Supervisory control of switched nonlinear systems

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SUMMARY

The objective of this paper is to describe recent progress in the field of hybrid control of nonlinear systems. We address the problem of controlling a nonlinear dynamical process by means of a hybrid controller combining continuous dynamics with discrete logics, the latter used to switch, from time to time, between several continuous control laws. The hybrid control system considered in this paper relies on the use of a switching logic, which generates online and in a fully adaptive fashion, 'sliding windows' of the monitoring signals used to select the control action. It is shown that, under suitable conditions, the resulting supervisory control scheme ensures $L_1$-induced gain to the disturbance-to-state map, whether the process dynamics are constant or not. Copyright © 2012 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The objective of this paper is to describe recent progress in the field of hybrid control of uncertain nonlinear systems. Typically, in hybrid control, one builds a bank of alternative candidate controllers and switches among them based on measurements collected online. The switching is orchestrated by a specially designed logic, called the supervisor, that uses the measurements to decide which controller should be placed in the feedback loop at each instant of time. Such switching schemes provide an alternative to traditional continuously tuned adaptive control laws and usually arise in applications where control methodologies based on a single continuous controller would not guarantee satisfactory performance. For an early overview on switching supervisory control, the reader is referred to [1, 2].

Over the last decade, switching supervisory control has attracted significant research efforts and various approaches have been developed. In this respect, the switching algorithms that seem to be the most promising are those which do not employ pre-routing, as they evaluate online the potential performance of each candidate controller without their actual insertion in the closed-loop. These type of algorithms can roughly be divided into two categories: those based on process estimation, which typically rely on the use of the ‘certainty equivalence’ principle (e.g., see [3–8]) and those which directly exploit measurements collected online to forecast performance of each candidate controller (e.g., see [9–12]).

Despite a copious research activity, some fundamental questions remain unanswered. In particular, although several methods are available for the control of uncertain linear time-invariant...
systems, analytical tools for studying nonlinear and/or time-varying processes are largely lacking. This is the primary motivation for the work described in this paper. The control architecture considered herein is adopted from [6] and consists of a pre-designed finite family of candidate controllers and a supervisory unit which orchestrates logic-based switching among the controllers. This supervisory control system has three other subsystems: the multi-estimator, the monitoring signal generator, and the switching logic. The task of the switching logic is to generate a switching signal which determines, from time to time, the candidate controller that is to be placed in the feedback loop. The selection of the candidate controllers is carried out online, based on the values of the monitoring signals produced by the multi-estimator. In essence, this type of supervisory scheme relies on the idea of ‘certainty equivalence’ in that it determines which of the output estimation errors is the smallest and selects the candidate controller designed for the corresponding parameter value.

With specific reference to nonlinear systems, switching supervisory control has been addressed in [6, 9, 13–19]. The main difficulty with the subject—a difficulty that also arises when the process to be controlled is linear—is that the resulting closed-loop system is of a switching nature, whether the process dynamics are constant or not. Unfortunately, it is known that a switching system may have unbounded solutions even if all its ‘frozen time’ individual subsystems are stable. Analysis of switched systems is typically difficult and this is basically the reason why most of the aforementioned contributions rely on switching stopping in finite time to provide stability guarantees. A notable contribution to the analysis of systems under persistent switching is given by [16], where it is shown that infinitely, many switches among input-to-state stable (ISS) systems yield an ISS stable system provided that the switching instants are separated by a sufficiently large dwell time. These results have been further extended in [19] to include ‘slow-on-the-average’ switching under the assumption that each subsystem is an ISS with a very special form. In this paper, we propose a scheme for supervising input-to-state stabilizing controllers where stability is ensured thanks to a specially devised switching logic. For the case where the process parameters are frozen in time, we show that if (i) the set of admissible parameters is finite and (ii) for each admissible parameters value an input-to-state stabilizing controller is available, then the plant states can be kept bounded for arbitrary initial conditions and persistent bounded disturbances. This result is achieved by incorporating, in the switching control scheme, a mechanism that adaptively selects a window of past measurements relevant to achieve the desired stability properties. We show that this approach not only provides a very simple means to ensure state boundedness when the process to be controller is time-invariant but also allows certain insights into the design of hybrid controllers for switched nonlinear systems.

Some of the ideas proposed have their roots in the work on supervisory control of linear time-varying systems reported in [20]. Although the control architecture considered here and the one in [20] originate from fundamentally different approaches and use different monitoring signals, they share the idea of using an ad-hoc mechanism to generate online (and in a fully adaptive fashion) a time-windowing of the monitoring signals.

A final point is worth mentioning. The stabilization of time-varying systems—including switched systems—by means of supervisory control schemes is still in an early stage and very few results are available. Concerning ‘non pre-routed’ algorithms, preliminary results on this topic, although restricted to linear systems, can be found in [20, 21]. The main contribution of this paper should not be viewed as a systematic procedure to design hybrid controllers for switched nonlinear systems, because the requirements placed on the process, the multi-estimator, and the controllers restrict the analysis to classes of nonlinear systems that possess special structure. Rather, this paper is intended to describe some of the difficulties in this area and illustrate how alternative design approaches can be used to overcome them.

The remainder of this paper is as follows. In Section 2, we define the problem of interest and we describe the supervisory control architecture. Some ISS-like stability and detectability properties used in the paper are discussed in Section 3. In Section 4, we describe the switching logic and explore some of its properties. The results are then applied to stabilization of uncertain nonlinear systems. The issue of designing hybrid controllers for switched nonlinear systems is addressed in Section 5. Finally, Section 6 ends the paper with concluding remarks.
Consider a family of continuous-time systems

\[ \dot{x} = f_p(x, u, d) \]
\[ y = h_p(x), \quad p \in \mathcal{P} \]  

(1)

where \( x \in \mathbb{R}^n \) is the state; \( u \in \mathbb{R}^m \) and \( d \in \mathbb{R}^l \) denote, respectively, the locally essentially bounded input and disturbance; \( y \in \mathbb{R}^q \) is the output, and \( \mathcal{P} \) is a finite index set. In this paper, we assume that for each \( p \in \mathcal{P} \), \( f_p \) and \( h_p \) are continuous and locally Lipschitz functions on \( x \), with \( f_p(0, 0, 0) = 0 \) and \( h_p(0) = 0 \). We also assume that for each \( p \in \mathcal{P} \), the functions \( f_p \) and \( h_p \) are known. Let \( \mathcal{P} \) be the switched system generated by the family (1) along with a switching signal \( \rho \), that is, \( \dot{x} = f_p(x, u, d) \) and \( y = h_p(x) \), where \( \rho : [0, \infty) \to \mathcal{P} \) is a piecewise constant (but otherwise unknown) switching signal, continuous from the right, specifying at every time the index of the active subsystem. We shall assume that there are no jumps in the state \( x \) at the switching instants, and that only a finite number of switches can occur in any bounded interval of time.

The problem of interest is to find, based on \( y \), feedback controls so as to ensure, under suitable conditions, \( L^\infty \)-induced gain to the disturbance-to-state map from \( d \) to \( x \), whether \( \rho \) is constant or not. The approach adopted in this paper consists of selecting a finite family of continuous time output-feedback controllers and a switching logic in such a way that the regulated plant have the stated stability properties.

2.1. Supervisory control system architecture

We now describe the basic ingredients of the supervisory control system considered in this paper. As anticipated, the control architecture adopted is essentially that of [6], to which the reader is referred for a more detailed treatment.

