Chapter 5

Realization of Positive Linear Systems

5.1 Introduction

The purpose of this chapter is to present results on the realization of positive linear systems.

The motivation for the study of positive realizations is the use of positive linear systems in biomathematics, economics, chemometrics and other research areas. A finite-dimensional positive linear system is a linear dynamic system in which the input, state, and output space are standard positive cones in real Euclidean spaces. Systems in this class are useful models in biomathematics, where they are called linear compartmental systems, see [76]. The identification problem for this class of systems is unsolved. No general conditions are known for global structural identifiability of such systems, see Chapter 3. To obtain such conditions requires the solution of the realization problem for positive linear systems.

This chapter will deal with time-invariant finite-dimensional positive linear systems mainly in discrete-time, for short they will be called positive linear systems. A positive realization of a given positive impulse response function is a positive linear system, such that its impulse response function equals the given one. A positive realization of an impulse response function is said to be minimal if the state space as a vector space over the positive real numbers is of minimal dimension. The positive realization problem is to show existence of a positive realization of a positive impulse response function and to classify all minimal positive realizations. To solve the problem techniques of the theory for polyhedral cones will be used. Kodama and co-workers have worked on this problem, [98, 99, 112]. Only in the last paper have they used polyhedral
cones explicitly. See for more comments Remark 5.3.3. Another reference on positive linear systems is [10], and other references on positive realizations are [109, 110, 147].

The realization problem for positive linear systems is closely related to the stochastic realization problem for finite-valued processes, which has been studied by G. Picci and J.H. van Schuppen [114, 117, 129]. Also M. Fliess has studied these systems in [47]. This paper presents a necessary and sufficient condition for the existence of a realization. In addition the paper characterizes systems that admit a transfer function with positive coefficients. An earlier reference on the characterization of finite state processes that can be realized as functions of a finite state Markov chain appeared in [67], the earliest reference seems to be [49], (thanks to L. Finesso). Up to now the positive realization problem is unsolved. In this chapter a necessary and sufficient condition for the existence of a positive realization and for the characterization of minimality of the state space will be presented. This leads to a factorization problem for positive matrices. For a special case the classification of all minimal positive realizations will be provided, but the complete classification is still unsolved.

The outline of this chapter is as follows. The positive realization problem is formulated in Section 5.2. The existence of a positive realization is proven in Section 5.3. In Section 5.4 results on the characterization of minimality are presented. In Section 5.5 something will be said about the relation between minimal positive realizations and about the classification of minimal positive realizations. Sections 5.2 to 5.5 are devoted to discrete-time positive linear systems. Finally, in Section 5.6, the realization problem for continuous-time positive linear systems will be treated. In Section 5.7 concluding remarks are made.

5.2 Problem formulation

In this section some notation is introduced and the problem is posed.

**Definition 5.2.1** A *positive linear system* (which denotes the more formal term time-invariant finite-dimensional positive linear system in discrete-time) is a linear dynamic system (see for example [82, page 5]) in which the input, state, and output space are respectively $U = R^m_+$, $X = R^n_+$, and $Y = R^k_+$, the time index set is $T = N$. The system will be represented by

\[
\begin{align*}
x(t+1) & = Ax(t) + Bu(t), \quad x(0) = x_0, \\
y(t) & = Cx(t) + Du(t),
\end{align*}
\]

in which $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{k \times n}$, and $D \in R^{k \times m}$. \hfill \Box

**Proposition 5.2.2** A linear dynamic system of the form (5.2.1) is a positive linear system if and only if $A \in R^{n \times n}_+$, $B \in R^{n \times m}_+$, $C \in R^{k \times n}_+$, and $D \in R^{k \times m}_+$.

A positive matrix may be regarded not only as a matrix, but also as a polyhedral cone, as has been seen in Chapter 4. The relationship between the matrix and the geometric approach is very useful. Because $R^n_+$ is not a vector space over
$R$, the usual linear algebra cannot be used. Therefore the positive realization problem will be treated with convex cone analysis.

**Problem 5.2.3** The *positive realization problem* for a positive impulse response function.

a. Formulate necessary and sufficient conditions for the existence of a positive linear system such that the impulse response function of this system equals the given impulse response function. If such a system exists, it is called a *positive realization* of the given impulse response function.

b. Determine the minimal dimension of the state space of a positive realization. If the state space of a positive realization is minimal, this realization is called a *minimal positive realization*.

c. Classify all minimal positive realizations of the given impulse response function.

d. If two positive realizations of the same impulse response function are minimal, then indicate the relation between them.

This problem asks for a realization in the class of positive linear systems. It is also possible to search for a realization in the class of ordinary linear systems, but this is not useful for compartmental systems, because in these systems the state also has to be positive.

A positive linear system is called a *minimal positive linear system* if it is a minimal positive realization of its impulse response function.

### 5.3 Existence of a positive realization

Necessary and sufficient conditions for the existence of a realization of a positive impulse response function as a positive linear system are presented in this section. First the case without input is considered, and then the general input-output case.

Let $T = \mathbb{N}$, $Y = R^k_+$. Let $\xi$ denote the *time shift operator*

$$(\xi y)(t) = y(t + 1), \quad \text{for } y : T \to Y.$$ 

Let

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \end{pmatrix} \in R^\infty_+.$$ 

Then define the shift operator $\sigma$ on $z \in R^\infty_+$ by

$$\sigma z = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \end{pmatrix}.$$
For \( r \in \mathbb{N} \), define \( \sigma^r \) recursively by

\[
\sigma^r z = \sigma(\sigma^{r-1} z), \quad \sigma^0 z = z.
\]

Thus \( \sigma^r z \) is the vector in \( \mathbb{R}_+^\infty \) obtained by deleting the first \( r \) elements of \( z \). A cone \( C_1 \subseteq \mathbb{R}_+^\infty \) is said to be \( k \)-shift invariant for a \( k \in \mathbb{N} \), if \( z \in C_1 \) implies \( \sigma^k z \in C_1 \).

5.3.1 Case of a system without input

First attention is restricted to a system without input. In that case the external behaviour is described by a set of trajectories \( y : T \rightarrow Y \).

**Theorem 5.3.1** Let \( T = \mathbb{N} \), \( Y = \mathbb{R}_+^k \). Consider a set \( V \) of trajectories, each of which is a function \( y : T \rightarrow Y \). Let

\[
\text{cone}(V) = \text{cone}\{(y(0))^T, y(1)^T, y(2)^T, \ldots\}^T \subseteq \mathbb{R}_+^\infty \mid y \in V \},
\]

where \( y^T \) denotes the transpose of \( y \). There exists a positive linear system

\[
\begin{align*}
x(t + 1) &= Ax(t), \quad x(0) = x_0, \\
y(t) &= Cx(t),
\end{align*}
\]

such that any element of \( V \) is represented by the output of this system for some \( x_0 \in \mathbb{R}_+^n \) if and only if there exists a set \( C_1 \subseteq \mathbb{R}_+^\infty \) satisfying

1. \( C_1 \) is a polyhedral cone;
2. \( \text{cone}(V) \subseteq C_1 \);
3. \( C_1 \) is \( k \)-shift invariant.

**Proof.** (\( \Rightarrow \)) Assume that the set \( V \) is represented by the positive linear system

\[
\begin{align*}
x(t + 1) &= Ax(t), \quad x(0) = x_0, \\
y(t) &= Cx(t),
\end{align*}
\]

with \( X = \mathbb{R}_+^n \), \( A \in \mathbb{R}_+^{n \times n} \), \( C \in \mathbb{R}_+^{k \times n} \), for \( n \in \mathbb{Z}_+ \). Let

\[
S = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \end{pmatrix} \in \mathbb{R}_+^{\infty \times n},
\]

\[
C_2 = \text{cone}(S) = SR_+^n \subseteq \mathbb{R}_+^\infty .
\]

Since \( C_2 \) has \( n \) spanning vectors, \( C_2 \) is a polyhedral cone. The set of trajectories generated by this system equals the cone \( C_2 \). Because the system is a realization,

\[\text{cone}(V) \subseteq C_2.\]
To prove that $C_2$ is $k$-shift invariant, let

$$y = \begin{pmatrix} y(0) \\ y(1) \\ \vdots \end{pmatrix} \in C_2, \quad \text{with } y(t) \in R^k_+, \text{ for } t \in N.$$  

From the definition of $C_2$ it follows that there exists an $x_0 \in R^k_+$ such that $y = Sx_0$, i.e., $y(t) = CA^t x_0$, for all $t \in T$. Then

$$(\xi y)(t) = y(t + 1) = CA^{t+1} x_0 = CA^t (Ax_0).$$

Because $Ax_0 \in R^k_+$, it follows that

$$\sigma^k y = \sigma^k \begin{pmatrix} y(0) \\ y(1) \\ \vdots \end{pmatrix} = \begin{pmatrix} y(1) \\ y(2) \\ \vdots \end{pmatrix} = \begin{pmatrix} \xi y(0) \\ \xi y(1) \\ \vdots \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \vdots \end{pmatrix} Ax_0 \in C_2.$$  

Therefore $C_2 = \text{cone}(S)$ is $k$-shift invariant. So $C_2$ satisfies the Conditions 1, 2, and 3.

$(\Leftarrow)$ a) Let $C_1 \subseteq R^n_+$ be a set satisfying the Conditions 1, 2, and 3. Because $C_1$ is a polyhedral cone there exist an $n \in Z_+ \text{ and an } S \in R^{n \times n}_+$ such that $C_1 = \text{cone}(S)$.

b) Define $C \in R^{k \times n}_+$ by

$$C_{ij} = S_{ij}, \quad i = 1, \ldots, k, \quad j = 1, \ldots, n.$$  

c) Let $e_i \in R^k_+$ be the $i$th unit vector. Define for $i \in Z_n$, $y_i = S e_i \in R^k_+$. Then $y_i \in C_1$. Because $C_1$ is $k$-shift invariant, $\sigma^k y_i \in C_1$. Then there exists an $x_i \in R^k_+$ such that

$$\sigma^k y_i = S x_i, \quad \text{for } i = 1, \ldots, n.$$  

Define $A \in R^{k \times n}_+$ by

$$A = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}.$$  

Then $\sigma^k y_i = S x_i = S A e_i$, for $i = 1, \ldots, n$.

d) Our task is now to show that if $y = S x$ for some $y \in \text{cone}(S) = C_1 \text{ and } x \in R^k_+$, then $\sigma^k y = S A x$. To this end, suppose $y \in \text{cone}(S) = C_1$. Then there exists an $x \in R^k_+$ such that

$$y = S x = S \sum_{i=1}^{n} \alpha_i e_i.$$  

Then

$$\sigma^k y = \sigma^k S x = \sigma^k S \sum_{i=1}^{n} \alpha_i e_i = \sum_{i=1}^{n} \alpha_i \sigma^k S e_i = \sum_{i=1}^{n} \alpha_i \sigma^k y_i = \sum_{i=1}^{n} \alpha_i S A e_i = S A \sum_{i=1}^{n} \alpha_i e_i = S A x.$$
e) Write $S$ as follows:

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ \vdots \end{pmatrix}, \quad \text{with } S_t \in R_+^{k \times n}, \quad \text{for all } t \in T.$$  

It will be proven that $S_t = CA^t, \forall t \in T$. By definition of $C$, $S_0 = C$. Suppose that $S_\tau = CA^\tau, \tau = 0, 1, \ldots, t$. It will be shown that $S_{t+1} = CA^{t+1}$. Let $y \in C_1$. Hence there exists an $x_0 \in R_+^n$ such that $y = Sx_0$. By step d) \[ \sigma h y = S Ax_0, \text{ or equivalently, } (\xi y)(t) = S_t Ax_0. \] Therefore

$$S_{t+1}x_0 = y(t+1) = (\xi y)(t) = S_t Ax_0 = (CA^t)Ax_0 = CA^{t+1}x_0.$$  

This holds for all $y \in C_1$, in particular for $y_i = Se_i$ as defined in c). Hence $S_{t+1} = CA^{t+1}$. It follows that

$$C_1 = \text{cone}(S) = \text{cone}\left( \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \end{pmatrix} \right).$$

f) Condition 2, cone$(V) \subseteq C_1$, implies that for any trajectory $y$ in $V \subseteq \text{cone}(V)$, hence in $C_1$, there exists an $x_0 \in R_+^m$ such that $y = Sx_0$, or equivalently, $y(t) = CA^tx_0$, for all $t \in T$. So $y$ is the output of a positive linear system due to $x_0 \in R_+^m$.

\[ \square \]

Remark 5.3.2 The realization problem for positive linear systems is closely related to the stochastic realization problem for finite-valued processes. The above theorem is inspired by the work of G. Picci [114].

Remark 5.3.3 This result is entirely different from the work of Maeda, Kodama et al. [98, 99, 112]. They have (1) a result on the relation between an ARMA representation and a positive realization and (2) a necessary and sufficient condition for an ordinary minimal realization to be a minimal positive realization. They do not use the notion of $k$-shift invariance. Also the work of Farina et al. [1, 44, 45, 42], doesn’t make use of the $k$-shift invariance.

