac{\rho_{ta}}{\rho_b} = a^3 \frac{\Omega_{ta,p}}{\Omega_{m,0}}.$$  \hspace{1cm} (A-3)

where $\Omega_{ta,p}$ is the mean density of the perturbation relative to the background at turn around, which is usually much larger than $\Omega_{m,0}$.

2.A.2 Virialization

After turn around, the spherical region starts to contract and collapse. Slight departures from the spherical symmetry will cause the kinetic energy of the collapse to be converted into random motions. The perturbation reaches some form of thermalized, bound, equilibrium state. At this time, the spherical region is relaxed and bound.
2.A. SPHERICAL COLLAPSE MODEL

We do not know how long virialization takes. The standard assumption it is that is roughly the collapse time. We can find the radius after virialization in terms of the turn around radius by using the virial theorem. We follow the derivation described in Lahav et al. (1991). At turn around, the velocity of the shell is zero, so the total energy is

\[ E_{ta} = U_{G,ta} + U_{A,ta} = -\frac{3}{5} \frac{GM^2}{r_{ta}} - \frac{1}{10} \Lambda M r_{ta}^2, \tag{A-4} \]

where we have integrated over the sphere. Since the energy is conserved, the uniform sphere has the same energy at turn around and at virialization, \( E_{ta} = E_{vir} \). Using the virial theorem, the energy at collapse can then be written as

\[ K_{vir} = -\frac{1}{2} U_{G,vir} + U_{A,vir}. \tag{A-5} \]

Conservation of energy tells us that \( E_{ta} = E_{vir} \), i.e., \( K_{vir} + U_{G,vir} + U_{A,vir} = U_{G,ta} + U_{A,ta} \). Using Eqn. A-5, we get

\[ \frac{1}{2} U_{G,vir} + 2 U_{A,vir} = U_{G,ta} + U_{A,ta}. \tag{A-6} \]

Assuming that the sphere remains uniform, we can define the effective virial radius of the system (in analogy to the initial energies). This leads to the a cubic equation for the ratio \( r_{vir}/r_{ta} \):

\[ 2\eta \left( \frac{R_{vir}}{R_{ta}} \right)^3 - (2 + \eta) \left( \frac{R_{vir}}{r_{ta}} \right) + 1 = 0, \quad \eta \equiv \frac{\Lambda r_{ta}^3}{3GM^3}. \tag{A-7} \]

Note that if \( \eta = 0 \) (\( \Lambda = 0 \)), \( R_{vir}/R_{ta} = 1/2 \). The density after virialization is then given by

\[ \Delta_{vir} = \frac{\rho_{vir}}{\bar{\rho}} = \frac{a^3 \Omega_{p,ta}}{\Omega_{m,0}} \left( \frac{R_{ta}}{R_{vir}} \right)^3, \tag{A-8} \]

Following the same procedure, it is possible to obtain the linearly extrapolated overdensity at virialization. This is given by (Gross 1997)

\[ \delta_f = \frac{3}{5} D(a) \left[ \frac{\Omega_{k,0}}{\Omega_{m,0}} - \frac{\Omega_{k,ta}}{\Omega_{p,ta}^{1/3} \Omega_{m,0}^{1/3}} \right], \tag{A-9} \]

where \( D(a) \) is the growth factor and \( \Omega_{k,ta} = -(\Omega_{p,ta} + \Omega_{\Lambda}) \).

2.A.3 Results

We compare our numerical results with the approximations given by Bryan & Norman (1998) for a flat universe and that of Pierpaoli et al. (2001), whose fit is for general cosmologies. The approximation given by Bryan & Norman (1998) is

\[ \Delta_{vir}(z) = \frac{18\pi^2 + 82x - 39x^2}{1 + x}, \tag{A-10} \]

where \( x = \Omega_m(z) - 1 \). This relation is accurate in the range \( \Omega(z) = 0.1 - 1 \). The fit by Pierpaoli et al. (2001) is given by

\[ \Delta_{vir}(z) = \Omega_m(z) \sum_{i,j=0}^{4} c_{ij} x^i y^j, \tag{A-11} \]

where \( x \equiv \Omega_m(z) - 0.2, y \equiv \Omega_{\Lambda}(z), \) and the coefficients \( c_{ij} \) are given in their Table 1. Their fit is accurate within 2% in the range \( 0.2 \leq \Omega_m \leq 1.1 \) and \( 0 \leq \Omega_{\Lambda} \leq 1 \).

As an example, Fig. 2.8 shows \( \Delta_{vir} \) versus \( \Omega_m \) for flat models using both fits and our numerical results. They agree quite well in the range \( 0.2 \leq \Omega_m \leq 1 \) (where the fit given by Pierpaoli et al. (2001)
is accurate). For lower values of $\Omega_m$ the fits presents problems in comparison with the numerical result. The restriction in the fit by Pierpaoli et al. (2001) is more evident in the right panel of the same figure. We plot the evolution of $\Delta_{\text{vir}}$ for one of our models ($\Omega_m + \Omega_\Lambda = 0.6$) using the numerical solution (solid line) and the fit given by Eqn. A-11 (dashed line). As expected, for lower values of the expansion factor, both curves agree quite well but as we approach $a = 1$, the value of $\Omega_m$ tends to 0.1, and the fit is no longer applicable.