We use a high-level controller, called the supervisor, which, from time to time, orchestrates the switching among a suitably defined family of candidate controllers \( \mathcal{C}_p \), \( p \in \mathcal{P} \). Each candidate \( \mathcal{C}_p \) is designed as a controller that would be used to solve the regulation problem for the \( p \)th subsystem in (1). In addition to the family of candidate controllers, the supervisor has other three subsystems:

**Multi-estimator \( \mathcal{E} \):** A dynamical system whose inputs are the process input \( u \) and output \( y \), whose state is denoted by \( x_\mathcal{E} \), and whose outputs are denoted by \( y_p \), \( p \in \mathcal{P} \).

**Monitoring signal generator \( \mathcal{M} \):** A dynamical system whose inputs are the estimation errors

\[ e_p := y_p - y, \quad p \in \mathcal{P} \]  

(2)

and whose outputs \( \pi_p \), \( p \in \mathcal{P} \) are suitable functions of the estimation errors, called monitoring signals.

**Switching logic \( \mathcal{S} \):** A dynamical system whose inputs are the monitoring signals \( \pi_p \), \( p \in \mathcal{P} \) and whose output is the switching signal \( \sigma \) taking values in \( \mathcal{P} \).

To maintain continuity, a precise definition of the monitoring signal generator and the switching logic is deferred to Section 4.

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*Notations:* The notation for this paper is, in the main, standard. For a vector \( x \in \mathbb{R}^n \), \( |x| \) denotes Euclidean norm and \( \|x\|_I \) the (essential) supremum norm restricted to the interval \( I \). We denote by \( L^\infty(I) \) the set of (essentially) bounded time functions on \( I \). When \( I = \mathbb{R}^+ \), we just write \( \|x\|_\infty \) and \( L^\infty \), respectively. A function \( \alpha : [0, \infty) \to [0, \infty) \) is said to be of class \( K \) if it is continuous, strictly increasing, and \( \alpha(0) = 0 \). In addition, it is said to be of class \( K_\infty \) if \( \alpha(s) \to \infty \) as \( s \to \infty \). A function \( \beta : [0, \infty) \times [0, \infty) \to [0, \infty) \) is said to be of class \( KL \) if \( \beta(\cdot, t) \) is of class \( K \) for each fixed \( t \geq 0 \) and \( \beta(r, t) \) decreases to 0 as \( t \to \infty \) for each fixed \( r \geq 0 \).

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We write the multi-estimator $E$ as
\[ \dot{x}_E = F(x_E, y, u) \]
\[ y_p = h_p(x_E), \quad p \in \mathcal{P} \]  
(3)
where we assume that $h_p(0) = 0$ for each $p \in \mathcal{P}$. Intuitively, (3) is such that $y_p$ would asymptotically converge to the process output $y$ if $\rho$ were constant and equal to $p$ and there were no noise. Because $\mathcal{P}$ is a finite set, $E$ can be realized simply as a parallel connection of individual estimators of the form $\dot{x}_p = f_p(x_p, y, u)$, $y_p = h_p(x_p)$, or it can be realized by means of shared-state architectures [1, 3].

For a given family of candidate controllers, we write the switched controller $C_\sigma$ as
\[ \dot{x}_C = g_\sigma(x_C, x_E, e_\sigma) \]
\[ u = r_\sigma(x_C, x_E, e_\sigma) \]  
(4)

with $r_p(0, 0, 0) = 0$, $\forall p \in \mathcal{P}$, and where $\sigma$ is a piecewise constant switching signal (the control switching signal) taking values in $\mathcal{P}$. As in [6], we therefore assume that the entire state $x_E$ of the multi-estimator $E$ is available for control, which is a mild requirement because $E$ is implemented by the control designer. The understanding, here, is that for each frozen value of $\sigma$ in $\mathcal{P}$, the aforementioned equations model the candidate controller $C_p$. Notice that because $y = h_p(x_E) - e_p$ for each $p \in \mathcal{P}$, this description encompasses the case where one switches between dynamic output-feedback controllers of the form $\dot{x}_p = g_p(x_C, y)$, $u_p = r_p(x_C, y)$. Further discussion above the design of controllers of the form (4) can be found in [6, 13].

The resulting supervisory control system is depicted in Figure 1. Task of the supervisor is to select, from time to time, the candidate controller index $q$ whose corresponding monitoring signal $\pi_q$ is currently the smallest. Intuitively, the idea behind this strategy is that the $q$th subsystem with the smallest monitoring signal is more likely to correspond to the current process configuration, and thus the associated candidate controller can be expected to provide the best control performance. This idea originates from parameter adaptive control and in essence, extends the principle of ‘certainty equivalence’ from tuning to switching. A more plausible justification for such a strategy will become clear in Section 4, where a precise definition of the monitoring signals is given.

Before proceeding to describe the control architecture, a preliminary observation is in order. Throughout the paper, when we consider intervals of the form $[\tau, t]$, we should, to be more precise, write $[\tau, \min\{t, T_{max}\}]$ if the finiteness of $T_{max}$—the maximal time for which the solution of the relevant system is defined—cannot be ruled out a priori. For simplicity of exposition, we will just write $[\tau, t]$ in view of the fact that the approach described next ensures that, under suitable conditions, each trajectory is defined for all $t \geq 0$.

![Figure 1. Supervisory control system architecture.](image-url)
3. DETECTABILITY OF THE SWITCHED SYSTEM

In this section, we introduce certain key assumptions upon which all next developments depend. We also provide a few basic results pertaining to the frozen-time analysis, that is, properties of the feedback loop assuming that both the process switching signal $\rho$ and the controller switching signal $\sigma$ are constant in time. In this respect, Theorem 1 provides a preliminary justification for the control strategy considered in this paper, which is based on choosing a candidate controller whose index minimizes suitable functions of the estimation error.

We make the following assumption.

**Assumption 1**

For each $p \in \mathcal{P}$, the process is input/output-to-state stable (IOSS), that is, for every $x(\tau)$ and every $u$ and $d$, the corresponding solution of (1) with $\rho(t) = p$, $\forall \ t \geq \tau \geq 0$, satisfies the inequality

$$|x(t)| \leq \theta(|x(\tau)|, t - \tau) + \gamma_2(\|d\|_{l(t, \tau)}^2) + \gamma_3(\|u\|_{l(t, \tau)}) + \gamma_3(\|y\|_{l(t, \tau)}^2), \quad \forall \ t \geq \tau$$

for some functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ and $\theta \in \mathcal{L}$.

Notice that for notational simplicity, the class $\mathcal{L}$ functions $\theta, \gamma_1, \gamma_2$ and $\gamma_3$ in the aforementioned inequality have been chosen to be independent on $p$. This can always be carried out because the index set $\mathcal{P}$ is assumed to be finite. In essence, IOSS means that the process state eventually becomes small if the inputs (disturbances) and outputs are small. This is the reason by which, within this framework, IOSS is usually referred to as a ‘detectability’ property [13].

At times, we will actually demand more of Assumption 1 (cf. Section 5).

**Assumption 1’**

For each $p \in \mathcal{P}$, the process is large-time final-state norm observable, that is, $\exists T_o > 0$ such that for every $u$ and $d$, the corresponding solution of (1) with $\rho(t) = p$, $\forall \ t \geq \tau \geq 0$, satisfies the inequality

$$|x(t)| \leq \gamma_2(\|d\|_{l(t, \tau)}^2) + \gamma_2(\|u\|_{l(t, \tau)}^2) + \gamma_3(\|y\|_{l(t, \tau)}^2), \quad \forall \ t \geq T_o + \tau$$

for some functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$.

Strengthening these notions, we say that the process is small-time final-state norm observable in case (6) is valid for all $T_o > 0$. As discussed in [22], large[small]-time final-state norm observability can be interpreted as saying that the process is IOSS and moreover, there exists an $a > 0$[for all $a > 0$], we have $\theta(r, a) < \nu(r)$ for every function $\nu \in \mathcal{K}$, where $\beta$ is the function appearing in the formula (5).