5.3.2 General case

Now it is assumed that an input is involved. The external behaviour is described by the input $u : T \to U = R_+^m$ and the output $y : T \to Y = R_+^k$, or equivalently, by the relation between $u$ and $y$, the impulse response $W : T \to R_+^{k \times m}$. For this case the following result can be stated.

\[ \text{Theorem 5.3.4 Let } T = N, Y = R_+^k, U = R_+^m. \text{ Consider a positive impulse response function } W : T \to R_+^{k \times m}. \text{ Define } \]

$$H = (W(1)^T \quad W(2)^T \quad W(3)^T \quad \cdots)^T \in R_+^{\infty \times m}$$

\[ \blacksquare \]
and consider $\text{cone}(H)$. There exists a positive linear system
\[ x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0, \]
\[ y(t) = Cx(t) + Du(t), \]
such that the impulse response function of this system equals $W$ if and only if there exists a set $C_1 \subseteq R^\infty_+$ satisfying
\begin{enumerate}
\item $C_1$ is a polyhedral cone;
\item $\text{cone}(H) \subseteq C_1$;
\item $C_1$ is $k$-shift invariant.
\end{enumerate}

Proof. ($\Rightarrow$) Assume that $W$ is the impulse response function of the positive linear system
\[ x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0, \]
\[ y(t) = Cx(t) + Du(t), \]
with $X = R_+^n$, $A \in R_+^{n \times n}$, $B \in R_+^{n \times m}$, $C \in R_+^{k \times n}$, $D \in R_+^{k \times m}$, for $n \in Z_+$. Let
\[ S = \begin{pmatrix}
C \\
CA \\
CA^2 \\
\vdots
\end{pmatrix} \in R_+^{\infty \times n}, \]
\[ C_2 = \text{cone}(S) = SR_+^n \subseteq R_+^\infty. \]
As in the proof of Theorem 5.3.1, $C_2$ is a polyhedral cone and $k$-shift invariant. Finally,
\[ \text{cone}(H) = \text{cone}(SB) \subseteq \text{cone}(S) = C_2. \]
($\Leftarrow$) Step a) to e) are as in the proof of Theorem 5.3.1. Step f) is slightly different.

f) Write $H$ as follows:
\[ H = (H_1 \ H_2 \ \cdots \ H_m) \quad \text{with} \quad H_i \in R_+^\infty, \quad i = 1, \ldots, m. \]
Condition 2, $\text{cone}(H) \subseteq C_1$, implies that for any $H_i$ with $i = 1, 2, \ldots, m$, $H_i \in \text{cone}(H)$, hence in $C_1$, there exists a $b_i \in R_+^n$ such that $H_i = Sb_i$. Define $B = (b_1 \ b_2 \ \cdots \ b_m)$. It follows that
\[ SB = S \begin{pmatrix}
b_1 \\
b_2 \\
b_m
\end{pmatrix} = (Sb_1 \ Sb_2 \ \cdots \ Sb_m) = (H_1 \ H_2 \ \cdots \ H_m) = H, \]
or equivalently, $CA^iB = W(t+1)$ for all $t \in T$. Finally, define $D = W(0)$. \(\square\)

The cone containing all states $x$ that can be reached from the origin by positive inputs $u \in R_+^m$ is denoted by
\[ \mathcal{R}(A, B) = \{ x \in R_+^n \mid x = \sum_{t=0}^{\infty} A^t Bu(t), \quad u : Z_+ \to R_+^m \}. \]
It is called the reachability cone, see for example [112]. Note that in general \( \mathbb{R}(A, B) \neq \mathbb{R}_+^n \), even if \((A, B)\) is a reachable pair in the sense of ordinary linear systems. In this thesis reachable cones have not been studied. For references on this subject, see for example [29, 40, 41, 108, 124].

### 5.4 Characterization of minimality

In this section results on the characterization of minimality will be presented.

Realization theory for linear dynamic systems over \( \mathbb{R}^n \) provides necessary and sufficient conditions for minimality, see for example [133]. For positive linear systems these conditions are sufficient, but not necessary! If a positive linear system is such that \((A, B)\) is a reachable pair and \((A, C)\) is an observable pair, then the system is a minimal positive realization of its impulse response. The converse is not true in general. At the end of Subsection 5.4.1, an example of a system will be presented that is a minimal positive linear system, but that is not minimal as a linear system over \( \mathbb{R}^n \). The problem is to derive necessary and sufficient conditions for a positive linear system to be a minimal positive realization of the impulse response function.

In Subsection 5.4.1 results on the minimality of positive linear systems, using positive rank, are presented. It turns out to provide a weaker sufficient condition for minimality than the reachability/observability condition, but still not necessary. Finally, in Subsection 5.4.2 a sufficient and necessary condition for the minimality of positive linear systems is presented.

#### 5.4.1 Positive rank

What can be said about the positive rank in relation to a positive linear system? For the definition of positive rank and its properties, see Subsection 4.5.1. First attention is restricted to this concept in positive linear algebra of which contribution to the characterization of minimality is expected.

Consider a positive linear system \((A, B, C)\) with \( A \in \mathbb{R}_+^{n \times n} \), \( B \in \mathbb{R}_+^{n \times m} \), \( C \in \mathbb{R}_+^{k \times n} \). For \( p, q \in \mathbb{Z}_+ \), define \( H(p, q) \) to be the Hankel matrix

\[
H(p, q) = \begin{pmatrix}
CB & CAB & \cdots & CA^{q-1}B \\
CAB & CA^2B & \cdots & \\
\vdots & \ddots & \ddots & \\
CA^{p-1}B & \cdots & CA^{p+q-2}B
\end{pmatrix}.
\]

**Proposition 5.4.1** Let the positive linear system \((A, B, C)\) be given as above. Then \( \text{pos-rank}(H(p, q)) \leq n \) for all \( p, q \in \mathbb{Z}_+ \).

**Proof.** The Hankel matrix \( H(p, q) \) can be factorized as follows:

\[
H(p, q) = \begin{pmatrix}
C \\
CA \\
\vdots \\
CA^{p-1}
\end{pmatrix}
\begin{pmatrix}
B & AB \cdots & A^{q-1}B
\end{pmatrix}.
\]
It follows that, for all \( p, q \in \mathbb{Z}_+ \),
\[
\text{pos-rank}(H(p, q)) \leq \min \left\{ \text{pos-rank} \left( \begin{array}{c}
C \\
CA \\
\vdots \\
C A^{p-1}
\end{array} \right), \text{pos-rank}(B \ \ AB \ \ \ldots \ \ A^{p-1}B) \right\} \\
\leq \min\{pk, n, qm, n\} \leq n.
\]

A sufficient condition for minimality will now be presented. This condition is weaker than the reachability/observability condition, as will be shown in Example 5.4.3.

**Proposition 5.4.2** Let the positive linear system \((A, B, C)\) be given as above. If there exist \( p, q \in \mathbb{Z}_+ \) such that \( \text{pos-rank}(H(p, q)) = n \), then \((A, B, C)\) is a minimal positive linear system.

The converse of Proposition 5.4.2 turns out to be not true, as will be seen in Subsection 5.4.2.

**Proof.** Assume there exist \( p, q \in \mathbb{Z}_+ \) such that \( \text{pos-rank}(H(p, q)) = n \). Suppose \((A, B, C)\) is not minimal. Then there exists a triple \((A_1, B_1, C_1)\), \( A_1 \in \mathbb{R}^{n_1 \times n_1}_+ \), \( B_1 \in \mathbb{R}^{n_1 \times m}_+ \), \( C_1 \in \mathbb{R}^{k_1 \times n_1}_+ \), with \( n_1 < n \), that has the same impulse response function as \((A, B, C)\), i.e., \( CA^rB = C_1 A_1^r B_1 \) for all \( r \in \mathbb{N} \). Then
\[
H(p, q) = \begin{pmatrix}
CB & \ldots & CA^{p-1}B \\
\vdots & \ddots & \vdots \\
CA^{p-1}B & \ldots & CA^{p+q-2}B
\end{pmatrix}
= \begin{pmatrix}
C_1 B_1 & \ldots & C_1 A_1^{r-1}B_1 \\
\vdots & \ddots & \vdots \\
C_1 A_1^{p-1}B_1 & \ldots & C_1 A_1^{p+q-2}B_1
\end{pmatrix}
= \begin{pmatrix}
C_1 \\
C_1 A_1 \\
\vdots \\
C_1 A_1^{p-1}
\end{pmatrix}
(B_1 \ A_1 B_1 \ \ldots \ \ A_1^{r-1}B_1),
\]
and the matrices after the last equality have sizes \( kp \times n_1, n_1 \times qm \), respectively. It follows that \( \text{pos-rank}(H(p, q)) \leq n_1 < n \). This is a contradiction. So \((A, B, C)\) is a minimal positive linear system. \( \square \)

**Example 5.4.3** Consider the positive linear system \((A, B, C)\) with
\[
A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad C = (1 \ 1 \ 0 \ 0).
\]
Then
\[
H(4,4) = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}
\]
and this is a prime in the positive matrices, hence \(\text{pos-rank}(H(4,4)) = 4\). With Proposition 5.4.2 it follows that the system is minimal as a positive linear system. On the other hand, note that the system is \textit{not} minimal as a linear system over \(R^4\), since \((A, C)\) is not an observable pair.

\[\square\]

\section*{5.4.2 Positive system rank}

In Proposition 5.4.2, a sufficient condition for the minimality of a positive linear system has been presented. This condition turns out to be \textit{not} necessary. This follows from the following example, in which a minimal positive linear system is presented, that has \(\text{pos-rank}(H(p,q))\) strictly smaller than the minimal dimension of the state space, for all \(p, q \in Z_+\). The example has been inspired by [45].

\textbf{Example 5.4.4} Consider the positive linear system \((A, B, C)\) with

\[
A = \begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}, \quad C = \begin{pmatrix}
3/2 & 1 & 1/3
\end{pmatrix}.
\]

The \(4 \times 4\) block Hankel matrix is

\[
H(4,4) = \begin{pmatrix}
3/2 & 1 & 1 & 3/2 \\
1 & 1 & 3/2 & 3/2 \\
1 & 3/2 & 3/2 & 1 \\
3/2 & 3/2 & 1 & 1
\end{pmatrix}.
\]

Note that, since \(\text{rank}(H(4,4)) = 3\), \((A, B, C)\) is not minimal as a linear system. With the positive matrix factorization

\[
\begin{pmatrix}
3/2 & 1 & 1 & 3/2 \\
1 & 1 & 3/2 & 3/2 \\
1 & 3/2 & 3/2 & 1 \\
3/2 & 3/2 & 1 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
1/2 & 1 & 0 \\
0 & 1 & 1 \\
1/2 & 0 & 1
\end{pmatrix} \begin{pmatrix}
3/2 & 1 & 1 & 3/2 \\
1/4 & 1/2 & 1 & 3/4 \\
3/4 & 1 & 1/2 & 1/4
\end{pmatrix},
\]

it follows that \(\text{pos-rank}(H(p,q)) \leq 3\) for all integers \(p, q \geq 4\), because the \(j\)th row (column) of \(H(p,q)\) for \(4 < j \leq p \leq q\), is equal to one of the first 4 rows (columns) of \(H(p,q)\). So \(\text{pos-rank}(H(p,q)) = 3\) for all integers \(p, q \geq 4\).

Consider the transfer function

\[
C(\lambda I - A)^{-1}B = \frac{3/2\lambda^3 + \lambda^2 + \lambda + 3/2}{\lambda^4 - 1} = \frac{3/2\lambda^2 - 1/2\lambda + 3/2}{(\lambda - 1)(\lambda^2 + 1)}.
\]
which has poles 1, i, and −i. Suppose there exists a realization of order 3. Then there exists a matrix \( A \in \mathbb{R}^{3 \times 3}_+ \) with eigenvalues 1, i, and −i. But with the Karpelevich theory, see [83] or for an English translation [105, Section 7.1], this is impossible. It follows that \((A, B, C)\) is a minimal positive linear system. Actually, also from Equation (2.1) in Chapter 4 of [11] it follows that there does not exist an \( \hat{A} \in \mathbb{R}^{3 \times 3}_+ \) with eigenvalues 1, i, and −i.

So Proposition 5.4.2 only provides a sufficient condition for a positive linear system to be minimal. In this subsection a necessary and sufficient condition for the minimality of positive linear systems is presented. First the shift, defined in Section 5.3, is extended to matrices. Positive matrices with an infinite number of rows are considered, e.g., \( F \in \mathbb{R}^{\infty \times m}_+ \) is a matrix with \( m \) columns and an infinite number of rows. Writing

\[
F = \begin{pmatrix}
F_1 \\
F_2
\end{pmatrix},
\]

with \( F_1 \in \mathbb{R}^{1 \times m}_+ \) the first row of \( F \), the shift operator \( \sigma : \mathbb{R}^{\infty \times m}_+ \to \mathbb{R}^{\infty \times m}_+ \) is defined as \( \sigma F = F_2 \). For \( r \in \mathbb{N} \), \( \sigma^r \) is defined recursively: \( \sigma^r F = \sigma(\sigma^{r-1} F) \), \( \sigma^0 F = F \). Thus, \( \sigma^r F \) is the matrix in \( \mathbb{R}^{\infty \times m}_+ \) obtained by deleting the first \( r \) rows of \( F \).

**Definition 5.4.5** Let \( E \in \mathbb{R}^{\infty \times m}_+ \), for \( m \in \mathbb{Z}_+ \). Let \( r \in \mathbb{N} \). The \textit{r-positive rank} of the matrix \( E \) is defined as the least integer \( n \in \mathbb{Z}_+ \) for which there exists a factorization

\[
E = FG,
\]

with the following properties:

1. \( F \in \mathbb{R}^{\infty \times n}_+, G \in \mathbb{R}^{n \times m}_+ \);
2. there exists a \( Q \in \mathbb{R}^{n \times n}_+ \) satisfying \( \sigma^r F = FQ \).

Let \( r\text{-pos-rank}(E) \) denote this integer.

An \textit{r-positive matrix factorization} of \( E \) is any factorization of \( E \) of the form (5.4.2) with properties 1 and 2, for arbitrary \( n \in \mathbb{R}_+ \). A \textit{minimal r-positive matrix factorization} of \( E \) is any \( r \)-positive matrix factorization of \( E \) of the form (5.4.2) with properties 1 and 2, in which \( n = r\text{-pos-rank}(E) \). □

Note that for \( r = 0 \) the \( r \)-positive rank coincides with the positive rank, as defined in Subsection 4.5.1.

