Convolution on homogeneous spaces
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Chapter III

Distributions Concentrated On Submanifolds

III.1 Induced Spaces of Distributions over Universal Enveloping Algebras

For \( g \) a real Lie-algebra, \( b \) a subalgebra, and \( \mathcal{W} \) a locally convex \( \mathcal{O}(b) \)-algebra, let \( \mathcal{O}(g) \otimes \mathcal{W} \) be the induced locally convex module as studied in Section II.6. In this chapter we take for \( \mathcal{W} \) spaces of distributions. This allows us to give a coordinate free way of describing the space of distributions concentrated on a closed subgroup of a Lie group as a \( \mathcal{O}(g) \)-module (Theorem III.3.1). To allow for greater generality and a better understanding of which topological properties are essential, we take for \( \mathcal{W} \) first the strong dual of a Fréchet space. One can think of the space \( \mathcal{E}'(\mathcal{X}) \) of compactly supported distributions on a manifold \( \mathcal{X} \). Next we take for \( \mathcal{W} \) the strong dual of an inductive limit of Fréchet spaces. The space \( \mathcal{D}'(\mathcal{X}) \) of distributions on a (non-compact) manifold is typically of this kind.

Let \( \mathcal{W} \) be the strong dual of a Fréchet space. Let \( I \) be a linear complement of \( b \) in \( g \). Then \( \mathcal{S}(I) \) is the direct sum of the subspaces \( \mathcal{S}(I)^m \) of homogeneous elements of degree \( m, m=0,1,\ldots \), so that, according to Grothendieck [29], the space \( \mathcal{S}(I) \otimes \mathcal{W} \) is canonically isomorphic to the direct sum \( \sum_{m=0}^{\infty} \left( \mathcal{S}(I)^m \otimes \mathcal{W} \right) \). Since \( \mathcal{S}(I)^m \) is finite dimensional, \( \mathcal{S}(I)^m \otimes \mathcal{W} \) is already complete. The topological direct sum of complete spaces being complete, it follows that \( \mathcal{S}(I) \otimes \mathcal{W} = \sum_{m=0}^{\infty} \left( \mathcal{S}(I)^m \otimes \mathcal{W} \right) \) is already complete.

In fact, instead of the subspaces \( \mathcal{S}(I)^m \) one can take any direct sum decomposition with finite dimensional summands. In particular, one can choose a basis \( x_1, x_2,\ldots,x_\ell \) for \( I \), and consider the one-dimensional subspaces spanned by monomials \( x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_\ell^{\alpha_\ell} \), \( \alpha \in \mathbb{N}^\ell \). Then, under the topological isomorphism \( \mathcal{S}(I) \otimes \mathcal{W} \rightarrow \mathcal{O}(g) \otimes \mathcal{W}, \mathcal{P} \otimes \mathcal{W} \rightarrow \mu(\mathcal{P}) \otimes \mathcal{W} \) (as discussed in the paragraph preceding Proposition II.6.2), one finds that \( \mathcal{O}(g) \otimes \mathcal{W} \) is the topological direct sum.

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The decompositions thus discussed depend on the choice of \( I \), and even on the choice of a basis \( x_1, x_2, \ldots, x_l \). One obtains a more intrinsic description of the topology on \( \tilde{U}(g) \otimes W \) by noting that the topology on \( S(I) \) is that of strict inductive limit with respect to the subspaces \( S(I^{(m)}) \), and that corresponding to this \( S(I) \otimes W = S(I^{(m)}) \otimes W \) carries the strict inductive limit with respect to the increasing sequence \( S(I^{(m)}) \otimes W = S(I^{(m)}) \otimes W \). By applying a suitable isomorphism, and Proposition II.6.2, one obtains intrinsic statements about \( \tilde{U}(g) \otimes W \). We sum up the results in