3.1. Injected systems

In the analysis of hybrid controllers of this type, it is convenient to define an auxiliary (and process independent) family of systems which we call injected systems: For an arbitrary fixed $q \in \mathcal{P}$, we first rewrite (3) as $\dot{x_E} = F(x_E, h_q(x_E) - e_q, u) =: \bar{F}(x_E, e_q, u)$. Accordingly, the $q$th injected system is defined by

$$\begin{align*}
\dot{x_E} &= \bar{F}_q(x_E, e_q, r_q(x_C, x_E, e_q)) \\
\dot{x_C} &= g_q(x_C, x_E, e_q), \quad q \in \mathcal{P}
\end{align*}$$

(7)

where $g_q$ is the function appearing in the formula (4) for $\sigma \equiv q$. The understanding, here, is that system (7) has the same solutions as the one that results when the $q$th candidate controller given by (4) with $\sigma \equiv q$ is connected to the multi-estimator given by (3). As will become clear in the results to follow, the motivation for resorting to this auxiliary family of systems is closely related to the requirement that, for every frozen interconnection of the uncertain process with the controller $C_q$, the resulting closed-loop be detectable through the estimation error $e_q$. Such a property relies on the following assumption.

**Assumption 2**

For each $q \in \mathcal{P}$, the injected system is ISS with respect to the estimation error $e_q$, that is, for every $\mathbf{x}_E(\tau), \mathbf{x}_C(\tau)$ and every $e_q$, the corresponding solution of (7) with $\sigma(t) = q$, $\forall \tau \geq \tau \geq 0$ satisfies the inequality

$$
\left\| \begin{bmatrix} \mathbf{x}_E(t) \\ \mathbf{x}_C(t) \end{bmatrix} \right\| \leq \beta_q \left( \left\| \begin{bmatrix} \mathbf{x}_E(\tau) \\ \mathbf{x}_C(\tau) \end{bmatrix} \right\|, t - \tau \right) + \gamma_q(\|e_q\|_{[\tau,t]}), \quad \forall \tau \geq \tau
$$

for some functions $\gamma_q \in \mathcal{K}_\infty$ and $\beta_q \in \mathcal{KL}$.

The requirement placed on the candidate controllers is dictated by the need to ensure robustness against persistent disturbances. In [6], the authors consider an integral variant of the aforementioned stability notion, namely the integral-ISS stability, which is less stringent but requires that the disturbances have finite energy. Examples of design of ISS injected systems for nonlinear systems can be found in [19], although a general characterization of nonlinear systems for which Assumption 2 holds is still the subject of ongoing research.

We now turn our attention to the system that results when the $q$th candidate controller is placed in the feedback loop with the process $P$ and the multi-estimator $E$. The dynamics of this system are described by the following equations:

\[
\begin{align*}
\dot{x} &= f(x, r_q(x_C, x_E, h_q(x_E) - h(x))) \\
\dot{x}_E &= F(x_E, h(x), r_q(x_C, x_E, h_q(x_E) - h(x))) \\
\dot{x}_C &= g_q(x_C, x_E, h_q(x_E) - h(x))
\end{align*}
\]  

(9)

The theorem that follows is the main result of this section and states that over each interval of time where the process is time-invariant and the switched-on controller is $C_q$, the resulting (time-invariant) closed-loop (9) is ISS with respect to the disturbance $d$ and the estimation error $e_q$. More precisely, by letting $\mathbf{x} := (x', x_E', x_C')$, the following result holds.

**Theorem 1**

Let Assumptions 1 and 2 hold and further assume that $\rho(t) = p$ and $\sigma(t) = q$, $\forall \tau \geq \tau \geq 0$. Then, for every $\mathbf{x}(\tau)$ and every $d$ and $e_q$, the corresponding solution of (9) satisfies the inequality

$$
|\mathbf{x}(t)| \leq \beta_0(|\mathbf{x}(\tau)|, t - \tau) + \gamma_0(\|d\|_{[\tau,t]}) + \hat{\gamma}_q(\|e_q\|_{[\tau,t]}), \quad \forall \tau \geq \tau
$$

(10)

for some functions $\gamma_0, \hat{\gamma}_q \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$.

For the sake of completeness, a proof of Theorem 1 is given in the Appendix. As can be seen, it is a simple variant of the proof given in [6] for the case where the term $d$ is zero and the rightmost term in (10) is replaced by a suitable integral norm of $e_q$.

Theorem 1 has some interesting implications: The first, and perhaps more obvious, consequence of Equation (10) is a justification for switching strategies that select the candidate controller whose index minimizes a suitable function of the estimation error. Another important implication of Theorem 1 is that if the switching logic ensures that the switching stops in a finite time, then, provided $e_q$ is bounded, then $\mathbf{x}$ will be bounded as well. As we will see, to obtain boundedness properties of this type, it will be sufficient to ensure that both the plant and control switching signals remain constant in time over sufficiently large windows. This issues are addressed in detail in the next section.

4. HYSTERESIS SWITCHING WITH ADAPTIVE TIME-WINDOWING

The procedure outlined after Theorem 1 indicates that when the process to be controlled is time-invariant and equal, say, to $p \in \mathcal{P}$, then the boundedness of the estimation error $e_p$ is sufficient to ensure stability. Accordingly, the following assumption is considered.
Assumption 3
For each $p \in \mathcal{P}$, the multi-estimator $E$ is such that if $\rho(t) = p, \forall t \geq 0$, then there exists a constant $C$ such that $|e_p(t)| \leq C$, for all $t \geq 0$, where $C$ only depends on the disturbance bound, the system parameters and on initial conditions $x(0)$ and $x_G(0)$.

When considering a process whose dynamics can vary in time, we need to strengthen Assumption 3 as indicated below.

Assumption 3’
For each $p \in \mathcal{P}$, the multi-estimator $E$ is such that if $\rho(t) = p, \forall t \geq \tau \geq 0$, then, for every $x(\tau)$, we have that

$$|e_p(t)| \leq \alpha(|x(\tau)|, t - \tau) + \eta(\|x(t)\|), \quad \forall t \geq \tau$$

for some functions $\eta \in K_\infty$ and $\alpha \in KL$.

Remark 1
Assumption 3 is a standard requirement when dealing with persistent disturbances (e.g., see [19]) and basically demands that at least one of the estimation errors remains bounded in the presence of such disturbances. Assumption 3’ is a more stringent requirement and basically corresponds to the existence of a global asymptotic observer. Similarly to Assumption 2 (cf. Remark 2), a general characterization of nonlinear systems for which Assumption 3’ holds is not straightforward and is a topic of ongoing research.

The remainder of this section is as follows. First, we introduce the switching logic and describe some of its properties. Then, we describe the monitoring signals which are used in the switching logic and we show how the resulting supervisory control scheme ensures, under time invariance of the process, $C^\infty$-induced gain to the disturbance-to-state map.