The \( r \)-positive matrix factorization (5.4.2) can be interpreted in geometric terms as

\[
\text{cone}(E) \subseteq \text{cone}(F),
\]

\[
\text{cone}(\sigma^r F) \subseteq \text{cone}(F), \quad \text{for } r \in \mathbb{N}.
\]

Note that the latter relation is formulated in terms of \( F \), not in terms of \( E \).

The following problem can be formulated:
Problem 5.4.6 Let $E \in \mathbb{R}_+^{\infty \times m}$, $r \in \mathbb{N}$. Determine the $r$-positive rank of $E$ and all minimal $r$-positive matrix factorizations of this matrix.

This problem has not been solved yet. The following can be said about the $r$-positive rank.

Proposition 5.4.7 Let $E \in \mathbb{R}_+^{\infty \times m}$, $F \in \mathbb{R}_+^{m \times n}$, $r \in \mathbb{N}$. Then

1. $\text{r-pos-rank}(E) \geq \text{pos-rank}(E) \geq \text{rank}(E)$;

2. $\text{r-pos-rank}(EF) \leq \text{r-pos-rank}(E)$.

Proof. Let $\text{r-pos-rank}(E) = q$. Then there exists a factorization of $E$, $E = GQ$ with

- $G \in \mathbb{R}_+^{\infty \times q}$, $Q \in \mathbb{R}_+^{q \times m}$,
- there exists an $S \in \mathbb{R}_+^{q \times q}$ satisfying $\sigma^r G = GS$.

1. Follows immediately.

2. $EF$ can be written as $EF = G(QF)$ with

- $G \in \mathbb{R}_+^{\infty \times q}$, $QF \in \mathbb{R}_+^{q \times n}$;
- $\sigma^r G = GS$ for $S \in \mathbb{R}_+^{q \times q}$.

Hence $\text{r-pos-rank}(EF) \leq q$. 

Proposition 5.4.9 Consider a positive linear system $(A, B, C)$ with $A \in \mathbb{R}_+^{n \times n}$, $B \in \mathbb{R}_+^{m \times m}$, $C \in \mathbb{R}_+^{k \times n}$. Define $H(p, q)$ to be the Hankel matrix defined in (5.4.1). Define the infinite matrix $H(q) := H(\infty, q)$.

Definition 5.4.8 Consider a positive impulse response function $W : T \to \mathbb{R}_+^{k \times m}$. Note that $k$ denotes the number of outputs. The positive system rank of $H(q)$ is defined to be the $k$-positive rank of $H(q)$. The notation that will be used is $\text{p-rank}(H(q)) = k$-pos-rank$H(q)$. 

Proposition 5.4.9 Consider a positive linear system $(A, B, C)$ with $A \in \mathbb{R}_+^{n \times n}$, $B \in \mathbb{R}_+^{m \times m}$, $C \in \mathbb{R}_+^{k \times n}$. Then for every $q \in \mathbb{Z}_+$, $\text{p-rank}(H(q)) \leq n$.

Proof. $H(q)$ can be factorized as

$$H(q) = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \end{pmatrix} (B \ AB \ \cdots \ A^{q-1}B),$$
with positive factors, and
\[
\sigma^k \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \end{pmatrix} = \begin{pmatrix} CA \\ CA^2 \\ CA^2 \\ CA^2 \\ \vdots \end{pmatrix} A, \tag{5.4.3}
\]
for \( A \in \mathbb{R}_+^{n \times n} \). It follows that \( p\text{-rank}(H(q)) \leq n \). \qed

The main result of this section is the following theorem.

**Theorem 5.4.10** Consider a positive linear system \((A, B, C)\) with \( A \in \mathbb{R}_+^{n \times n} \), \( B \in \mathbb{R}_+^{n \times m} \), \( C \in \mathbb{R}_+^{n \times n} \). This positive linear system \((A, B, C)\) is minimal if and only if there exists a \( q \in \mathbb{Z}_+ \) such that \( p\text{-rank}(H(q)) = n \).

Note that the condition involves an infinite test, namely the determination of the positive system rank of \( H(q) \) over all \( q \in \mathbb{Z}_+ \).

**Proof.** \((\Leftarrow)\) Assume there exists a \( q \in \mathbb{Z}_+ \) such that \( p\text{-rank}(H(q)) = n \). Suppose \((A, B, C)\) is not minimal. Then there exists a triple \((A_1, B_1, C_1)\), with matrices \( A_1 \in \mathbb{R}_+^{n_1 \times n} \), \( B_1 \in \mathbb{R}_+^{n_1 \times m} \), \( C_1 \in \mathbb{R}_+^{n \times n_1} \), for \( n_1 < n \), having the same impulse response function as \((A, B, C)\), i.e., \( CA^rB = C_1A_1^rB_1 \), for all \( r \in \mathbb{N} \). Then
\[
H(q) = \begin{pmatrix} CB & \cdots & CA^{q-1}B \\ \vdots & \ddots & \vdots \\ CA^{q-1}B & \cdots & CA^{2q-2}B \\ \vdots & \ddots & \vdots \\ C_1B_1 & \cdots & C_1A_1^{q-1}B_1 \\ \vdots & \ddots & \vdots \\ C_1A_1^{q-1}B_1 & \cdots & C_1A_1^{2q-2}B_1 \\ \vdots & \ddots & \vdots \\ C_1 & \cdots & C_1A_1^{q-1} \\ \vdots & \ddots & \vdots \\ C_1A_1^{q-1} & \cdots & C_1A_1^{q-1}B_1 \\ \vdots & \ddots & \vdots \end{pmatrix}
\]
gives a positive matrix factorization of \( H(q) \) with the properties

1. \( \begin{pmatrix} C_1 \\ C_1A_1 \\ \vdots \end{pmatrix} \in \mathbb{R}_+^{n \times n_1}, \quad (B_1 \quad A_1B_1 \cdots \quad A_1^{q-1}B_1) \in \mathbb{R}_+^{n_1 \times m} \);

2. \( \sigma^k \begin{pmatrix} C_1 \\ C_1A_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} C_1A_1 \\ C_1A_1^2 \\ \vdots \end{pmatrix} = \begin{pmatrix} C_1 \\ C_1A_1 \\ \vdots \end{pmatrix} A_1, \quad \text{for } A_1 \in \mathbb{R}_+^{n_1 \times n_1} \).
from which it follows that $\text{p-rank}(H(q)) \leq n_1 < n$. This is a contradiction. So $(A, B, C)$ is a minimal positive linear system.

($\Rightarrow$) Assume $(A, B, C)$ is a minimal positive linear system. From Proposition 5.4.9 it follows that $\text{p-rank}(H(p)) \leq n$, for all $p \in Z_+$. Suppose for all $p \in Z_+$, $\text{p-rank}(H(p)) < n$, i.e.,

$$\max_{p \in Z_+}\{\text{p-rank}(H(p))\} = n_1 < n.$$ 

Then there exists a $q \in Z_+$ satisfying $\text{p-rank}(H(q)) = n_1$, so $H(q)$ can be factorized as $H(q) = PQ$ with the properties

1. $P \in R_+^{n \times n_1}, Q \in R_+^{n_1 \times m}$;
2. there exists an $A_1 \in R_+^{n_1 \times n_1}$ satisfying $\sigma^k P = PA_1$.

Define $C_1 \in R_+^{k \times n_1}$ by

$$(C_1)_{ij} = P_{ij}, \quad i = 1, \ldots, k, \quad j = 1, \ldots, n_1,$$

and $B_1 \in R_+^{n_1 \times n}$ by

$$(B_1)_{ij} = Q_{ij}, \quad i = 1, \ldots, n_1, \quad j = 1, \ldots, m.$$ 

Then $H(1) = PB_1$ with

$$P = \begin{pmatrix} C_1 & \sigma^k P \\ \sigma^k P \\ \vdots \end{pmatrix} = \begin{pmatrix} C_1 \\ \sigma^k P \\ \vdots \end{pmatrix} = \begin{pmatrix} C_1 \\ \sigma^k PA_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} C_1 \\ \sigma^k PA_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} C_1 A_1^{s+1} B_1 \\ \sigma^k PA_1 A_1^{s+1} B_1 \\ \vdots \end{pmatrix},$$

from which it follows that $CA^{s+1}B = C_1 A_1^{s+1} B_1$. It follows that for all $r \in Z_+$, $CA^rB = C_1 A_1^r B_1$, so $(A_1, B_1, C_1)$ is a positive linear system with state space dimension $n_1 < n$. This is a contradiction. Hence there exists a $p \in Z_+$ such that $\text{p-rank}(H(p)) = n$. \hfill $\Box$

The geometric version of this theorem is as follows (compare with the existence theorem in Section 5.3, Theorem 5.3.4). The positive linear system $(A, B, C)$ is minimal if and only if there exists a set $J \subseteq R_+^{\infty}$ satisfying

1. $J$ is a polyhedral cone;
2. cone $\begin{pmatrix} CB \\ CAB \\ CA^2 B \\ \vdots \end{pmatrix} \subseteq J$;
3. $J$ is $k$-shift invariant;

4. the size of the frame of $J$ is as small as possible within the Conditions 1, 2, and 3.

For the definition of size and frame, see Definition 4.4.7. The minimal dimension of the matrix $A$ equals the minimal size of $C_1$. The $k$-shift invariance appears in the second property of the $k$-positive rank in Definition 5.4.5. As can be seen from Example 5.4.4, this condition is necessary for minimality. Note that all conditions are phrased in terms of $J$ rather than in terms of $H(1)$. In general cone$(H(1)) \neq J$. Although cone$(H(q)) \subseteq J$, it is also possible that cone$(H(q)) \neq J$ for all $q \in \mathbb{Z}_+$, which can be seen in Example 5.6.10. This is in contrast to the realization of ordinary linear systems, where there exists a $q \in \mathbb{Z}_+$ such that span$(H(q)) = \mathbb{R}^n$. This makes the positive realization problem much more difficult than that for ordinary linear systems.

The problem is now reduced to the problem of the determination of the positive system rank of a positive matrix. This latter problem has not been solved yet. Compare this problem with the problem of the determination of the positive rank, which theoretically has been solved by Procedure 4.5.19. A similar procedure must be formulated for the positive system rank. As for ordinary linear systems, tests have to be performed on an infinite matrix. The characterization of a minimal positive linear system covers both the external characterization in terms of the impulse response function, or equivalently, the Hankel matrix, and the internal characterization in terms of $A$, $B$, and $C$. The external characterization will always be a test for an infinitely long Hankel matrix.

The internal characterization may possibly be simplified because in this case the dimension of the system is known, so the test may be applied to a finite matrix. A search must be performed over all polyhedral cones satisfying Conditions 2, 3, and 4 to determine the minimal dimension of the state space. These issues require further research.

Theorem 5.4.10 translates the condition of minimality to the geometric formulation as stated above. This formulation is useful, since it admits a direct analysis.

### 5.5 Equivalence and classification

In this section relations between minimal positive realizations of a positive impulse response function are presented, together with a classification of minimal positive realizations for a special class of Hankel matrices.

#### 5.5.1 A relation between minimal positive realizations

For a relation between minimal positive realizations of a positive impulse response function, the following can be stated.

**Proposition 5.5.1** Let the positive linear system of the form (5.2.1), with matrices $(A, B, C, D)$, be a minimal positive realization of a positive impulse response
function. For any monomial matrix \( M \in \mathbb{R}_+^{n \times n} \), the positive linear system of the form (5.2.1), with matrices \((MAM^{-1}, MB, CM^{-1}, D)\), is also a minimal positive realization of the same impulse response function.

**Proof.** This immediately follows from the fact that for a monomial matrix \( M \in \mathbb{R}_+^{n \times n} \), also \( M^{-1} \in \mathbb{R}_+^{n \times n} \).

The converse of this proposition does not hold. That is, let the positive linear system of the form (5.2.1), with matrices \((A, B, C, D)\), be a minimal positive realization of a positive impulse response function. The set \( S \) of matrices, given by

\[
S = \{ T \in \mathbb{R}^{n \times n} \mid T \text{ nonsingular }, \quad TAT^{-1} \in \mathbb{R}_+^{n \times n}, \\
TB \in \mathbb{R}_+^{n \times m}, \quad CT^{-1} \in \mathbb{R}_+^{k \times n} \}
\]

contains the set of monomial matrices \( M \in \mathbb{R}_+^{n \times n} \), but it is possible that \( S \) is not equal to the set of monomial matrices. Indeed, consider a positive linear system of the form (5.2.1), with matrices

\[
A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 0.
\]

This system is minimal. Take

\[
T = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.
\]

Then \((TAT^{-1}, TB, CT^{-1}, D)\), with

\[
TAT^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}, \quad TB = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad CT^{-1} = \begin{pmatrix} 1 & 2 \end{pmatrix},
\]

is also a minimal positive realization with the same impulse response function as \((A, B, C)\), but \( T \) is not a monomial matrix.

So, in general, two minimal positive realizations of the same impulse response function, say \((A_1, B_1, C_1, D_1)\) and \((A_2, B_2, C_2, D_2)\), may be related by a nonsingular matrix \( T \), that can have negative elements. Also other relations are possible, even with singular matrices \( T \), as will be shown in the next subsection.

### 5.5.2 Classification

In this subsection a classification of minimal positive linear systems for a special class will be presented. Depending on a factorization condition of the Hankel matrix one or two classes of minimal positive realizations exist.

The special class that will be considered in this subsection is the class in which the Hankel matrix \( H(s, t) \), see (5.4.1), is not strictly factorizable and for all \( p \geq s, q \geq t \), pos-rank\((H(p, q)) = n \). The reason for this restriction is that for not strictly factorizable matrices the complete classification of minimal positive matrix factorizations is known. Indeed, for \( k \geq m \), any minimal positive matrix factorization is given by one of the two following forms:
1. \( P = MB \), with \( M \in R_+^{k \times m} \), \( B \in R_+^{m \times m} \), (5.5.1) 
in which \( M \) is part of a monomial;

2. \( P = A \), with \( A \in R_+^{k \times m} \), \( N \in R_+^{m \times m} \), (5.5.2) 
in which \( N \) is a monomial.