**Theorem III.1.1**

Let \( g \) be a complex Lie algebra, and \( b \) a subalgebra. Let \( W \) be a complete Hausdorff locally convex \( \tilde{U}(b) \)-module that is the strong dual of a Fréchet space. Then \( \tilde{U}(g) \otimes W \) is complete when equipped with the projective tensor product topology, so

\[
\tilde{U}(g) \otimes W = \tilde{U}(g) \otimes_{\tilde{U}(b)} W.
\]

Moreover, the projective tensor product topology on \( \tilde{U}(g) \otimes W \) coincides with the strict inductive limit topology defined by the increasing sequence \( \tilde{U}(g) \otimes W^{(m)} \) of elements of transversal order not exceeding \( m \) \( (m=0,1,2, \ldots) \).

Note that this description as inductive limit defines the topology, because the topology on \( \tilde{U}(g) \otimes W^{(m)} \) has an independent definition, cf. Proposition II.6.2.

**Corollaries III.1.2 (of the proof)**

i) Let \( F' \) be the strong dual of a Fréchet space \( F \). Then the space \( \mathcal{C}(\mathbb{N}) \otimes F' \) is complete, and isomorphic to \( \sum_{n} F' \). In particular, \( \mathcal{C}(\mathbb{N}) \otimes F' \) can be seen as the strong dual of the Fréchet space \( F^{\mathbb{N}} \), the space of sequences in \( F \) equipped with the product topology \( \prod_{\mathbb{N}} F \).

ii) Under the assumptions of Theorem III.1.1, let \( I \) be a linear complement of \( b \) in \( g \), and \( x_1, x_2, \ldots, x_l \) a basis for \( I \). Then \( \tilde{U}(g) \otimes W = \tilde{U}(g) \otimes_{\tilde{U}(b)} W \) equals the topological direct sum

\[
\sum_{I} \mathcal{E}_{I},
\]

where \( \mathcal{E}_{I} \) is the strong dual of \( \prod_{I} F \), direct sum of strong duals (A. Grothendieck, *Produits Tensoriels Topologiques et Espaces Nucléaires*, Chapitre IV, § 1°, n°3, Proposition 7, Corollaire); one can also prove that \( \prod_{I} F \) is isomorphic to \( \mathcal{C}(\mathbb{N}) \otimes F \) (Fréchet).
We now come to the more interesting case where the space $\hat{\mathcal{O}}(\mathfrak{g})$, $\otimes \mathcal{W}$ is not complete when carrying the projective tensor product topology.

Let $\mathcal{W}$ be the strong dual of an $\mathcal{L}_\mathcal{F}$-space $\mathcal{F}$. So, $\mathcal{F} = \lim_{p \to \infty} \mathcal{F}_p$ for a sequence of Fréchet spaces $\mathcal{F}_p$ with homomorphic imbeddings $\mathcal{F}_p \subseteq \mathcal{F}_{p+1}$. Let $\mathcal{W}_p$ be the strong dual of $\mathcal{F}_p$, then $\mathcal{W}$ can be described as the projective limit $\mathcal{W} = \lim_{p \to \infty} \mathcal{W}_p$, that is, $\mathcal{W}$ carries the coarsest locally convex topology entailing continuity of the transpositions $\pi_p: \mathcal{W} \to \mathcal{W}_p$ of the homomorphic imbeddings $\mathcal{F}_p \subseteq \mathcal{F}$. Also, $\mathcal{W}$ is complete, because it is the strong dual of a bornological space $\mathcal{W}_p$. In general for every locally convex space $\mathcal{E}$ the completed tensor product $\mathcal{E} \hat{\otimes} \mathcal{W}$ is the projective limit with respect to the maps $\Pi_p: \mathcal{E} \hat{\otimes} \mathcal{W} \to \mathcal{E} \mathcal{W}_p$, $\Pi_p = \varepsilon \otimes \pi_p$ (this means $\Pi_p(\varepsilon \otimes \mathcal{W}) = \varepsilon \otimes \pi_p(\mathcal{W})$, $\varepsilon \in \mathcal{E}$, $\mathcal{W} \in \mathcal{W}$), provided the maps $\mathcal{E} \to \mathcal{W}_p$ have dense image. In this case they are in fact onto.