4.1. Hysteresis switching logic
As for the switching logic, we consider the scale-independent hysteresis switching logic proposed in [5,6,13], whose functioning can be described as follows. Pick a positive number $h > 0$ and select $\sigma(0) \in \mathcal{P}$ arbitrarily. Assume now that at a certain time $t_i$, the value of $\sigma$ has switched to some $p \in \mathcal{P}$. Then, we keep $\sigma$ fixed until a time $t_{i+1} > t_i$ such that $(1 + h) \min_{q \in \mathcal{P}} \{\pi_q(t_{i+1})\} \leq \pi_p(t_{i+1})$, at which point we set $\sigma(t_{i+1}) = \arg \min_{q \in \mathcal{P}} \{\pi_q(t_{i+1})\}$. Repeating this procedure, we generate a piecewise constant signal $\sigma$ that is continuous from the right everywhere. Formally, by letting $\pi := \{\pi_q\}_{q \in \mathcal{P}}$ and $i_*(t) := \arg \min_{q \in \mathcal{P}} \pi_q(t)$, $\sigma$ is given by

$$\sigma(t) = l(\sigma(t^-), \pi(t)), \quad \sigma(0) = i_0 \in \mathcal{P}$$

$$l(i, \pi(t)) = \begin{cases} i, & \text{if } \pi_i(t) < (1 + h) \pi_{i_*(t)}(t) \\ i_*(t), & \text{otherwise} \end{cases}$$

Notice that when the minimum is not unique, a particular value for $\sigma$ among those that achieve the minimum (e.g., the smallest element in $\mathcal{P}$, which minimizes $\pi_q$), can be chosen arbitrarily.

In the next lemma, whose easy proof is omitted, we establish a useful property of the switching logic considered in this paper.

Lemma 1
Consider an arbitrary time interval $I$ and assume that for every $q \in \mathcal{P}$, we have that $\pi_q(t) \geq \mu, \forall t \in I$ and for some $\mu > 0$. Further assume that each $\pi_q$ is monotonically nondecreasing over

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For the sake of completeness, notice that in place of (12), one could alternatively use an additive hysteresis logic with no change in the conclusions of the results to follow.

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and that \( \min_{q \in \mathcal{P}} \pi_q(t) \leq \pi^*(\mathcal{I}), \forall t \in \mathcal{I} \). Then, the total number of switches \( N_\sigma(\mathcal{I}) \) over \( \mathcal{I} \) is bounded as follows

\[
N_\sigma(\mathcal{I}) \leq N \left\lfloor \log \left( \frac{\pi^*(\mathcal{I})}{\mu} \right) \frac{1}{\log(1+h)} \right\rfloor
\]

where \([a]\) denotes the smallest positive integer greater than or equal to \( a \geq 0 \). \( \square \)

Before proceeding any further, we digress briefly to discuss a simple but useful implication of the aforementioned lemma. Assume that the process \( \mathcal{P} \) is time-invariant. Further, assume that Assumptions 1–3 hold and that we have selected \( \pi_q(t) := \max\{\mu; \|e_q\|_{(0,t)}\}, p \in \mathcal{P} \). Then, by virtue of the hysteresis logic, Assumption 3 yields

\[
\pi_{\sigma(t)}(t) \leq (1+h) \max\{\mu, C\}, \quad \forall t \geq 0
\]

By virtue of Lemma 1, the switching stops in finite time (more precisely, there exists a time \( T^* < T_{\text{max}} \) such that \( \sigma(t) = q^* \in \mathcal{P} \) for all \( t \geq T^* \)). In addition, \( \pi_{q^*} \) is bounded on \([0, T_{\text{max}}]\) because of Assumption 3. Then, \( \hat{\gamma}_{q^*}(\|e_{q^*}\|_{0,T_{\text{max}}}) \) is finite in view of the continuity of \( \gamma_{q^*} \). Using Theorem 1, we have therefore that \( x, x_\mathcal{E}, \) and \( x_\mathcal{C} \) are bounded on \([0, T_{\text{max}}]\), and hence, the estimation error \( e_p = y_p - y \) is also bounded for each \( p \in \mathcal{P} \). Then, all the monitoring signals \( \pi_p \) remain bounded because they are obtained from bounded inputs \( e_p \). Thus, we have that \( T_{\text{max}} = \infty \), that is, the solution is defined for all \( t \geq 0 \).

Although the aforementioned approach provides a simple means to establish boundedness of the closed-loop signals, a switching scheme based on the maximum value of the estimation errors can cause degradation in performance because of the presence of disturbances. Even more importantly, the ability of the supervisor to adapt to processes that vary in time may be severely degraded because there is no mechanism to forget large estimation errors that occurred in the past. One way to remedy the situation is to consider monitoring signals which are based on the maximum value taken on by the feedback loop.

### 4.2. Monitoring signals with adaptive time-windowing

For each \( p \in \mathcal{P} \), we define the monitoring signals as

\[
\pi_q(t) := \max\{\mu; \|e_q\|_{(t-T(t),t)}\}, \quad t \geq 0
\]

where \( \mu > 0 \) and \( T(t) \) is a non-negative timing signal to be defined shortly. Before proceeding, some comments are in order regarding the meaning of (15)

(i) The parameter \( \mu \) is used to satisfy the conditions of Lemma 1. In essence, it is needed to avoid Zeno-behavior, that is, the possibility of an infinite number of discontinuities of \( \sigma \) in a finite interval of time.

(ii) The function \( T(t) \) determines, from time to time, the interval of past records of the monitoring signals used by the supervisor to select the candidate controller that is to be placed in the feedback loop.

The function \( T(t) \) can be constructed as follows: Consider an arbitrary partition of the time axis, \( \mathbb{R}_+ = \bigcup_k \mathcal{I}_k, k \in \mathbb{Z}_+ := \{0, 1, \ldots\} \), where \( \mathcal{I}_k := [t_k, t_{k+1}) \), with \( t_0 := 0 \) by convention. The function \( T(t) \) is given by

\[
T(t) = \begin{cases} t, & t \leq T_*, \\ T_* + t - t_k, & t > T_*, k \in \mathbb{Z}_+, t \in \mathcal{I}_k \end{cases}
\]

where \( T_* \) is a non-negative constant. One can regard \( T_* \) as a design parameter that permits a trade-off between readiness of the algorithm in discarding past information versus sensitivity to disturbances, that is, false-alarm rate. In words, before time \( T_* \), all the monitoring signals are given by \( \pi_q(t) = \max\{\mu; \|e_q\|_{(0,t)}\} \), which is in order to let the switching be based on a sufficiently large
observation window. Thereafter, depending on $t_{k+1}$, the observation window can either increase according to $[t_k - T, t)$, or it can be 'reset', which leads to $\pi_q(t) = \max\{\mu; \|e_q \|_{[t_k+1 - T, t)}\}, t \in \mathcal{I}_{k+1}$. Intuitively, we would like to select the sequence $\{t_k\}$ in such a way that stability is not destroyed. As shown below, this problem is equivalent to selecting $\{I_k\}$ in such a way that the state $x_{EC}$ of the injected system is uniformly bounded wherever past data are discarded.