See Theorem 4.5.8. Both \( A \) and \( B \) are uniquely determined by \( A = PN^{-1} \) and \( B = (S_RM)^{-1}S_MP \), with \( S_R \) as defined in (4.5.2). For other positive matrices the classification is not yet known. Of course, they have positive factorizations of the form (5.5.1) or (5.5.2), but this does not imply that Proposition 4.5.10 holds, see the example below Corollary 4.5.12.

For a positive impulse response function \( W : T \rightarrow R_+^{k \times m} \), consider the following block-Hankel matrices of \( W \), for \( r, s, t \in Z_+ \),

\[
H_s(t) = \begin{pmatrix}
W(r) & W(r + 1) & \cdots & W(r + t - 1) \\
W(r + 1) & W(r + 2) & \cdots & W(r + t) \\
& \ddots & \ddots & \ddots \\
W(r + s - 1) & W(r + s) & \cdots & W(r + s + t - 2)
\end{pmatrix}.
\]

Assume \( H_1(s, t) \) is not strictly factorizable and assume for all \( p \geq s, q \geq t \), \( \text{pos-rank}(H_1(p, q)) = \text{pos-rank}(H_1(s, t)) \). Without loss of generality assume \( sk \geq tm \), since for \( sk < tm \), the dual positive impulse response function \( W^T : T \rightarrow R_+^{m \times k} \) can be considered. Since \( H_1(s, t) \) is not strictly factorizable, it follows from Proposition 4.5.8 that \( \text{pos-rank}(H_1(s, t)) = tm = n \). If \( H_1(s, t) \) contains \( l = sk - tm \) zero rows, say row \( \tilde{r}_1, \ldots, \tilde{r}_l \), then it can be shown that also row \( \tilde{r}_1, \ldots, \tilde{r}_l \) of \( H_k(s, q) \) are zero for all \( q \in Z_+ \). For \( q \leq t \) and \( k = 1 \) this is obvious. For \( q > t \) and \( k = 1 \), \( \text{pos-rank}(H_1(s, q)) = n \), so there exist \( X \in R_+^{sk \times tm} \), \( Y_1 \in R_+^{pm \times tm} \), and \( Y_2 \in R_+^{pm \times (s-t)m} \), such that

\[
H_1(s, q) = \begin{pmatrix} H_1(s, t) & H_{t+1}(s, q - t) \end{pmatrix} = X \begin{pmatrix} Y_1 & Y_2 \end{pmatrix}.
\]

With Proposition 4.5.9.b, it follows that row \( i \) in \( H_1(s, t) = XY_1 \) is zero if and only if row \( i \) in \( X \) is zero. But then also row \( i \) in \( H_1(s, q) \) is zero. For \( k > 1 \), it follows from the Hankel structure that row \( \tilde{r}_1, \ldots, \tilde{r}_l \) of \( H_1(s, q) \) are zero for all \( q \in Z_+ \), also row \( \tilde{r}_1, \ldots, \tilde{r}_l \) of \( H_k(s, q) \) are zero for all \( q \in Z_+ \) and \( k > 1 \).

If \( H_1(s, t) \in R_+^{sk \times tm} \) contains \( sk - tm \) zero rows, define its selector matrix \( S_R \in R_+^{tm \times sk} \) as in (4.5.2).

**Theorem 5.5.2** Consider a positive impulse response function \( W : T \rightarrow R_+^{k \times m} \), with Hankel matrix \( H_1(p, q) \), such that \( H_1(s, t) \) is not strictly factorizable for some \( s, t \in Z_+ \) and \( \text{pos-rank}(H_1(p, q)) = n \) for all \( p \geq s, q \geq t \). Assume \( sk \geq tm \).

1. If \( sk > tm \) and \( H_1(s, t) \) does not contain exactly \( sk - tm \) zero rows, then
Chapter 5. Realization of Positive Linear Systems

there exists a $V_i \in R_+^{n \times m}$ such that $(A_{V_i}, B_1, C_1)$, with

$$A_{V_i} = \begin{pmatrix} 0 & \cdots & 0 \\ I_m & 0 & \cdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} V_i, \quad B_1 = \begin{pmatrix} I_m \\ \vdots \\ 0 \end{pmatrix}, \quad (5.5.3)$$

$$C_1 = H_1(1,t),$$

is a minimal positive realization of $W$. Moreover, $(A_{V_i}, B_1, C_1)$ is a minimal positive realization of $W$ for every $V_i \in R_+^{n \times m}$ satisfying the matrix equation $H_1(s + 1, t)V_i = H_{t+1}(s + 1, 1)$.

2. If $sk = tm$ or $H_1(s, t)$ contains exactly $sk - tm$ zero rows, then at least one of the following hold.

(a) There exists a $V_i \in R_+^{n \times m}$ such that $(A_{V_i}, B_1, C_1)$ is a minimal positive realization of $W$. Moreover, $(A_{V_i}, B_1, C_1)$ is a minimal positive realization of $W$ for every $V_i \in R_+^{n \times m}$ satisfying the matrix equation $H_1(s + 1, t)V_i = H_{t+1}(s + 1, 1)$.

(b) There exists a $V_2 \in R_+^{k \times n}$ such that $(\tilde{A}_{V_2}, \tilde{B}_2, \tilde{C}_2)$, with

$$\tilde{A}_{V_2} = \begin{pmatrix} O_{nx1} & S_R \\ \end{pmatrix} \begin{pmatrix} S_R^T \\ \end{pmatrix}, \quad \tilde{B}_2 = S_R H_1(s,1), \quad (5.5.4)$$

$$\tilde{C}_2 = \begin{pmatrix} I_k & O_{k \times (sk-k)} \end{pmatrix} S_R^T,$$

is a minimal positive realization of $W$. Moreover, $(\tilde{A}_{V_2}, \tilde{B}_2, \tilde{C}_2)$ is a minimal positive realization of $W$ for every $V_2 \in R_+^{k \times n}$ satisfying the matrix equation $V_2 S_R H_1(s,t+1) = H_{s+1}(1,t + 1)$.

3. If both (5.5.3) and (5.5.4) are minimal positive realizations of $W$, then their relation is given by

$$S_R H_1(s,t) A_{V_i} = \tilde{A}_{V_2} S_R H_1(s,t)$$

$$S_R H_1(s,t) B_1 = \tilde{B}_2$$

$$C_1 = \tilde{C}_2 S_R H_1(s,t). \quad (5.5.5)$$

Remark 5.5.3 Note that the matrix $V_i$ in $A_{V_i}$ need not be unique. This can happen if $\text{rank}(H_1(s,t)) < \text{pos-rank}(H_1(s,t))$, since then the kernel of $H_1(s,t)$, $\mathcal{N}(H_1(s,t)) \neq \{0\}$. Also the matrix $V_2$ in $A_{V_2}$ need not be unique, in case $\text{rank}(S_R H_1(s,t)) < \text{pos-rank}(S_R H_1(s,t))$.

Remark 5.5.4 The assumptions that $H_1(s,t)$ is not strictly factorizable and that $\text{pos-rank}(p,q) = n$ for all $p \geq s$, $q \geq t$ are restrictive. With Proposition 5.4.2 it follows that this is a sufficient condition for minimality. This restriction is imposed to use results for positive matrices on positive dependence. The extension to factorizability with shift invariance has not yet been made.
Proof. Let $P = H_1(s, t)$ and

$$H_1(s + 1, t + 1) = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}.$$  

Since $H_1(s, t)$ is not strictly factorizable, it follows from Proposition 4.5.8 that $n = \text{pos-rank}(H_1(s, t)) = \min\{sk, tm\} = tm$.

1. Suppose $sk > tm$ and $H_1(s, t)$ does not contain $sk - tm$ zero rows. Because pos-rank($H_1(s + 1, t + 1)) = n$, with Proposition 4.5.10 it follows that there exists a matrix $V_1 \in \mathbb{R}^{n \times m}$ such that

$$H_{i+1}(s + 1, 1) = \begin{pmatrix} Q \\ S \end{pmatrix} = \begin{pmatrix} P \\ R \end{pmatrix} V_1 = H_1(s + 1, t)V_1,$$  

or equivalently,

$$H_1(s + 1, t + 1) = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} P \\ R \end{pmatrix} (I_n \ V_1) = H_1(s + 1, t)(I_n \ V_1).$$

It follows that a positive realization of $W$ is given by

$$A_{V_i} = \begin{pmatrix} I_n & V_1 \end{pmatrix} \begin{pmatrix} O_{m \times n} \\ I_n \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & I_m \\ 0 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_m \end{pmatrix},$$

$$B_1 = \begin{pmatrix} I_m \\ O_{(n-m) \times m} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} I_k & O_{k \times (sk-k)} \end{pmatrix} H_1(s, t) = H_1(1, t).$$

Indeed,

$$C_1 A_{V_i} = H_1(1, t) \begin{pmatrix} I_n & V_1 \end{pmatrix} \begin{pmatrix} O_{m \times n} \\ I_n \end{pmatrix} = H_1(1, t + 1) \begin{pmatrix} O_{m \times n} \\ I_n \end{pmatrix} = H_2(1, t),$$

and if $C_1 A_{V_i}^i = H_{i+1}(1, t)$ for $i \in \mathbb{N}$, then

$$C_1 A_{V_i}^{i+1} = H_{i+1}(1, t) \begin{pmatrix} I_n & V_1 \end{pmatrix} \begin{pmatrix} O_{m \times n} \\ I_n \end{pmatrix} = H_{i+1}(1, t + 1) \begin{pmatrix} O_{m \times n} \\ I_n \end{pmatrix} = H_{i+2}(1, t).$$

By induction it follows that $C_1 A_{V_i}^i = H_{i+1}(1, t)$ for all $i \in \mathbb{N}$. Therefore

$$\begin{pmatrix} C_1 \\ C_1 A_{V_i} \\ \vdots \\ C_1 A_{V_i}^s \end{pmatrix} = \begin{pmatrix} H_1(1, t) \\ H_2(1, t) \\ \vdots \\ H_{s+1}(1, t) \end{pmatrix} = H_1(s + 1, t).$$
On the other hand,

\[
A_V B_1 = \begin{pmatrix} I_n & V_1 \end{pmatrix} \begin{pmatrix} O_{m \times n} & I_m \\ I_n & O_{(n-m) \times m} \end{pmatrix} = \begin{pmatrix} I_n \\ V_1 \end{pmatrix}.
\]

and repeating this calculation gives

\[
A_V^i B_1 = \begin{pmatrix} O_{m \times m} \\ I_m \end{pmatrix} I_m \begin{pmatrix} O_{(n-(i+1)m) \times m} \end{pmatrix},
\]

for \(i = 0, 1, 2, \ldots, t-1\), and

\[
A_V^t B_1 = \begin{pmatrix} I_n & V_1 \end{pmatrix} \begin{pmatrix} O_{m \times m} \\ I_m \end{pmatrix} = \begin{pmatrix} I_n & V_1 \end{pmatrix}.
\]

Therefore

\[
\begin{pmatrix} B_1 & A_V B_1 & \cdots & A_V^{t-1} B_1 & A_V^t B_1 \end{pmatrix} =
\begin{pmatrix} I_m & O_{m \times n} & O_{(t-1)m \times n} \\ O_{(n-m) \times m} & I_m & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ O_{(n-2m) \times m} & O_{(n-2m) \times m} & \cdots & I_m \end{pmatrix} = \begin{pmatrix} I_n & V_1 \end{pmatrix}.
\]

Since \(\text{pos-rank}(H_1(p, q)) = n\) for all \(p \geq s, q \geq t\), it follows from Proposition 5.4.2 that \((A_V, B_1, C_1)\) is a minimal positive realization of \(W\). With the same reasoning it follows that for every solution \(V_1 \in \mathbb{R}_{+}^{m \times m}\) of (5.5.6), \((A_V, B_1, C_1)\) is a minimal positive realization of \(W\).

2. If \(sk = tm\), then the non strict factorizability of \(H_1(s, t)\) implies that \(H_1(s, t)\) is either a prime or a monomial, so \(H_1(s, t)\) contains \(sk - tm = 0\) zero rows. Therefore it is sufficient to suppose that \(H_1(s, t)\) contains exactly \(sk - tm \geq 0\) zero rows. With Proposition 4.5.10, at least one of the following can happen.

(a) There exists a matrix \(V_1 \in \mathbb{R}_{+}^{m \times m}\) satisfying (5.5.6), in which case the triple \((A_V, B_1, C_1)\) is a minimal positive realization of \(W\), and for every \(V_1 \in \mathbb{R}_{+}^{m \times m}\) satisfying (5.5.6), \((A_V, B_1, C_1)\) is a minimal positive realization of \(W\).