For $\mathcal{E}$ take the space $C^\infty(\mathfrak{N})$, and consider one of the spaces $\mathcal{W}_p$. According to the kind of argument leading up to Theorem III.1.1, $C^\infty(\mathfrak{N}) \hat{\otimes} \mathcal{W}_p = C^\infty(\mathfrak{N}) \otimes \mathcal{W}_p$, since $\mathcal{W}_p$ is the strong dual of a Fréchet space. On $C^\infty(\mathfrak{N})$ define the coordinate form $l_n(x) = x_n$. Let $e_n, n \in \mathfrak{N}$, denote the sequence that is 1 at the point $n$, and 0 elsewhere. Then obviously $x = \sum l_n(x) e_n, x \in C^\infty(\mathfrak{N})$. Define $l_{P_n}: C^\infty(\mathfrak{N}) \hat{\otimes} \mathcal{W}_p \to C^\infty(\mathfrak{N}) \otimes \mathcal{W}_p$ as $l_{P_n} = l_n \otimes l_{P_n}$, $l_{P_n}$ the identity on $\mathcal{W}_p$, so $l_{P_n}(x \otimes y) = (l_n(x)) y$. The map $l_{P_n}$ is continuous. On the space $C^\infty(\mathfrak{N}) \hat{\otimes} \mathcal{W}_p$ one has the decomposition

\[(III.1.2.a) \quad s = \sum_n e_n \otimes l_{P_n}(s), \quad s \in C^\infty(\mathfrak{N}) \hat{\otimes} \mathcal{W}_p.\]

It is easy and sufficient to check (III.1.2.a) for elementary tensors $x \otimes y, x \in C^\infty(\mathfrak{N}), y \in \mathcal{W}_p$.

We want to establish an analogue of (III.1.2.a) for elementary tensors $x \otimes y, x \in C^\infty(\mathfrak{N}), y \in \mathcal{W}_p$. Define $l_n: C^\infty(\mathfrak{N}) \hat{\otimes} \mathcal{W} \to C^\infty(\mathfrak{N}) \hat{\otimes} \mathcal{W} \cong \mathcal{W}$ by $l_n = l_n \otimes l_{I_n}$, $l_{I_n}$ the identity on $\mathcal{W}$ (as always, the 'tensor product' $A \hat{\otimes} B$ of two linear maps $A$ and $B$ is continuous for the projective tensor product topologies, and extends to the completions of the spaces involved). Then for $t \in C^\infty(\mathfrak{N}) \hat{\otimes} \mathcal{W}$ one has

\[31 \text{Let } \mathcal{W} = \mathcal{F}_b \text{ (strong dual of } \mathcal{F}), \text{ and } \tilde{\mathcal{W}} = \lim_{n \to \infty} \mathcal{W}_n. \text{ The maps } P_n \text{ are continuous, the topology on } \tilde{\mathcal{W}} \text{ is the coarsest for which they are so, and so the topology on } \mathcal{W} \text{ finer than he topology on } \tilde{\mathcal{W}}, \text{ therefore the identity } \tilde{\mathcal{W}} \to \mathcal{W} \text{ is continuous (this is true for every strong dual of a, not necessarily strict, inductive limit). Conversely, let } (\xi_n)_{\alpha \in I} \text{ be a net tending to zero in } \tilde{\mathcal{W}}. \text{ Then for every } n \in \mathcal{N} \text{ the net } (P_n(\xi_n))_{\alpha \in I}, \text{ that is, the net of restrictions to } F_n, \text{ tends to zero in } \mathcal{W}_n, \text{ that is, uniformly on bounded subsets in } F_n. \text{ Let } B \text{ be a bounded subset of } F, \text{ then } (F \text{ being a strict inductive limit) } B \text{ is contained in one of the } F_n, \text{ and bounded there, and so } (\xi_n)_{\alpha \in I} \text{ tends to } 0 \text{ uniformly on } B. \text{ But this means precisely that } (\xi_n)_{\alpha \in I} \text{ tends to } 0 \text{ in } W = F_b. \]