To see this, we first derive a bound on $x_{EC}$ over each interval $I_k$. Let $\gamma_0 := \max_{q \in \mathcal{P}} \{\gamma_q\}$. From (14), we have $\gamma_0 \gamma_0 := \max_{q \in \mathcal{P}} \{\gamma_q\} \leq \gamma_0 \gamma_0 (1 + \mathbb{h}) \max\{\mu, C\} =: \kappa_0$, for all $t \geq 0, k \in \mathbb{Z}_+$. Let now $I_{k_1}$ denote the $i$th subinterval of $I_k$, where the control switching signal $\sigma$ is constant, that is, $I_k = \bigcup_{i=1}^{k_1} I_{k_i}$ where $I_{k_i} := [t_{k_i}, t_{k_i+1})$, $t_{k_0} = t_k$, and $\sigma(t) = q \in \mathcal{P}$ for all $t \in I_{k_i}$, Using (8) and letting $\beta_0 := \max_{q \in \mathcal{P}} \{\beta_q\}$, we obtain

$$|x_{EC}(t)| \leq \beta_0 (|x_{EC}(t_k)|, t - t_k) + \kappa_0 \leq \beta_0 (|x_{EC}(t_k)|, t - t_k) + \kappa_0, \quad \forall t \in I_{k_0}$$

$$|x_{EC}(t)| \leq \beta_0 (|x_{EC}(t_{k_1})|, t - t_{k_1}) + \kappa_0$$

$$\leq \beta_0 (\beta_0 (|x_{EC}(t_{k_1})|, t_{k_1} - t_k) + \kappa_0, t - t_{k_1}) + \kappa_0$$

$$\leq \beta_0 (2\beta_0 (|x_{EC}(t_{k_1})|, t_{k_1} - t_k), t - t_{k_1}) + \beta_0 (2\kappa_0, t - t_{k_1}) + \kappa_0$$

$$< \beta_1 (|x_{EC}(t_{k_1})|, t - t_{k_1}) + \kappa_1(\kappa_0) \quad \forall t \in I_{k_1}$$

where the $\mathcal{K}$ class function $\beta_1$ is defined by $\beta_1(t, r) := \beta_0 (2\beta_0 (r, 0), t)$ and $\mathcal{K}$ class function $\kappa_1$ is defined by $\kappa_1(t) := \beta_0 (2\kappa_0 (r, 0), t) + r$. By induction, it is simple to verify that

$$|x_{EC}(t)| < \beta_1 (|x_{EC}(t_{k_1})|, t - t_{k_1}) + \kappa_1(\kappa_0), \quad \forall t \in I_{k_i}$$

where, for every $i > 0$, the $\mathcal{K}$ class function $\beta_i$ and the $\mathcal{K}$ class function $\kappa_i$ are defined recursively by $\beta_i(t, r) := \beta_0 (2\beta_{i-1}(r, 0), t)$, with $\beta_0 = \beta_0$, and $\kappa_i(t) := \beta_0 (2\kappa_{i-1}(r, 0) + r$, respectively.

Observe now that under the Assumptions 1–3, Lemmas 1 and (15) ensure that the total number of switches $N_\sigma (I_k)$ over each interval $I_k$ is bounded by

$$N_\sigma (I_k) \leq N^* := N \left[ \log \left( \frac{(1 + \mathbb{h}) \max\{\mu, C\}}{\mu} \right) \cdot \frac{1}{\log(1 + \mathbb{h})} \right]$$

Accordingly, $i \leq N^*$ in the formula (17). Thus, one sees that $x_{EC}$ is (uniformly) bounded provided that $x_{EC}(t_k)$ is such for all $k \in \mathbb{Z}_+$. Consider now an arbitrary $t_k$ and let $t_{k*} \in I_k$ denote the time instant such that $\sigma(t) = q \in \mathcal{P}$ for all $t \in [t_{k*}, t)$. In essence, $t_{k*}$ can be thought of as the last (currently) switching time of the control signal on $I_k$. Let us now pick a number $\epsilon > 0$. The following definition characterizes the logic underlying the choice of $\{I_k\}$.

**Algorithm 1**

Let $t_0 := 0$. The sequence $\{t_k\}$ is defined recursively by

$$t_{k+1} := \inf \{t > T_* + t_{k*}: \beta_\sigma(t_{k*}) (|x_{EC}(t_{k*})|, t - t_{k*}) \leq \epsilon_1 \}, \quad k \in \mathbb{Z}_+$$

where $T_*$ is as in (15).

It is straightforward to verify that stability cannot be destroyed under this mechanism. To see this, assume that Assumptions 1–3 hold. By virtue of Lemma 1, the number of switches over any interval $I_k$ is bounded by $N^*$. Accordingly, let $\beta_\star (r, t) := \beta_{N^*} (r, t)$ and $\kappa_\star (k_0) := \kappa_{N^*} (k_0)$, where $\beta_\star$ and $\kappa_\star$ are the functions appearing in the formula (17). Then, if $I_k$ is selected according to (19), we have that $|x_{EC}(t_k)| \leq \max \{ |x_{EC}(t_0)|, \epsilon + \kappa_\star (k_0) \}$, for all $k \geq 0$, that is, the state of the injected system $x_{EC}$ is bounded upon $\{t_k\}$. Using the fact that the number of switches is finite over any interval $I_k$, we can apply (17) with $\beta_\star$ and $\kappa_\star$ to conclude that $x_{EC}$ is bounded. Then, $y_p = h_p (x_{EC})$ is also bounded for each $p \in \mathcal{P}$. By Assumption 3, there exists an index $p^* \in \mathcal{P}$ such that $e_{p^*}$ is bounded by $C$. Hence, $y = y_{p^*} - e_{p^*}$ is bounded and therefore the estimation error $e_{p^*} = y_{p^*} - y$ is also bounded. Then, all the monitoring signals $\pi_p$ remain bounded because they are obtained from bounded inputs $e_p$. Moreover, $u = r_\sigma (x_{EC}, x_E, e_0)$ is bounded. Because $d, u, y$ and $y$
are bounded, the state $x$ is also bounded in view of Assumption 1. Thus, as before, we have that the solution is defined for all $t \geq 0$.

**Theorem 2**
Let $\mathcal{P}$ be a finite set and consider the supervisory control system defined by (1)–(4) under the switching logic (12) with monitoring signals (15), (16), and (19). Under Assumptions 1–3, and provided that the plant is time-invariant, all the closed-loop signals remain bounded for arbitrary initial conditions and bounded disturbances.

**Remark 2**
The analysis carried out previously is quite different from the one existing in the relevant literature. Unlike the results in [14], our scheme does not rely on dwell-time switching which can be a stringent requirement from the control point of view. Similar to [13], our analysis is based on the property that, under a suitable choice of the monitoring signals, the switching is sufficiently slow on the average. Nonetheless, while in [13] such a property is obtained by imposing additional design constraints on the injected system in the present paper stability in ensured thanks to a specially devised mechanism which adjust (in a fully adaptive fashion) the length of the observation window. Such an approach not only requires less stringent design constraints but as we will see, also allows us to deal with persistent variations of the process dynamics.

Before proceeding with the analysis of the feedback system in the presence of variations of the process dynamics, we would like to show that, when the process is time-invariant, the condition involving $\bar{\beta}$ in (19) always hold after some finite time after $t_k$ and therefore, the sequence produced by Algorithm 1 is always infinite. To this end, let $\xi := \max\{x_{EC}(t_k), \epsilon + \kappa_*(\kappa_0)\}$ and recall that under (19), we have that $|x_{EC}(t_k)| \leq \xi$ for all $k \in \mathbb{Z}_+$. Furthermore, in view of (17) and using the fact that the number of switching over each interval $\mathcal{I}_k$ is upper bounded by $N^*$, we have

$$|x_{EC}(t)| < \beta_*(\xi, 0) + \kappa_*(\kappa_0) := \Xi, \quad t \geq 0 \tag{20}$$

where, using the notation previously introduced, $\beta_*(r, t) := \beta_{N^*}(r, t)$ and $\kappa_*(\kappa_0) := \kappa_{N^*}(\kappa_0)$. Now, for a given $\mathcal{KL}$ class function $\bar{\beta}(\cdot, t)$ we define

$$\bar{\beta}(r_1, r_2) := \inf \{t \geq 0 : \beta(r_1, \tau) \leq r_2 \text{ for all } \tau \geq t\}, \quad r_1, r_2 > 0 \tag{21}$$

It is therefore immediate to conclude that, provided that the same candidate controller is kept in the loop, condition (19) is attained after at most $\max\{T_*, \bar{\beta}_*(\Xi, \epsilon)\}$ instants, where $\bar{\beta}_*(\Xi, \epsilon)$ is finite since $\Xi$ is (uniformly) bounded. Then, recalling that by virtue of Assumption 3 the number of switches if bounded by $N^*$, we have at once the following.