(b) There exists a matrix \(V_1 \in \mathbb{R}_{+}^{k \times n}\) such that

\[
H_{s+1}(1, t+1) = \begin{pmatrix} R & S \end{pmatrix} = V_2 \begin{pmatrix} S_R \quad P & S_R Q \end{pmatrix}
= V_2 S_R H_1(s, t+1),
\]

or equivalently,

\[
H_1(s+1, t+1) = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} S_R & T \\ V_2 \end{pmatrix} \begin{pmatrix} S_R P & S_R Q \end{pmatrix}
= \begin{pmatrix} S_R & T \\ V_2 \end{pmatrix} S_R H_1(s, t+1).
\]
It follows that a positive realization of $W$ is given by

$$\tilde{A}_{V_2} = (O_{n \times k} \ S_R) \left( S_R^T \frac{V_2}{1} \right).$$

$$\tilde{B}_2 = S_R H_1(s, t) \left( \begin{bmatrix} I_m \\ O_{(n-m) \times m} \end{bmatrix} \right) = S_R H_1(s, 1),$$

$$C_2 = (I_k \ O_{k \times (sk-k)}) S_R^T = S_R^T (1 : k),$$
in which $S_R^T (i : j)$ denotes rows $i, i + 1, \ldots, j$ of $S_R^T$. Indeed,

$$\tilde{A}_{V_2} \tilde{B}_2 = (O_{n \times k} \ S_R) \left( S_R^T \frac{V_2}{1} \right) S_R H_1(s, 1) = (O_{n \times k} \ S_R) H_1(s + 1, 1) = S_R H_2(s, 1),$$

and continuing this it follows that $\tilde{A}_{V_2} \tilde{B}_2 = S_R H_{i+1}(s, 1)$. Therefore

$$\begin{pmatrix} \tilde{B}_2 & \tilde{A}_{V_2} \tilde{B}_2 & \cdots & \tilde{A}_{V_2}^{i-1} \tilde{B}_2 \end{pmatrix} = (S_R H_1(s, 1) \ S_R H_2(s, 1) \cdots S_R H_{i+1}(s, 1)) = S_R H_1(s, t + 1).$$

On the other hand,

$$C_2 \tilde{A}_{V_2} = S_R^T (1 : k) (O_{n \times k} \ S_R) \left( S_R^T \frac{V_2}{1} \right) = (O_{k \times k} \ S_R) S_R^T (1 : k) S_R \left( S_R^T \frac{V_2}{1} \right) = S_R^T (k + 1 : 2k).$$

By induction this implies $C_2 \tilde{A}_{V_2} = S_R^T (ik + 1, (i+1)k)$, for $i = 0, 1, \ldots, s - 1$, and using this

$$C_2 \tilde{A}_{V_2} = S_R^T ((s - 1)k + 1, sk) (O_{n \times k} \ S_R) \left( S_R^T \frac{V_2}{1} \right) = (O_{k \times k} \ S_R) S_R^T ((s - 1)k + 1, sk) S_R \left( S_R^T \frac{V_2}{1} \right) = V_2.$$

Therefore

$$\begin{pmatrix} C_2 \tilde{A}_{V_2} \\ C_2 \tilde{A}_{V_2}^{i=1} \\ \vdots \\ C_2 \tilde{A}_{V_2}^{i=s-1} \\ C_2 \tilde{A}_{V_2}^{i=s} \end{pmatrix} = \begin{pmatrix} S_R^T (1 : k) \\ S_R^T (k + 1 : 2k) \\ \vdots \\ S_R^T ((s - 1)k + 1, sk) \end{pmatrix} \left( S_R^T \frac{V_2}{1} \right).$$

Again, pos-rank($H_1(p, q)$) = $n$ for all $p \geq s$, $q \geq t$, which implies that $(\tilde{A}_{V_2}, \tilde{B}_2, C_2)$ is a minimal positive realization of $W$. With the same reasoning it follows that for every solution $V_2 \in \mathbb{R}^{k \times n}$ of (5.5.7), $(\tilde{A}_{V_2}, \tilde{B}_2, C_2)$ is a minimal positive realization of $W$. 
3. Suppose both \((A_{1}, B_{1}, C_{1})\) and \((\bar{A}_{2}, B_{2}, C_{2})\) are minimal positive realizations of \(W\). Then

\[
S_{H}H_{1}(s,t)B_{1} = S_{H}H_{1}(s,t)\left(\begin{array}{c}
I_{m} \\
0
\end{array}\right) = S_{H}H_{1}(s,1) = \bar{B}_{2},
\]

\[
\bar{C}_{2}S_{H}H_{1}(s,t) = (I_{k} \ 0) S_{H}^{T}S_{H}H_{1}(s,t) = (I_{k} \ 0) H_{1}(s,t)
= H_{1}(1,t) = C_{1},
\]

\[
S_{H}H_{1}(s,t)A_{V_{1}} = S_{H}H_{1}(s,t)\left(\begin{array}{c}
I_{n} \\
V_{1}
\end{array}\right) = S_{H}H_{1}(s,t+1)\left(\begin{array}{c}
0 \\
I_{n}
\end{array}\right)
= S_{H}H_{2}(s,t),
\]

\[
\bar{A}_{V_{2}}S_{H}H_{1}(s,t) = (0 \ S_{R})\left(\begin{array}{c}
S_{H}^{T} \\
V_{2}
\end{array}\right)S_{H}H_{1}(s,t)
= (0 \ S_{R}) H_{1}(s+1,t) = S_{H}H_{2}(s,t).
\]

So \(S_{H}H_{1}(s,t)A_{V_{1}} = \bar{A}_{V_{2}}S_{H}H_{1}(s,t)\). \(\square\)

Theorem 5.5.2 presents examples of minimal positive realizations of the given positive impulse response function \(W\). In the following theorem all minimal positive realizations of \(W\) will be presented.

**Theorem 5.5.5** Consider a positive impulse response function \(W : T \to R_{+}^{k \times m}\), with Hankel matrix \(H_{1}(p,q)\), such that \(H_{1}(s,t)\) is not strictly factorizable for some \(s,t \in Z_{+}\) and \(\text{pos-rank}(H_{1}(p,q)) = n\) for all \(p \geq s, q \geq t\). Assume \(sk \geq tm\). \((\bar{A}, \bar{B}, \bar{C})\) is a minimal positive realization of \(W\) if and only if it has one of the following forms:

1. \((MA_{\phi}, M^{-1}, MB_{1}, C_{1}M^{-1})\), with \(M \in R_{+}^{n \times n}\) a monomial and the triple \((A_{\phi}, B_{1}, C_{1})\) given in (5.5.3) for \(\Psi_{1} \in R_{+}^{k \times m}\) satisfying the matrix equation \(H_{1}(s+1,t)\Psi_{1} = H_{t+1}(s+1,1)\).

2. \((N^{-1}, A_{\psi_{2}}, N^{-1}B_{2}, C_{2}N)\), with \(N \in R_{+}^{n \times n}\) a monomial and the triple \((A_{\psi_{2}}, B_{2}, C_{2})\) given in (5.5.4) for \(\Psi_{2} \in R_{+}^{k \times n}\) satisfying the matrix equation \(\Psi_{2}S_{H}H_{1}(s,t+1) = H_{s+1}(1,t+1)\).

**Proof.** As in Theorem 5.5.2, pos-rank\((H_{1}(s,t)) = tm = n\), since \(H_{1}(s,t)\) is not strictly factorizable.

(\(\Leftarrow\)) This follows from Theorem 5.5.2 and Proposition 5.5.1.

(\(\Rightarrow\)) Let \((\bar{A}, \bar{B}, \bar{C})\) be a minimal positive realization of \(W\). Define

\[
E = \left(\begin{array}{c}
\bar{C} \\
\bar{C}A \\
\vdots \\
\bar{C}A^{n-1}
\end{array}\right) \in R_{+}^{n \times n}, \quad F = \left(\begin{array}{cccc}
\bar{B} & \bar{A}B & \cdots & \bar{A}^{t-1}B
\end{array}\right) \in R_{+}^{n \times m}.
\]

Since \(H_{1}(s,t) = EF\) is not strictly factorizable, either \(E\) is part of a monomial or \(F\) is a monomial. Suppose \(F =: M\) is a monomial, then \(M^{-1} \in R_{+}^{n \times n}\).
Define $\Psi_1 = M^{-1} \tilde{A}^t \tilde{B}$. Then $\Psi_1 \in \mathbb{R}^{n \times m}$ and

\[
H_1(s+1, t)\Psi_1 = \left( \begin{array}{c}
\tilde{C} \\
\frac{\tilde{C}}{\tilde{A}} \\
\vdots \\
\frac{\tilde{C}}{\tilde{A}^s}
\end{array} \right) \left( \begin{array}{cccc}
\tilde{B} & \tilde{AB} & \cdots & \tilde{A}^{t-1} \tilde{B}
\end{array} \right) M^{-1} \tilde{A}^t \tilde{B}
\]

\[
= \left( \begin{array}{c}
\tilde{C} \\
\frac{\tilde{C}}{\tilde{A}} \\
\vdots \\
\frac{\tilde{C}}{\tilde{A}^s}
\end{array} \right) M M^{-1} \tilde{A}^t \tilde{B} = \left( \begin{array}{c}
\tilde{C} \\
\frac{\tilde{C}}{\tilde{A}} \\
\vdots \\
\frac{\tilde{C}}{\tilde{A}^s}
\end{array} \right) \tilde{A}^t \tilde{B}
\]

\[
= H_{t+1}(s+1, 1).
\]

From Theorem 5.5.2, item 1, it follows that $(A_{\Psi_1}, B_1, C_1)$ is a minimal positive realization of $W$ for $\Psi_1 = M^{-1} \tilde{A}^t \tilde{B}$. Note that

\[
\left( \begin{array}{c}
C_1 \\
C_1 A_{\Psi_1} \\
\vdots \\
C_1 A_{\Psi_1}^{t-1}
\end{array} \right) = H_1(s, t)
\]

and $(B_1, A_{\Psi_1} B_1, \cdots, A_{\Psi_1}^{t-1} B_1, A_{\Psi_1}^t B_1) = (I_n, \Psi_1)$, so

\[
F = M = M \left( \begin{array}{cccc}
B_1 & A_{\Psi_1} B_1 & \cdots & A_{\Psi_1}^{t-1} B_1
\end{array} \right), \quad (5.5.8)
\]

\[
E = H_1(s, t) F^{-1} = \left( \begin{array}{c}
C_1 \\
C_1 A_{\Psi_1} \\
\vdots \\
C_1 A_{\Psi_1}^{t-1}
\end{array} \right) M^{-1}, \quad (5.5.9)
\]

and

\[
M^{-1} \tilde{A}^t \tilde{B} = \Psi_1 = A_{\Psi_1}^t B_1. \quad (5.5.10)
\]

From (5.5.9) it follows that $\tilde{C} = C_1 M^{-1}$ and from (5.5.8) that $\tilde{B} = M B_1$. From

\[
\tilde{A} M = \tilde{A} \left( \begin{array}{cccc}
\tilde{B} & \tilde{AB} & \cdots & \tilde{A}^{t-1} \tilde{B}
\end{array} \right) = \left( \begin{array}{cccc}
\tilde{A} B & \tilde{A}^2 B & \cdots & \tilde{A}^t B
\end{array} \right)
\]

\[
= M \left( \begin{array}{cccc}
A_{\Psi_1} B_1 & A_{\Psi_1}^2 B_1 & \cdots & A_{\Psi_1}^t B_1
\end{array} \right)
\]

\[
= MA_{\Psi_1} \left( \begin{array}{cccc}
B_1 & A_{\Psi_1} B_1 & \cdots & A_{\Psi_1}^{t-1} B_1
\end{array} \right) = MA_{\Psi_1},
\]

in which the third equality follows from (5.5.8) and (5.5.10), it follows that $\tilde{A} = MA_{\Psi_1} M^{-1}$.

On the other hand, suppose $E$ is part of a monomial. Let $S_R$ be the selector matrix that selects the nonzero rows of $E$, defined in (4.5.2), so $N := S_R E$ is a monomial, and $N^{-1} \in \mathbb{R}^{m \times n}$. From Proposition 4.5.9.b it follows that row $i$ in
$H_1(s,t)$ is zero if and only if row $i$ in $E$ is zero, so $S_R$ selects also the nonzero rows of $H_1(s,q)$ for all $q \in Z_+$. Define $\Psi_2 = \bar{C} \bar{A}^t N^{-1}$. Then $\Psi_2 \in R^{k \times n}$ and

$$
\Psi_2 S_R H_1(s,t+1) = \bar{C} \bar{A}^t N^{-1} S_R \begin{pmatrix} 
\bar{C} \\
\bar{C} \bar{A} \\
\vdots \\
\bar{C} \bar{A}^{t-1}
\end{pmatrix} \begin{pmatrix} 
\bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{t-1}\bar{B}
\end{pmatrix}
$$

$$
= \bar{C} \bar{A}^t N^{-1} S_R E \begin{pmatrix} 
\bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{t-1}\bar{B}
\end{pmatrix}
$$

$$
= \bar{C} \bar{A}^t N^{-1} N \begin{pmatrix} 
\bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{t-1}\bar{B}
\end{pmatrix}
$$

$$
= \bar{C} \bar{A}^t \begin{pmatrix} 
\bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{t-1}\bar{B}
\end{pmatrix} = H_{s+1}(1,t+1).
$$

From Theorem 5.5.2, item 2, it follows that $(\bar{A}_{q_2}, \bar{B}_2, \bar{C}_2)$ is a minimal positive realization of $W$ for $\Psi_2 = \bar{C} \bar{A}^t N^{-1}$. Note that

$$
\begin{pmatrix}
\bar{C}_2 \\
\bar{C}_2 \bar{A}_{q_2} \\
\vdots \\
\bar{C}_2 \bar{A}^{q_2-1} \\
\bar{C}_2 \bar{A}^{q_2}
\end{pmatrix} = \begin{pmatrix} S_R^T \Psi_2 \end{pmatrix}, \quad \begin{pmatrix} \bar{B}_2 & \bar{A}_{q_2} \bar{B}_2 & \cdots & \bar{A}^{q_2-1} \bar{B}_2 
\end{pmatrix} = S_R H_1(s,t),
$$

so

$$
E = S_R^T S_R E = \begin{pmatrix} \bar{C}_2 \\
\bar{C}_2 \bar{A}_{q_2} \\
\vdots \\
\bar{C}_2 \bar{A}^{q_2-1} \\
\bar{C}_2 \bar{A}^{q_2}
\end{pmatrix} N, \quad (5.5.11)
$$

$$
F = (S_R E)^{-1} S_R H_1(s,t)
$$

$$
= N^{-1} \begin{pmatrix} \bar{B}_2 & \bar{A}_{q_2} \bar{B}_2 & \cdots & \bar{A}^{q_2-1} \bar{B}_2 
\end{pmatrix}, \quad (5.5.12)
$$

and

$$
\bar{C} \bar{A}^t N^{-1} = \Psi_2 = \bar{C}_2 \bar{A}_q.
$$

From $(5.5.11)$ it follows that $\bar{C} = \bar{C}_2 N$ and from $(5.5.12)$ that $\bar{B} = N^{-1} \bar{B}_2$.

From

$$
N \bar{A} = S_R \begin{pmatrix}
\bar{C} \\
\bar{C} \bar{A} \\
\vdots \\
\bar{C} \bar{A}^{t-1}
\end{pmatrix} \bar{A} = S_R \begin{pmatrix} \bar{C} \bar{A} \\
\bar{C} \bar{A}^2 \\
\vdots \\
\bar{C} \bar{A}^{t-1}
\end{pmatrix} = S_R \begin{pmatrix} \bar{C}_2 \bar{A}_{q_2} \\
\bar{C}_2 \bar{A}_{q_2}^2 \\
\vdots \\
\bar{C}_2 \bar{A}_{q_2}^{t-1}
\end{pmatrix} N
$$

$$
= S_R \begin{pmatrix} \bar{C}_2 \\
\bar{C}_2 \bar{A}_{q_2} \\
\vdots \\
\bar{C}_2 \bar{A}^{q_2-1} \\
\bar{C}_2 \bar{A}^{q_2}
\end{pmatrix} = S_R S_R^T \bar{A}_{q_2} N = \bar{A}_{q_2} N,
$$

and

$$
\bar{A}_q = S_R \begin{pmatrix} \bar{C}_2 \\
\bar{C}_2 \bar{A}_{q_2} \\
\vdots \\
\bar{C}_2 \bar{A}^{q_2-1} \\
\bar{C}_2 \bar{A}^{q_2}
\end{pmatrix} = S_R S_R^T \bar{A}_q N = \bar{A}_q N.
$$
in which the third equality follows from (5.5.11) and (5.5.13), it follows that 
\[ A = N^{-1}A_{q_x}N. \]

\[ \square \]

To illustrate the theory, consider the following examples.

**Example 5.5.6** Consider a positive impulse response function \( W : T \to R_+^{2 \times 2} \), with
\[
W(1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad W(2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]
\[
W(3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad W(4) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Then
\[
H_1(3, 2) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Take \( S_R = (e_1 \ e_2 \ e_3 \ e_4)^T \), then Example 4.2.11 implies that the matrix \( S_RH_1(3, 2) \) is prime. Therefore \( H_1(3, 2) \) is not strictly factorizable, which follows from Proposition 4.5.9.c. First let
\[
W(5) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \quad W(6) = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}.
\]

Then
\[
H_1(4, 3) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Because \( H_1(3, 2) \) has two zero rows, it follows from Proposition 4.5.10 that a necessary and sufficient condition for pos-rank(\( H_1(4, 3) \)) = 4 is that at least one of the following is satisfied

(a) There exists a \( V_1 \in R_+^{4 \times 2} \) satisfying \( H_3(4, 1) = H_1(4, 2)V_1 \).

(b) There exists a \( V_2 \in R_+^{2 \times 4} \) satisfying \( H_4(1, 3) = V_2S_RH_1(3, 3) \).

Solving these equations yields for \( V_1 \in R_+^{4 \times 2} \) and \( V_2 \in R_+^{2 \times 4} \) the unique solutions
\[
V_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]
From Theorem 5.5.5 it follows that all minimal positive realizations of the given impulse response function can be represented by \( (MA, MB, C_1, M^{-1}) \) and \( (NA_2N^{-1}, NB_2, C_2N^{-1}) \) for monomials \( M, N \in R^+_{4 \times 4} \) and

\[
A_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\]

On the other hand, if

\[
W(5) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad W(6) = \begin{pmatrix} 1 & 4/3 \\ 0 & 0 \end{pmatrix},
\]

then there exists no \( V_1 \in R^+_{4 \times 2} \) satisfying \( H_3(4, 1) = H_1(4, 2)V_1 \), but still \( \text{pos-rank}(H_3(4, 3)) = 4 \), since there exists a \( V_2 \in R^+_{2 \times 4} \) satisfying the equation \( H_4(1, 3) = V_2S_RH_1(3, 3) \). Solving this equation yields the unique solution

\[
V_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Now it follows from Theorem 5.5.5 that all minimal positive realizations of the given impulse response function are given by \( (NA_2N^{-1}, NB_2, C_2N^{-1}) \) for a monomial \( N \in R^+_{4 \times 4} \) and

\[
A_2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\]

What happens in the case \( k = m = 1 \)? This will be shown by the following example.

**Example 5.5.7** Consider a single-input-single-output positive impulse response function \( W : T \to R_+ \), with Hankel matrix \( H_1(p, q) \), such that for all \( p, q \geq 4 \) \( \text{pos-rank}(H_1(p, q)) = 4 \) and

\[
H_1(4, 4) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.
\]

Note that \( \text{rank}(H_1(4, 4)) = 3 \), but \( \text{pos-rank}(H_1(4, 4)) = 4 \), since \( H_1(4, 4) \) is a prime. Let \( P = H_1(4, 4) \). For \( p = q = 5 \), there exist \( a, b \in R_+ \) such that
\(H_1(p, q)\) has the following form.

\[
H_1(5, 5) = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & a \\
1 & 1 & 0 & a & b
\end{pmatrix} = \begin{pmatrix}
P & Q \\
Q^T & S
\end{pmatrix}.
\]

From Corollary 4.5.12 it follows that pos-rank(\(H_1(5, 5)\)) = 4 if and only if there exists a \(B \in R_+^{4 \times 4}\) such that \((Q^T S) = B \begin{pmatrix} P & Q \end{pmatrix}\), that is, there exist \(b_1, b_2, b_3, b_4 \in R_+\), such that

\[
\begin{pmatrix}
1 & 1 & 0 & a & b
\end{pmatrix} = \begin{pmatrix}
b_1 & b_2 & b_3 & b_4
\end{pmatrix} \begin{pmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & a
\end{pmatrix}.
\]

Solving this equation for \(b_1, b_2, b_3, b_4 \in R_+\) yields the unique solution \(b_1 = 1, b_2 = b_3 = b_4 = 0\). It follows that \(a = 0\) and \(b = 1\). So pos-rank(\(H_1(5, 5)\)) = 4 if and only if

\[
H_1(5, 5) = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1
\end{pmatrix}.
\]

The matrix \(V_2 \in R_+^{4 \times 4}\) mentioned in Theorem 5.5.2 is \(V_2 = (1 \ 0 \ 0 \ 0)\) and \(S_R = I_4\). Note that \(V_2\) is the unique solution of (5.5.7) and \(V_1 = V_2^T\) is the unique solution of (5.5.6). Hence all minimal positive realizations of the given impulse response function are given by \((MA_1 M^{-1}, MB_1, C_1 M^{-1})\) and \((NA_2 N^{-1}, NB_2, C_2 N^{-1})\), for monomials \(M, N \in R_+^{4 \times 4}\) and

\[
\begin{align*}
A_1 &= \begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, & B_1 &= \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}, & C_1 &= \begin{pmatrix}
1 & 1 & 0 & 0
\end{pmatrix}, \\
A_2 &= \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}, & B_2 &= \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}, & C_2 &= \begin{pmatrix}
1 & 0 & 0 & 0
\end{pmatrix}.
\end{align*}
\]

Note that \((A_1, B_1, C_1)\) and \((A_2, B_2, C_2)\) are related by

\[
H_1(4, 4)A_1 = A_2 H_1(4, 4), \quad H_1(4, 4)B_1 = B_2, \quad C_1 = C_2 H_1(4, 4),
\]
even though \(H_1(4, 4)\) is singular! \(\square\)
5.6 Realization of continuous-time positive linear systems

In this section time-invariant finite-dimensional continuous-time positive linear systems will be treated. It will be shown that the continuous-time case can be deduced from the discrete-time case by a transformation.

5.6.1 Problem formulation

A matrix $A \in \mathbb{R}^{n \times n}$ is said to be a Metzler matrix if all its off-diagonal elements are in $\mathbb{R}_+$, see [77]. Metzler matrices can be characterized as follows.

**Proposition 5.6.1** A matrix $A \in \mathbb{R}^{n \times n}$ is a Metzler matrix if and only if there exists an $\alpha \in \mathbb{R}$ satisfying $(A + \alpha I) \in \mathbb{R}^{n \times n}_+$.

**Definition 5.6.2** Consider a continuous-time linear dynamic system

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t),
\end{align*}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^k$, $t \in T = [t_0, \infty)$. The system (5.6.1) is said to describe a (continuous-time) positive linear system if for all $x_0 \in \mathbb{R}^n_+$ and for all $u(t) \in \mathbb{R}^m_+$, $t \in T$, we have $x(t) \in \mathbb{R}^n_+$ and $y(t) \in \mathbb{R}^k_+$ for $t \in T$.

The following proposition presents a characterization of continuous-time positive linear systems.

**Proposition 5.6.3** The continuous-time linear dynamic system of the form (5.6.1) is a positive linear system if and only if

$$B \in \mathbb{R}^{n \times m}_+, \quad C \in \mathbb{R}^{n \times k}_+, \quad D \in \mathbb{R}^{k \times m}_+, \quad \text{and} \quad A \text{ is a Metzler matrix}.$$

**Proof.** Suppose $u(t) = 0$ for all $t \in T$. For $i \in \mathbb{Z}_+$, $x_i(t) \geq 0$ if and only if $x_i \geq 0$ whenever $x_i = 0$ and $x_j \geq 0$ for all $j \neq i$. This is equivalent to $a_{ij} \geq 0$ for all $j \neq i$. Now the conditions for $B$, $C$, and $D$ follow. \qed

Consider the impulse response function $W : [0, \infty) \rightarrow \mathbb{R}^{k \times m}_+$ of the system (5.6.1), given by

$$W(0) = D; \quad W(t) = Ce^{At}B, \quad t > 0.$$

The claim is that $e^{At} \geq 0$ if and only if $A$ is a Metzler matrix. Indeed, if $A$ is a Metzler matrix, there exists an $\alpha \in \mathbb{R}$ satisfying $A + \alpha I \in \mathbb{R}^{n \times n}_+$. From this it follows that $e^{(A + \alpha I)t} \in \mathbb{R}^{n \times n}_+$ for all $t \geq 0$, and the relation

$$e^{At} = e^{(A + \alpha I)t}e^{-\alpha t}$$

implies $e^{At} \in \mathbb{R}^{n \times n}_+$ for all $t \geq 0$. The other way round, if $e^{At} \geq 0$, then $e^{At}x_0 \in \mathbb{R}^n_+$ whenever $x_0 \in \mathbb{R}^n_+$ for all $t \geq t_0$, so $\dot{x} = Ax$ implies $x(t) \geq 0$ whenever $x_0 \geq 0$. It follows that $A$ is a Metzler matrix. So for continuous-time positive linear systems, besides $B$, $C$, and $D$, also $e^{At}$ is a positive matrix for $t \geq 0$, which implies $W(t) \in \mathbb{R}^{k \times m}_+$ for all $t \geq 0$. On the other hand, the Markov
parameters corresponding to $W(t)$ are not necessarily positive. However, for
\( \alpha \in \mathbb{R} \) satisfying $A + \alpha I \in \mathbb{R}_{++}^{n \times n}$,
\[
D, \quad C(A + \alpha I)^{j-1}B, \quad j = 1, 2, \ldots
\]
are elements of $\mathbb{R}_{++}^{k \times m}$. This follows from Proposition 5.6.1. It can be shown
that these matrices are the Markov parameters corresponding to the impulse
response function $e^{\alpha t}W(t)$. This fact will be used in the sequel.

**Problem 5.6.4** The *continuous-time positive realization problem* for a positive
impulse response function.

a. Formulate necessary and sufficient conditions for the existence of a continu-
ous-time positive linear system such that the impulse response function
of this system equals the given impulse response function. If such a system
exists, it is called a *positive realization* of the given impulse response
function.

b. Determine the minimal dimension of the state space of a positive realization.
   If the state space of a positive realization is minimal, this realization is
called a minimal positive realization.

c. Classify all minimal positive realizations of the given impulse response
   function.

d. If two positive realizations of the same impulse response function are mini-
   mal, then indicate the relation between them.

A positive linear system is called a *minimal positive linear system* if it is a
minimal positive realization of its impulse response function.

### 5.6.2 Existence of a positive realization

In this subsection necessary and sufficient conditions for the existence of a positive realization of a continuous-time positive impulse response function will
be presented. As in the discrete-time case, convex cone analysis will be used,
so the terminology used is as in Section 5.3. New is the notion of a Metzler
impulse response function, which is defined below.

**Definition 5.6.5** A continuous-time impulse response function $W : \mathbb{R}_+ \to \mathbb{R}_{++}^{k \times m}$
is said to be a *Metzler impulse response function* if there exists an $\alpha \in \mathbb{R}$ such
that the Markov parameters corresponding to $e^{\alpha t}W(t)$,
\[
M_a(0) = W(0);
M_a(j) = \left. \frac{d^{j-1}}{dt^{j-1}}e^{\alpha t}W(t) \right|_{t=0}, \quad j = 1, 2, \ldots
\]
are positive matrices.

For the existence of a positive realization, the following result can be stated.
Theorem 5.6.6. Let $T = R_+, Y = R_+^k$, $U = R_+^m$. Consider a continuous-time positive impulse response function $W : T \to R_+^{k \times m}$. There exists a positive linear system

$$
\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0,
$$

$$
y(t) = Cx(t) + Du(t),
$$

such that the impulse response function of this system equals $W$ if and only if $W$ is a Metzler impulse response function and there exists a set $C_1 \subseteq R_+^{\infty}$ satisfying

1. $C_1$ is a polyhedral cone;
2. $\text{cone}(H_a) \subseteq C_1$;
3. $C_1$ is $k$-shift invariant,

with

$$
H_a = (M_a(1)^T \ M_a(2)^T \ M_a(3)^T \ldots)^T
$$

for an $\alpha \in R$ satisfying $M_a(j) \in R_+^{k \times m}$ for all $j \in Z_+$. 