**Lemma 2**
Under the same assumptions and conditions as in Theorem 2,

$$t_{k+1} - t_k \leq (N^* + 1) \max\{T_*, \bar{\beta}_*(\Xi, \epsilon)\}, \quad \forall k \in \mathbb{Z}_+ \tag{22}$$

where $\epsilon$ and $T_*$ are as in (19).

**Remark 3**
Lemma 2 has another interesting implication. Assuming that the disturbance is vanishing and that in Assumption 3 we replace $C$ by a vanishing bound $C(t)$ (e.g., as in Assumption 3'), then it is straightforward to verify that there exists a finite time after which $\pi_\rho^*$ is bounded by $\mu$. Then, (cf. (13)), there will also be a finite time after which, over each interval $\mathcal{I}_k$, the total number of switching will be bounded by $N$. Following the same lines of Theorem 2, we therefore have that the plant state $x$ can be driven arbitrarily close to the origin by decreasing $\epsilon$, $\mu$, and $h$. 

5. STABILIZATION OF SWITCHED NONLINEAR SYSTEMS

In this section, we analyze the behavior of the proposed supervisory control scheme in the presence of time variations of the plant dynamics. To this end, let \( \{ \ell_c \}, c \in \mathbb{Z}_+ \) denote the sequence of time instants at which a variation in the plant dynamics occurs, with \( \ell_0 := 0 \) by convention. Accordingly, we let \( \mathcal{L}_c := [\ell_c, \ell_{c+1}) \), \( c \in \mathbb{Z}_+ \), define the \( c \)th time interval over which the signal \( \rho \) is constant and say, equal to \( \rho_c \in \mathcal{P} \).

As should be clear from the previous analysis, stability of the switched system can be preserved if, over each \( \mathcal{I}_k \), there is an index \( p \in \mathcal{P} \) such that \( \rho_p \) is bounded. We show next that, under certain conditions, this property holds provided that, over each \( \mathcal{L}_c \), the estimation error \( e_{\rho_c} \) associated to the current plant configuration remains bounded. In order to ensure boundedness of \( e_{\rho_c} \), Assumption 3 is of little help in the presence of plant variations. Moreover, even Assumption 3’ alone cannot guarantee the existence of a uniform bound on \( e_{\rho_c} \). Nonetheless, such a property can be shown to hold provided that a uniform bound on the norm \( |x(\ell_c)| \) of the switched system state can be derived. In fact, Assumption 3’ implies that for any \( t \in \mathcal{L}_c \) and any \( c \in \mathbb{Z}_+ \),

\[
|e_{\rho_c}(t)| \leq \alpha(|x(\ell_c)|, t - \ell_c) + \eta(||d||_{\ell_c, t}) .
\]  

(23)

The purpose of the proposed data-reset mechanism is precisely to ensure that such a uniform bound exists. In order to provide a formal proof of this, we strengthen Assumption 1 by requiring that, for each frozen-time subsystem, the norm of the state can be bounded in terms of the norms of the past outputs and inputs on a suitably large time interval. This can be carried out by resorting to the norm observability notion introduced in Assumption 1’ which allows to state the following result.

\[\text{Lemma 3}\]

Suppose that Assumption 1’ and 2 hold and let \( T_* \geq T_o \). Then, there exist class \( K \) functions \( \varphi_1, \varphi_2 \) such that, for any \( k \in \mathbb{Z}_+ \),

\[
|\mathbf{x}(t)| \leq \varphi_1(||e_{\sigma}(t)||_{\ell_k, t}) + \varphi_2(||d||_{\ell_k + 1 - T_o, t}) + \Phi , \quad \forall t, \tau \in \mathcal{I}_{k+1}, \ t \geq \tau
\]  

(24)

for some constant \( \Phi \), provided that system mode \( \rho \) is constant in \( \mathcal{I}_k \cup \mathcal{I}_{k+1} \).

In words, equation (24) allows us to bound the state of the switched system in terms of the norms of the estimation error and of the exogenous disturbance on suitable intervals. Notice that the interval on which the norm of estimation error \( e_{\sigma} \) is computed comprises two consecutive data-reset instants \( t_k \) and \( t_{k+1} \). Then, equations (23) and (24) can be combined to derive a bound on \( e_{\rho} \) provided that a suitably large dwell time exists between successive plant variations.

To see this, suppose that at time \( \ell_c \) when the \( c \)th system mode variation occurs, the system state is bounded by some constant. Thus, for any given accuracy \( \nu \) and provided that the next plant variation instant \( \ell_{c+1} \) is far enough, the vanishing term \( \alpha(|x(\ell_c)|, t - \ell_c) \) in the right-hand side of (23) eventually enters a neighborhood of the origin of amplitude \( \nu \). For the sake of compactness, we let

\[
\ell_c^\nu := \inf \{ t \geq \ell_c + T_* : \alpha(|x(\ell_c)|, \tau - \ell_c) \leq \nu , \quad \forall \tau \geq t - T_* \}.
\]

Then, if at least two resets occur in the time interval \( [\ell_c^\nu, \ell_{c+1}) \), that is, there exists at least one index \( k \) such that \( \mathcal{I}_k \subseteq [\ell_c^\nu, \ell_{c+1}) \), we can exploit Lemma 3 in order to derive a bound on \( |x(\ell_{c+1})| \). We note that one single reset would not be sufficient because, in view of Equation (24), the state amplitude in the \(( k + 1 \)th interval \( \mathcal{I}_{k+1} \) depends on the amplitudes of \( e_{\sigma} \) and \( d \) also in the previous interval \( \mathcal{I}_k \). Because the aforementioned reasoning does not depend on the index \( c \), by means of simple induction arguments, the following result can be proved.

\[\text{Lemma 4}\]

Suppose that Assumptions 1’, 2, and 3’ hold and let \( T_* \geq T_o \). Then, if

\[
\forall c \in \mathbb{Z}_+ \quad \exists k \in \mathbb{Z}_+ \quad \text{such that} \quad \mathcal{I}_k \subseteq [\ell_c^\nu, \ell_{c+1}) ,
\]

(25)
and the exogenous disturbance is bounded, the solution of the switched system is defined for all \( t \geq 0 \) and its state \( x \) is bounded.

The bound on \( x \) will depend in general on the amplitude of the exogenous disturbance, the initial condition \( x(0) \), and the constants \( e, \mu, \) and \( h \). Clearly, boundedness of \( x \) is equivalent to the boundedness of all the signals in the switched system and in particular, of all the estimation errors. In the light of Lemma 4, it is immediate to see that a sufficient condition for stability is that the plant dwell time is large enough to allow the fulfillment of condition (25). Moreover, it is straightforward to verify that if condition (25) is satisfied up to a certain \( \ell_c \), then the existence of a uniform bound on \( x \) implies that \( \ell^p_c - \ell_c \) is bounded and that the time necessary for two resets to occur after \( \ell^p_c \) is bounded as well (cf. Lemma 2). If we denote by \( \tau_{\text{dwell}} \) the sum of such two bounds (which is independent of the index \( c \) since so does the bound on \( x \)), then condition (25) can be satisfied in finite time also up to \( \ell_{c+1} \) provided that \( \ell_{c+1} - \ell_c \geq \tau_{\text{dwell}} \). Then, the following stability result can be claimed which follows again from simple induction arguments.