Proof. ($\Rightarrow$) Assume that $W$ is the impulse response function of the positive linear system

$$
\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0,
$$

$$
y(t) = Cx(t) + Du(t),
$$
in which $X = R_+^n$, $B \in R_+^{n \times m}$, $C \in R_+^{k \times n}$, $D \in R_+^{k \times m}$, and $A \in R_+^{n \times n}$ is a Metzler matrix, for $n \in Z_+$. It follows that there exists an $\alpha \in R$ satisfying $A + \alpha I \in R_+^{n \times n}$. Then

$$
M_a(0) = W(0) = D,
$$

$$
M_a(j) = \frac{d^{j-1}}{dt^{j-1}} e^{\alpha t} W(t)|_{t=0} = \frac{d^{j-1}}{dt^{j-1}} e^{\alpha t} C e^{A t} B|_{t=0}
$$

$$
= \frac{d^{j-1}}{dt^{j-1}} C e^{(A + \alpha I) t} B|_{t=0} = C (A + \alpha I)^{j-1} B, \quad j = 1, 2, \ldots,
$$

with $M_a(j) \in R_+^{k \times m}$, since $A + \alpha I \in R_+^{n \times n}$. So $W$ is a Metzler impulse response function. This provides $A_a (= A + \alpha I), B, C$, and $D$ satisfying

$$
M_a(0) = D,
$$

$$
M_a(j) = C A_a^{j-1} B, \quad j = 1, 2, \ldots,
$$

and with Theorem 5.3.4, 1, 2, and 3 follow.

($\Leftarrow$) Because $W$ is a Metzler impulse response function there exists an $\alpha \in R$ such that for all $j \in N$, $M_a(j) \in R_+^{k \times m}$. Step a) to f) in the proofs of Theorem 5.3.1 and Theorem 5.3.4 provide $A \in R_+^{n \times n}, B \in R_+^{n \times m}, C \in R_+^{k \times n},$ and $D \in R_+^{k \times m}$ satisfying

$$
M_a(0) = D,
$$

$$
M_a(j) = C \bar{A}^{j-1} B, \quad j = 1, 2, \ldots
$$
The matrices $M_a(j)$ are the Markov parameters corresponding to the impulse response function $\hat{W}(t) = Ce^{At}B$, and because $W$ is a Metzler impulse response function, $\alpha \in R$ satisfies $\hat{W}(t) = e^{\alpha t}W(t)$. So, with $A = A - \alpha I$, there exists a positive linear system

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0, \\
y(t) &= Cx(t) + Du(t),
\end{align*}
$$

with $W(t) = Ce^{At}B$ for $t > 0$, and $W(0) = D$.

\subsection*{5.6.3 Characterization of minimality}

In this subsection results on the characterization of minimality for the continuous-time case are presented, which turn out to be related to the results for the discrete-time case, see Section 5.4. The problem is to derive sufficient and necessary conditions for a continuous-time positive linear system to be a minimal positive realization of the impulse response function. About the positive rank in relation to a continuous-time positive linear system the following can be said.

Consider a positive linear system $(A, B, C)$ with $B \in R_{+}^{m \times m}$, $C \in R_{+}^{k \times n}$, and $(A + \alpha I) \in R_{+}^{n \times n}$ for some $\alpha \in R$. Let $A_{\alpha} = A + \alpha I$. For $p, q \in Z_{+}$, define $H_{\alpha}(p, q)$ to be the Hankel matrix

$$
H_{\alpha}(p, q) = 
\begin{pmatrix}
CB & CA_{\alpha}B & \cdots & CA_{\alpha}^{n-1}B \\
CA_{\alpha}B & CA_{\alpha}^{2}B & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
CA_{\alpha}^{n-1}B & \cdots & \cdots & CA_{\alpha}^{p+q-2}B
\end{pmatrix}.
$$

\begin{proposition}
Consider a positive linear system $(A, B, C)$ with $A \in R_{+}^{n \times n}$ a Metzler matrix, $B \in R_{+}^{m \times m}$, and $C \in R_{+}^{k \times n}$. For all $p, q \in Z_{+}$ and for all $\alpha \in R$ satisfying $A + \alpha I \in R_{+}^{n \times n}$, the necessary conditions for a continuous-time positive linear system to be a minimal positive realization of the impulse response function, is the Hankel matrix $H_{\alpha}(p, q)$ to be the Hankel matrix

$$
H_{\alpha}(p, q) = 
\begin{pmatrix}
CB & CA_{\alpha}B & \cdots & CA_{\alpha}^{n-1}B \\
CA_{\alpha}B & CA_{\alpha}^{2}B & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
CA_{\alpha}^{n-1}B & \cdots & \cdots & CA_{\alpha}^{p+q-2}B
\end{pmatrix}.
$$

Proof. Analogous to discrete-time case, Proposition 5.4.1. \hfill \square
\end{proposition}

For the positive system rank, the analogue of Proposition 5.4.9 is

\begin{proposition}
Consider a positive linear system $(A, B, C)$ with $A \in R_{+}^{n \times n}$ a Metzler matrix, $B \in R_{+}^{m \times m}$, and $C \in R_{+}^{k \times n}$. For all $q \in Z_{+}$ and for all $\alpha \in R$ satisfying $A + \alpha I \in R_{+}^{n \times n}$, the positive system rank of $H_{\alpha}(q)$ is less than or equal to $n$.

The proof is again analogous to discrete-time case, Proposition 5.4.9.

Below the relation between minimal discrete-time positive linear systems and continuous-time positive linear systems is presented.

\begin{theorem}
Let the continuous-time positive linear system $(A, B, C)$ be given as above. This continuous-time positive linear system $(A, B, C)$ is minimal if and only if the discrete-time positive linear system $(A + \beta I, B, C)$ is minimal for all $\beta \in R$ satisfying $A + \beta I \in R_{+}^{n \times n}$.
\end{theorem}
Proof. $\implies$ Assume $(A, B, C)$ is a minimal continuous-time positive linear system. Suppose there exists a $\beta \in \mathbb{R}$ such that $(A + \beta I, B, C)$ is a discrete-time positive linear system that is not minimal. Then there exists a discrete-time positive linear system $(A, B, C)$, with $A \in \mathbb{R}^{n \times n}_{+}, B \in \mathbb{R}^{n \times m}_{+}, C \in \mathbb{R}^{k \times n}_{+}$, for $n_1 < n$, with the same impulse response function as $(A + \beta I, B, C)$. But then $(A - \beta I, B, C)$ is a continuous-time positive linear system with state-space dimension $n_1$ and the same impulse response function as $(A, B, C)$, so $(A, B, C)$ is not minimal. This is a contradiction. So $(A + \beta I, B, C)$ is a minimal discrete-time positive linear system for all $\beta \in \mathbb{R}$ satisfying $A + \beta I \in \mathbb{R}^{n \times n}_{+}$.

$\iff$ Assume $(A + \beta I, B, C)$ is a minimal discrete-time positive linear system for all $\beta \in \mathbb{R}$ satisfying $A + \beta I \in \mathbb{R}^{n \times n}_{+}$. Suppose $(A, B, C)$ is not a minimal continuous-time positive linear system. Then there exists a continuous-time positive linear system $(\hat{A}, \hat{B}, \hat{C})$, with $\hat{A} \in \mathbb{R}^{n \times n}_{+}$ a Metzler matrix, $\hat{B} \in \mathbb{R}^{n \times m}_{+}, \hat{C} \in \mathbb{R}^{k \times n}_{+}$, for $n_1 < n$, with the same impulse response function as $(A, B, C)$. Since $\hat{A}$ is, like $A$, a Metzler matrix, there exists an $\alpha \in \mathbb{R}$, satisfying both $\hat{A} + \alpha I \in \mathbb{R}^{n \times n}_{+}$, and $A + \alpha I \in \mathbb{R}^{n \times n}_{+}$. So $(\hat{A} + \alpha I, \hat{B}, \hat{C})$ is a discrete-time positive linear system with the same impulse response function as $(A + \alpha I, B, C)$, but with a smaller state space dimension, hence $(A + \alpha I, B, C)$ is not minimal. Contradiction. It follows that $(A, B, C)$ is a minimal continuous-time positive linear system. $\square$

If $(A + \beta I, B, C)$ is a minimal discrete-time positive linear system for only one $\beta \in \mathbb{R}$ satisfying $A + \beta I \in \mathbb{R}^{n \times n}_{+}$, this is not sufficient for $(A, B, C)$ to be minimal as continuous-time positive linear system, as the following example shows.

Example 5.6.10 Consider the continuous-time positive linear system of the form (5.6.1) with

\[
A = \begin{pmatrix}
-0.8 & 0.25 & 0 & 0 \\
1 & -0.8 & 0 & 0 \\
0 & 0.39 & -0.8 & 0.8 \\
0 & 0 & 0.8 & -0.8
\end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 1.1 \\ 0 \\ 2 \end{pmatrix}.
\]

For $\alpha = 0.8$, $A + \alpha I = A_\alpha \in \mathbb{R}^{4 \times 4}_{+}$. The discrete-time positive linear system $(A_\alpha, B, C)$ is minimal. To see this, consider the transfer function

\[
C(I - A_\alpha)^{-1}B = \frac{\lambda^3 + 1.1\lambda^2 - 0.64\lambda - 0.8}{(\lambda + 1)(\lambda - 0.25)(\lambda^2 - 0.64)} = \frac{\lambda_1 + 1.6\lambda + 0.16}{(\lambda + 0.5)(\lambda^2 - 0.64)}
\]

Suppose there exists a positive realization of order 3. Then there should exist a matrix $A \in \mathbb{R}^{3 \times 3}_{+}$ with eigenvalues $0.8$, $-0.8$, and $-0.5$. Because $A \in \mathbb{R}^{3 \times 3}_{+}$, trace($A$) $\geq$ 0. But trace($A$) equals the sum of the eigenvalues of $A$, which is $0.8 + (-0.8) + (-0.5) = -0.5$. This is a contradiction. So $(A_\alpha, B, C)$ is a
minimal discrete-time positive linear system. Now consider the Hankel matrix

\[
H_a(4, 4) = \begin{pmatrix}
CB & CAB & CA^2B & CA^3B \\
CAB & CA^2B & CA^3B & CA^4B \\
CA^2B & CA^3B & CA^4B & CA^5B \\
CA^3B & CA^4B & CA^5B & CA^6B \\
1 & 1.1 & 0.25 & 0.899 \\
1.1 & 0.25 & 0.899 & 0.0625 \\
0.25 & 0.899 & 0.0625 & 0.62411 \\
0.899 & 0.0625 & 0.62411 & 0.015625
\end{pmatrix}.
\]

The claim is that \(H_a(4, 4)\) has positive rank 4. This will be proven in Appendix 5.A.

But, also for \(\beta = 1.6\), \(A + \beta I = A_\beta \in \mathbb{R}_{+}^{4 \times 4}\). Now \((A_\beta, B, C)\) is a discrete-time positive linear system, which is not minimal. Indeed, the discrete-time positive linear system \((\hat{A}, \hat{B}, \hat{C})\), with

\[
\hat{A} = \begin{pmatrix}
0 & 0 & 0 \\
1.6 & 1.6 & 0 \\
0.3 & 0 & 0.3
\end{pmatrix}, \quad \hat{B} = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad \hat{C} = \begin{pmatrix}
1 & 1 & 1
\end{pmatrix},
\]

is a discrete-time positive linear system with the same impulse response function as \((A_\beta, B, C)\). It is minimal, since it is minimal as linear system, i.e., \((\hat{A}, \hat{B})\) is reachable and \((\hat{A}, \hat{C})\) is observable. So \((\hat{A} - \beta I, \hat{B}, \hat{C})\), with

\[
\hat{A} - \beta I = \begin{pmatrix}
-1.6 & 0 & 0 \\
1.6 & 0 & 0 \\
0.3 & 0 & -1.3
\end{pmatrix}, \quad \hat{B} = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad \hat{C} = \begin{pmatrix}
1 & 1 & 1
\end{pmatrix},
\]

is a minimal continuous-time positive linear system for \((A, B, C)\).

So, while \((A_\alpha, B, C)\) is a minimal discrete-time positive linear system, and \(\text{pos-rank}(H_a(4, 4)) = 4\), for \(\alpha = 0.8\), \((A, B, C)\) is not a minimal continuous-time positive linear system.