**Theorem 3**

Let \( \mathcal{P} \) be a finite set and consider the supervisory control system defined by (1)–(4) under the switching logic (12) with monitoring signals (15), (16), and (19), with \( T_* \geq T_o \). Let Assumptions 1', 2, and 3' hold. Then, for every initial conditions and bounded disturbances, there exists a dwell time \( \tau_{\text{dwell}} \) such that, if \( \ell_{c+1} - \ell_c \geq \tau_{\text{dwell}} \) for any \( c \in \mathbb{Z}_+ \), all the closed-loop signals remain bounded.

**Remark 4**

Before concluding, we briefly review the main results of the paper along with the assumptions considered. For the case of systems with constant dynamics, only Assumptions 1–3 are involved. The assumptions in question are quite standard in the relevant literature and can be seen as the natural counterparts of the assumptions used for the design of supervisory control schemes for linear systems. In particular, Assumption 1 expresses the requirement that the process state converges to zero when its inputs and outputs are zero. Thus generalizing the linear concept of detectability in a natural way [23]; Assumption 2, along with Assumption 1, ensures that each frozen-time interconnection consisting of the process, multi-estimator, and multi-controller is detectable through the corresponding estimation error (Theorem 1). This is again a quite natural requirement as it stipulates that boundedness of the estimation error will result in boundedness of the overall state of the switched system (the result being also known in the linear case as the Certainty Equivalence Stabilization Theorem [24]). As previously noted, the design of ISS injected systems, and more generally that of control laws achieving ISS, is nontrivial. Contributions to this topic, centered around the concept of ISS-control Lyapunov functions, can be found in [25–27] (see also [19] for specific design examples). Assumption 3 complements previous assumptions as it simply demands that at least one estimator provides a bounded estimation error in the presence of disturbances. A number of examples of multi-estimator design for nonlinear systems can be found in [28]. In case the process has time-varying dynamics, the analysis becomes more complicated and we require stronger assumptions. Assumption 3' basically requires the existence of a global asymptotic observer for each \( p \in \mathcal{P} \). Such a requirement is used to ensure that when the plant dynamics is constant, one of the \( y^p \)'s provides a bounded estimation error \( e_p \) which is asymptotically independent on the initial conditions. For recent advances on global observers design, see, for example, [29,30] and the references therein. Assumption 1' strengthens Assumption 1 ensuring that boundedness of the process state only depends on boundedness of the process input/output data. For a detailed discussion on the relationship between Assumption 1' and the IOSS property, the reader is referred to [22].

6. **CONCLUSIONS**

In this paper, we have described a new framework for supervisory control of nonlinear systems. The proposed solution consists of a supervisory switching control logic whereby a controller, selected from a family of pre-designed candidate controller, is switched-on in feedback to the plant. The
controller selection is based on a family of monitoring signals which quantifies, from time to time, the suitability of each candidate controller to control the uncertain process. Unlike the previous approaches, the hybrid control system considered in this paper relies on the use of a special logic unit responsible for generating online, and in a fully adaptive fashion, ‘sliding windows’ of the estimation errors that are used to select the control action. It was shown that the resulting supervisory control scheme can ensure $L^\infty$-induced gain to the disturbance-to-state map, whether the process dynamics are constant or not.

Several issues remain open to investigation. The requirements placed on the process and the candidate controllers in the time-invariant case seem to be natural and dictated by the need of handling persistent disturbances. As for the time-varying case, a natural question, to which no answer can yet be given, is whether the large-time norm observability assumption can be relaxed. Second, dealing with the presence of process unmodeled dynamics constitutes an important area for further investigation.

7. APPENDIX

Proof of Theorem 1

Fix an arbitrary $q \in \mathcal{P}$ and let $x_E := (x_E', x_C')'$. Without loss of generality, we let $\tau = 0$ in (10) for clarity of exposition. In view of Assumption 2, we have

$$|x_E(t)| \leq \beta_q(|x_E(0)|, t) + \gamma_q(\|e_q\|_{[0,t]})$$

(26)

for some functions $\gamma_q \in K_{\infty}$ and $\beta_q \in K_{\mathcal{L}}$. As for the process, Assumption 1’ along with $e_p = y_p - y$ yields

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma_1(\|q\|_{[0,t]}) + \gamma_2(\|u_q\|_{[0,t]})$$

$$\leq \beta(|x(0)|, t) + \gamma_1(\|d\|_{[0,t]}) + \gamma_2(\|u_q\|_{[0,t]}) + \gamma_3(\|2y\|_{[0,t]}) + \gamma_3(\|2e_p\|_{[0,t]})$$

Here, we used the fact that for class $K$ functions $\alpha$ and arbitrary positive numbers $r_1, r_2, \ldots, r_k$, we have $\alpha(r_1 + \cdots + r_k) \leq \alpha(k) + \cdots + \alpha(k_r)$. Recall that we have $u_q = r_q(x_E, x_C, e_q)$ with $r_q(0, 0, 0) = 0$ and $y_q = h_q(x_E)$ with $h_q(0) = 0$. In view of this, it is easy to check that for a suitable class $K_{\infty}$ functions $\gamma_1$, $\gamma_2$ and $\gamma$ and suitable class $K_L$ function $\beta$, we can rewrite the aforementioned inequality as

$$|x(t)| \leq \tilde{\beta}(|x(0)|, t) + \chi(\|x_E\|_{[0,t]}) + \tilde{\gamma}_1(\|d\|_{[0,t]}) + \tilde{\gamma}_2(\|e_q\|_{[0,t]})$$

(27)

which amounts to saying that the subsystem corresponding to $P$, when viewed as a system with inputs $x_E, x_C, e_q, d$ and state $x$, is ISS.

By time-invariance, the aforementioned inequality can be rewritten as

$$|x(t)| \leq \tilde{\beta}(|x(t/2)|, t/2) + \chi(\|x_E\|_{[t/2,t]}) + \tilde{\gamma}_1(\|d\|_{[t/2,t]}) + \tilde{\gamma}_2(\|e_q\|_{[t/2,t]})$$

(28)

In view of (26), it is straightforward to verify that $\chi(\|x_E\|_{[t/2,t]})$ can be upper bounded as

$$\chi(\|x_E\|_{[t/2,t]}) \leq \tilde{\beta}_q(|x_E(0)|, t) + \tilde{\gamma}_q(\|e_q\|_{[0,t]})$$

(29)

where the functions $\tilde{\beta}_q \in K_{\mathcal{L}}$ and $\tilde{\gamma}_q \in K_{\infty}$ are defined by $\tilde{\beta}_q(r, t) := \chi(2\beta_q(r, t/2))$ and $\tilde{\gamma}_q(r) := \chi(2\gamma_q(r))$.

As for $\tilde{\beta}(|x(t/2)|, t/2)$, using (27), we get

$$|x(t/2)| \leq \tilde{\beta}(|x(0)|, t/2) + \chi(\|x_E\|_{[0,t/2]}) + \tilde{\gamma}_1(\|d\|_{[0,t/2]}) + \tilde{\gamma}_2(\|e_q\|_{[0,t/2]})$$

$$\leq \tilde{\beta}(|x(0)|, t/2) + \tilde{\beta}_q(|x_E(0)|, 0) + \tilde{\gamma}_q(\|e_q\|_{[0,t/2]})$$

$$+ \tilde{\gamma}_1(\|d\|_{[0,t/2]}) + \tilde{\gamma}_2(\|e_q\|_{[0,t/2]})$$

$$\leq \tilde{\beta}(|x(0)|, t/2) + \tilde{\beta}_q(|x_E(0)|, 0) + \tilde{\gamma}_q(\|d\|_{[0,t/2]}) + \tilde{\gamma}_2(\|e_q\|_{[0,t/2]})$$

where $\tilde{\gamma}_q(r) := \tilde{\gamma}_q(r) + \tilde{\gamma}_2(r)$.
Then, we have
\[ \tilde{\beta}((x|t/2),t/2) \leq \beta_2((x|0),t) + \alpha_1(d||t/2) + \alpha_2(\epsilon_\sigma||t/2) \] (30)
where the functions $\beta_1, \beta_2 \in \mathcal{K}$ are defined by $\beta_1(r,t) := \beta_2(4\beta(r,t),t/2)$ and $\beta_2(r,t) := \beta(4\beta_2(r),t/2)$, whereas the functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ are defined by $\alpha_1(r) := \beta(4\tilde{\gamma}_1(r))$ and $\alpha_2(r) := \beta(4\tilde{\gamma}_2(r))$.