Note that with Proposition 5.6.7, \(\text{pos-rank}(H_\beta(p, q)) \leq 3\) for all \(p, q \in \mathbb{Z}_{+}\), so

\[
\text{pos-rank}(H_\beta(4, 4)) \leq 3 < 4 = \text{pos-rank}(H_a(4, 4)). \quad \Box
\]

To show that there exists a continuous-time positive linear system that is not minimal as an ordinary linear system, but is minimal as a positive linear system, consider the example in [112]. Let

\[
A = \begin{pmatrix}
-2 & 0 & 0 & 1 \\
1 & -2 & 0 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 1 & -2
\end{pmatrix}, \quad B = \begin{pmatrix}
1 \\
0 \\
0 \\
1
\end{pmatrix}, \quad C = \begin{pmatrix}
1 & 1 & 0 & 1
\end{pmatrix}
\]

be a continuous-time positive linear system. Note that this system is not minimal as an ordinary linear system. In [112] it has been shown that the system is minimal as a continuous-time positive linear system. Another way of showing
this is using the theory of this section as follows. With Theorem 5.6.9 and Example 5.6.10 it is not sufficient to check whether \((A + 2I, B, C)\) is minimal as a discrete-time positive linear system. For \((A, B, C)\) to be a minimal continuous-time positive linear system, it has to be shown that \((A + \beta I, B, C)\) is minimal as a discrete-time positive linear system for all \(\beta \in R\) satisfying \(A + \beta I \in R^{4 \times 4}_+\). Consider for (5.6.3) the discrete-time positive linear systems \((A + \beta I, B, C)\) for arbitrary \(\beta \geq 2\). The poles of the transfer function \(C(\lambda I - A)^{-1} B\) are \(\{\beta - 1, \beta - 2 + i, \beta - 2 - i\}\). If \(\{\beta - 1, \beta - 2 + i, \beta - 2 - i\}\) were the eigenvalues of a positive matrix \(A \in R^{3 \times 3}_+\), then Equation (4.2.1) in [11] must hold. For \(k = 1\) and \(m = 2\) this equation reads

\[
(\lambda_1 + \lambda_2 + \lambda_3)^2 \leq 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2),
\]

in which \(\lambda_1, \lambda_2, \lambda_3\) are the eigenvalues. Substituting

\[
\lambda_1 = \beta - 1, \quad \lambda_2 = \beta - 2 + i, \quad \lambda_3 = \beta - 2 - i,
\]

it can be seen that (5.6.4) does not hold for any \(\beta \in R\). So there does not exist a positive matrix \(A \in R^{3 \times 3}_+\) with eigenvalues \(\{\beta - 1, \beta - 2 + i, \beta - 2 - i\}\) for any \(\beta \in R\). It follows that \((A + \beta I, B, C)\) is a minimal discrete-time positive linear system for all \(\beta \geq 2\), so with Theorem 5.6.9 \((A, B, C)\) is a minimal continuous-time positive linear system.

This section will be closed with analogues for Proposition 5.4.2 and Theorem 5.4.10.

**Proposition 5.6.11** Let \((A, B, C) \in (R^{n \times n}_+ \times R^{n \times m}_+ \times R^{k \times n}_+)\) be a continuous-time positive linear system. If there exist \(p, q \in Z_+\) such that for all \(\alpha \in R\) satisfying \(A + \alpha I \in R^{n \times n}_+\), \(\text{pcr-rank}(H_\alpha(p, q)) = n\), then \((A, B, C)\) is a minimal positive linear system.

**Theorem 5.6.12** Let \((A, B, C) \in (R^{n \times n}_+ \times R^{n \times m}_+ \times R^{k \times n}_+)\) be a continuous-time positive linear system. This positive linear system \((A, B, C)\) is minimal if and only if there exist \(a q \in Z_+\) such that for all \(\alpha \in R\) satisfying \(A + \alpha I \in R^{n \times n}_+\), \(\text{pr-rank}(H_\alpha(q)) = n\).

The proofs follow from Theorem 5.6.9 and Proposition 5.4.2, Theorem 5.4.10, respectively.

### 5.7 Conclusions

The technique of the theory for polyhedral cones seems to be a useful way to deal with the realization problem for positive linear systems. Necessary and sufficient conditions have been presented for the existence of a positive realization. For the characterization of minimality a sufficient condition has been found, that is weaker than the reachability/observability condition, and also a necessary and sufficient condition has been found. A necessary and sufficient condition for a positive linear system to be minimal is that the positive system rank of the Hankel matrix equals the number of states. This condition shows resemblance to the McMillan degree of the Hankel matrix for ordinary
linear systems. For the latter systems, knowledge of linear algebra can be used, but for positive linear systems the knowledge of positive linear algebra is not mature enough to be applied. Preliminary results on positive linear algebra have been presented in Chapter 4. Further research has to be done on the positive rank of matrices and the positive system rank.

If a minimal positive realization has been given, a class of minimal positive realizations can be presented that are equivalent to it, but this is not the complete class. So the open problem here is to find a class of matrices $T$ such that all positive realizations $(TAT^{-1}, TB, CT^{-1}, D)$, equivalent to the positive realization $(A, B, C, D)$, can be given. The classification of minimal positive realizations could follow from the classification of positive matrix factorizations, which is, together with the accompanying determination of positive rank, still in its infancy.

For a special class of positive impulse response functions, all minimal positive realizations can be determined. Although the class is restricted, it shows a way of treating the problem in general. But as long as the problem of the classification of positive matrix factorizations has not been solved, the general solution cannot be presented.

As in the discrete-time case, the condition of reachability and observability is only sufficient for a continuous-time positive linear system to be minimal, but not necessary, as has been shown by the example described by (5.6.3). Theorem 5.6.12 presents a sufficient and necessary condition for minimality of positive linear systems.

Appendix

5.A Proof of pos-rank$(H_0(4, 4)) = 4$ in Example 5.6.10

As in Section 3.4 of [11], the following notation will be used. For $A \in R^{k \times m}$, let $a_j$ denote the $j$th column of $A$. $a_j^0$ denotes the $\{0, 1\}$ vector defined by $a_{ij}^0 = 1$ if $a_{ij} > 0$ and $a_{ij}^0 = 0$ if $a_{ij} = 0$.

**Theorem 5.A.1** Let $A \in R^{k \times m}$, with $k \geq m$. Let $1 \leq i, j \leq m$. If $a_i^0 \geq a_j^0$, then $A$ is strictly factorizable.

**Proof.** The proof is analogous to the proof of Theorem 3.4.19 in [11].

**Corollary 5.A.2** If $A \in CE_{k,m} \cup F_m(R^k_+)$, with $k \geq m$, i.e., if $A$ is not strictly factorizable, then $A$ contains a zero and a strictly positive element in every column, and a strictly positive element in every row. It also contains a zero in at least $m$ rows.

Consider the matrix

$$H_0(4, 4) = \begin{pmatrix} 1 & 1.1 & 0.25 & 0.899 \\ 1.1 & 0.25 & 0.899 & 0.0625 \\ 0.25 & 0.899 & 0.0625 & 0.62411 \\ 0.899 & 0.0625 & 0.62411 & 0.015625 \end{pmatrix}.$$
Suppose pos-rank$(H_a(4,4)) = 3$. From Theorem 4.5.18 it follows that there exist matrices $B \in R^3_{+}^{1 \times 3}$ and $C \in R^{3 \times 4}$ such that $H_a(4,4) = BC$ and $\text{cone}(B) \in CE_{4,3} \cup F_3(R^4_+)$. If $\text{cone}(B) \in CE_{4,3} \cup F_3(R^4_+)$, then from Corollary 5.A.2 it follows that $B$ contains at least one zero and one nonzero element in every column, and in at least 3 rows a zero. Note that, with $r_1, r_2, r_3$, and $r_4$ denoting the four rows of $H_a(4,4)$, that
\[
\frac{8}{25} r_1 + \frac{16}{25} r_2 = \frac{1}{2} r_3 + r_4.
\]
So this relation should also hold for the rows of $B$. With a post-multiplication by a monomial $M \in R^3_{+}^{1 \times 3}$, $B$ can contain one in every column. So $B$ has, without loss of generality, one of the following forms:

\[
B_1 = \begin{pmatrix}
\frac{25}{32} a + \frac{25}{16} b + \frac{25}{25} c & 0 & 1 \\
\frac{1}{25} a & 1 & 0 \\
1 & b & 16 c + \frac{8}{25}
\end{pmatrix},
\]

\[
B_2 = \begin{pmatrix}
\frac{25}{32} a + \frac{25}{16} b & 0 & 1 \\
\frac{1}{25} a & 1 & 0 \\
1 & 16 c + \frac{8}{25} & 0
\end{pmatrix},
\]

\[
B_3 = \begin{pmatrix}
\frac{25}{32} a + \frac{25}{16} c & 0 & 1 \\
\frac{1}{25} a & 1 & 0 \\
1 & b & 0
\end{pmatrix},
\]

\[
B_4 = \begin{pmatrix}
\frac{25}{32} a + \frac{25}{16} c & 1 & c \\
0 & 0 & 1 \\
1 & 16 c + \frac{8}{25} & 0
\end{pmatrix},
\]

with $a \geq 0$, $b \geq 0$, $c \geq 0$. Since $H_a(4,4) = BC$, $C$ can be calculated by $C = B^* H_a(4,4)$, with $B^*$ a left inverse of $B$. Below $C$ has been calculated with the help of Maple V, see [24]. Consider $C_1 = B_1^* H_a(4,4)$. All elements of $C_1$ must be positive. Consider the third and the fourth element of the third row of $C_1$:

\[
C_{1,(3,3)} = \frac{1.95034375 a + 0.3046875 + 3.4006875 a b - 0.390625 b}{a + 2 + 4 a c b + 2 a c},
\]

\[
C_{1,(3,4)} = \frac{0.48828125 a - 0.15234375 - 1.70034375 a b - 3.9006875 b}{a + 2 + 4 a c b + 2 a c}.
\]

Now

\[
C_{1,(3,3)} \geq 0 \text{ if and only if } b \leq \frac{1.95034375 a + 0.3046875}{0.390625 - 3.4006875 a} =: x,
\]

\[
\text{for } a \neq \frac{0.390625}{3.4006875}.
\]

\[
C_{1,(3,4)} \geq 0 \text{ if and only if } b \leq \frac{0.48828125 a - 0.15234375}{1.70034375 a + 3.9006875} =: y.
\]
So \( b \leq \min \{x, y\} \). But \( 0 \leq a \leq 1 \) implies \( y < 0 \), and \( a \geq 1 \) implies \( x < 0 \), thus for all \( a \geq 0 \) we have \( b < 0 \). This contradicts that \( H_a(4, 4) \) can be written as \( B_1C_1 \) with \( B_1 \) and \( C_1 \) positive matrices and \( B_1 \) given above.

Analogously, consider the second and the fourth element of the first row of \( C_2 = B_2^2 H_a(4, 4) \):

\[
C_{2(1,2)} = \frac{0.39cb + 0.32c - 2.816b - 1.158}{2acb + ac + 2c + 4},
C_{2(1,4)} = \frac{0.0975cb + 0.08c - 2.30144b - 1.08822}{2acb + ac + 2c + 4}.
\]

Now

\[
C_{2(1,2)} \geq 0 \quad \text{if and only if} \quad b \leq \frac{0.32c - 1.158}{2.816 - 0.39c} =: x, \quad \text{for} \quad c \neq \frac{2.816}{0.39},
\]

\[
C_{2(1,4)} \geq 0 \quad \text{if and only if} \quad b \leq \frac{0.08c - 1.08822}{2.30144 - 0.0975c} =: y, \quad \text{for} \quad c \neq \frac{2.30144}{0.0975}.
\]

So \( b \leq \min \{x, y\} \). But \( 0 \leq c \leq 10 \) implies \( y < 0 \), and \( c \geq 10 \) implies \( x < 0 \), thus for all \( c \geq 0 \) we have \( b < 0 \).

In the same way, consider the third and the fourth element of the third row of \( C_3 = B_3^3 H_a(4, 4) \):

\[
C_{3(3,3)} = \frac{-1.70034375ab + 0.1953125b - 0.15234375 - 0.975171875}{acb + 2cb + 4b + 2},
C_{3(3,4)} = \frac{0.85017188ab + 1.9503438 + 0.07617188 - 0.024414063a}{acb + 2cb + 4b + 2}.
\]

Now

\[
C_{3(3,3)} \geq 0 \quad \text{if and only if} \quad a \leq \frac{0.1953125b - 0.15234375}{0.975171875 + 1.70034375b} =: x, \quad \text{for} \quad b \neq \frac{0.975171875}{1.70034375},
\]

\[
C_{3(3,4)} \geq 0 \quad \text{if and only if} \quad a \leq \frac{1.9503438 + 0.07617188}{0.024414063 - 0.85017188} =: y, \quad \text{for} \quad b \neq \frac{0.024414063}{0.85017188}.
\]

So \( a \leq \min \{x, y\} \). But \( 0 \leq b \leq 0.1 \) implies \( x < 0 \), and \( b \geq 0.1 \) implies \( y < 0 \), thus for all \( b \geq 0 \) we have \( a < 0 \).

For the last possibility, consider the first and the third element of the first row of \( C_4 = B_4^4 H_a(4, 4) \):

\[
C_{4(1,1)} = \frac{0.39cb - 0.704c + 1.28b - 1.158}{2b + 1 + 2acb + 4ab},
C_{4(1,3)} = \frac{0.0975cb - 0.57536c + 0.32b - 1.08822}{2b + 1 + 2acb + 4ab}.
\]

Now

\[
C_{4(1,1)} \geq 0 \quad \text{if and only if} \quad c \leq \frac{1.28b - 1.158}{0.704 - 0.39b} =: x, \quad \text{for} \quad b \neq \frac{0.704}{0.39}.
\]

For the last possibility, consider the first and the third element of the first row of \( C_4 = B_4 H_a(4, 4) \):

\[
C_{4(1,1)} = \frac{0.39cb - 0.704c + 1.28b - 1.158}{2b + 1 + 2acb + 4ab},
C_{4(1,3)} = \frac{0.0975cb - 0.57536c + 0.32b - 1.08822}{2b + 1 + 2acb + 4ab}.
\]

Now

\[
C_{4(1,1)} \geq 0 \quad \text{if and only if} \quad c \leq \frac{1.28b - 1.158}{0.704 - 0.39b} =: x, \quad \text{for} \quad b \neq \frac{0.704}{0.39}.
\]
\[ C_{4(1,3)} \geq 0 \text{ if and only if } c \leq \frac{0.32b - 1.08822}{0.57536 - 0.0975b} =: y, \]

for \( b \neq \frac{0.57536}{0.0975}. \)

So \( c \leq \min\{x, y\}. \) But \( 0 \leq b \leq 3 \) implies \( y < 0, \) and \( b \geq 3 \) implies \( x < 0, \) thus for all \( b \geq 0 \) we have \( c < 0. \)

Conclusion: \( H_a(4, 4) \) cannot be written as \( BC \) with \( B \in R_+^{4\times 3}, \) \( C \in R_+^{3\times 4}, \) so \( \text{pos-rank}(H_a(4, 4)) = 4. \) \qed