Combining the aforementioned inequalities, we have at once (10). \[ \square \]

**Proof of Lemma 3**
By virtue of (8), when a reset occurs, the state of the injected system can be bounded as
\[ |x_\infty(t_{k+1})| \leq \epsilon + \gamma_0 (\|e_\sigma(\cdot)\|_{\tilde{x}_k}) \] (31)
Recall now that, when $\sigma(\cdot) = \sigma$, the injected system is ISS with respect to the estimation error $e_\sigma$. This in turn implies that, if $T_\sigma > T_\sigma$, there exist class $\mathcal{K}$ functions $\psi_1$ and $\psi_2$ such that
\[ |x_\infty(t)| \leq \psi_1(|x_\infty(t_{k+1})|) + \psi_2 (\|e_\sigma\|_{t_{k+1} - T_\sigma, t_{k+1}}) \] (32)
for any $t \in [t_{k+1} - T_\sigma, t_{k+1}]$. Combining (31) and (32), we readily obtain a bound on $x_\infty$ in the whole interval $[t_{k+1} - T_\sigma, t_{k+1}]$ as a function of $\|e_\sigma\|_{\tilde{x}_k}$. Further, because $y = e_\sigma + \gamma_\sigma = e_\sigma + h_\sigma(x_\infty)$ and $u = \sigma(x_\infty, x_\infty, e_\sigma, y|t_{k+1} - T_\sigma, t_{k+1})$, the function of the monitoring signal $\|e_\sigma\|_{t_{k+1} - T_\sigma, t_{k+1}}$ (cf. Lemma 1). The derivation parallels the one used to obtain the bound on $x(t_{k+1})$. Summing up, when a reset occurs, we have that the state $x$ of the switched system can be bounded as
\[ |x(t_{k+1})| \leq \psi (\|e_\sigma(\cdot)\|_{\tilde{x}_k}, d|_{t_{k+1} - T_\sigma, t_{k+1}}) \] (33)
for some suitable function $\psi : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ increasing in its arguments. We can now exploit Theorem 1 to derive a bound on $x$ at a generic time instant $t \in \mathcal{I}_{k+1}$ noting that the maximum number of switches of the index $\sigma(\cdot)$ in the interval $[t_{k+1}, t]$ is a function of the monitoring signal $\|e_\sigma\|_{t_{k+1} - T_\sigma, t}$ (cf. Lemma 1). The derivation parallels the one used to obtain the bound on $x_\infty$ in (17) and is omitted for the sake of the brevity. As a result, we have that
\[ |x(t)| \leq \varphi (\|e_\sigma(\cdot)\|_{t_{k+1}}, d|_{t_{k+1} - T_\sigma, t}) \] (34)
for some suitable function $\varphi : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ increasing in its arguments. Then, the proof is concluded by noting that the latter inequality can be written as (24) with standard manipulations. \[ \square \]

**Proof of Lemma 4**
The proof is given by induction. Let $D$ be a bound on $\|d\|_\infty$. Recall first that in the interval $\mathcal{L}_0$, before the first system mode variation occurs, we can bound the switched system state $x$ and the monitoring signal $\pi_\sigma$ by some suitable quantities $\tilde{X}$ and $\tilde{\Pi}$, respectively (whose dependence on the initial condition $x(0)$ and on the amplitude of the disturbance $d$ is omitted for the sake of compactness).

Suppose now that that the solution of the switched system exists up to a certain time $t_c$ (this is clearly true for $c = 1$) and let now $\mathcal{I}_{k_c}$ denote the reset interval wherein $t_c$ falls. Then, under condition (25), it follows that both $t_{k_c}$ and $t_{k_c - 1}$ fall in the interval $[t_{k_c - 1} - T_\sigma, t_c)$. This implies that
\[ |e_{\rho_{c - 1}}(t)| \leq v + \mu(D), \quad t \in [t_{k_c - 1} - T_\sigma, t_c) \]
and consequently that
\[ \|e_\sigma\|_{t_{k_c - 1} - T_\sigma, t_c} \leq (1 + h) \max \{\mu; v + \mu(D)\} \] (34)
Thus, in view of Lemma 3, the switched system state at time $t_c$ is bounded as
\[ |x(t_c)| \leq \varphi_1 (|1 + h| \max \{\mu; v + \mu(D)\}) + \varphi_2(D) + \Phi. \] (35)
Hence, proceeding like in the time-invariant case, we can conclude that the solution of the switched system will exist at least up to time $\ell_{c+1}$ and that the state $x$ in the interval $L_c$ can be bounded by some quantity $\bar{X}$. The only difference with respect to the time-invariant case is that, because of the memory of the monitoring signals, the bound on $\pi_{\rho_c}$ (with $\rho_c$ the index corresponding to the active system mode) in the interval $L_c$ will depend also on the values taken on by $e_{\rho_c}$ in the interval $[t_{k_c}, T_x, \ell_c)$. In fact, we have

$$\|\pi_{\rho_c}\|_{L_c} = \max \{\mu; \| e_{\rho_c} \|_{[t_{k_c}, T_x, \ell_{c+1})}\}.$$  

However, because $e_{\rho_c} = h_{\rho}(x) - h_{\rho_c}(x_{\bar{X}})$ is a function of $x$, along the same lines of the proof of Lemma 3, it could be shown that

$$\|e_{\rho_c}\|_{[t_{k_c}, T_x, \ell_c)} \leq \hat{\vartheta} \left( \|e_d\|_{[t_{k_c}, T_x, \ell_c)}, \|d\|_{[t_{k_c}, T_x, \ell_c)} \right)$$

for some function $\hat{\vartheta} : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ increasing in its arguments. Then, (34) implies

$$\|e_{\rho_c}\|_{[t_{k_c}, T_x, \ell_c)} \leq \hat{\vartheta} \left( (1 + h) \max \{\mu; \nu + \eta(D)\}, D \right) =: \hat{\mathcal{E}}.$$  

Further, thanks to (35), under Assumption 3', we have that

$$\|e_{\rho_c}\|_{L_c} \leq \alpha \left( \varphi_1 \left( (1 + h) \max \{\mu; \nu + \eta(D)\} \right) + \varphi_2(D) + \Phi, 0 \right) + \eta(D) =: \hat{\mathcal{E}}.$$  

The two latter inequalities can be used to bound $\pi_{\sigma(\cdot)}$ in the interval $L_c$ as follows:

$$\|\pi_{\sigma(\cdot)}\|_{L_c} \leq (1 + h)\|\pi_{\rho_c}\|_{L_c} \leq (1 + h) \max \{\mu; \hat{\mathcal{E}}, \hat{\mathcal{E}}\}.$$  

Because such a bound on the monitoring signal does not depend on the index $c$, we can conclude that neither the bound $\bar{X}$ does. This completes the induction arguments. The bound on the switched system state $x$ can then be taken as the maximum between $\bar{X}$ and $\bar{X}$.

REFERENCES