\[32 \text{Helmut H. Schaefer, } \textit{Topological Vector Spaces} \text{ (New York: the Macmillan Company, 1966), Chapter IV, } \S \text{ 6.1.} \]
This is proved by first checking for elements $t$ in $C^{(\mathbb{N})} \otimes \mathcal{W}$. Formally, under the identifications $C \otimes \mathcal{W}_p = \mathcal{W}_p$ and $C \otimes \mathcal{W} = \mathcal{W}$, (III.1.2.b) boils down to $(l_n \otimes l_p \otimes \pi_p)(l_n \otimes \pi_p) = (l \otimes \pi_p)(l_n \otimes l)$. 

By substitution (III.1.2.a) leads to

(III.1.2.c) \[ \Pi_p t = \sum_n e_n \otimes [l_n \Pi_p t] \quad t \in C^{(\mathbb{N})} \otimes \mathcal{W}. \]

Since $\Pi_p t$ belongs to $C^{(\mathbb{N})} \otimes \mathcal{W}_p$, the sum on the right has only a finite number of non-zero summands, that is, for fixed $t$, and fixed $p$, one has $l_n \Pi_p t = \pi_p l_n t = 0$ for almost all $n \in \mathbb{N}$. Invoking (III.1.2.b) one obtains

(III.1.2.d) \[ \Pi_p t = \sum_n l_n \Pi_p (e_n \otimes (l_n t)) \]

Fix $t$, and set $l_n = e_n \otimes (l_n t)$. Then $l_n$ is a sequence in $C^{(\mathbb{N})} \otimes \mathcal{W}$ such that for every $p \in \mathbb{N}$

(III.1.2.e) \[ \Pi_p t = \sum_n l_n \Pi_p l_n t \]

with $\Pi_p l_n t = 0$ for almost all $n \in \mathbb{N}$. Since $C^{(\mathbb{N})} \otimes \mathcal{W}$ has the initial topology with respect to the maps $\Pi_p$ formula (III.1.2.e) means that the sequence $\sum_n l_n$ is convergent to $t$. So, one obtains

(III.1.2.f) \[ t = \sum_n l_n \otimes (l_n t) \]

and, for every $p$, one has $\pi_p l_n t = 0$ for almost all $n$. The fact that $\pi_p l_n t = 0$ for almost all $n$ can be expressed more intrinsically by introducing the following

**Definition III.1.3**

A sequence $(w_n)_{n \in \mathbb{N}}$ in a locally convex vector space $\mathcal{W}$ will be called locally finite, if for every continuous semi-norm $\beta$ on $\mathcal{W}$ the sequence $\beta(w_n)$ is finite, that is, $\beta(w_n) = 0$ for all except a finite number of indices $n$. When the sequence $(w_n)_{n \in \mathbb{N}}$ is locally finite, the series $\sum_n w_n$ will also be called locally finite.

33 To avoid misunderstandings: a finite sequence will be one that is zero except for a finite number of indices, not a sequence that takes only a finite number of values.
This definition of local finiteness is a generalization of the usual concept of local finiteness of a sequence of distributions (see Corollary III.1.8). A locally finite sequence is also weakly finite, that is, for every \( \xi \in \mathcal{F}' \) the sequence \( \xi(w_n) \) is finite. More in general, it is obvious that if a sequence \( \{w_n\} \) in \( w \) is locally finite, its image \( \{u(w_n)\} \) under a continuous map \( u : w \to x \) is a locally finite sequence in \( x \).

In Definition III.1.3 it is enough for the sequence \( \beta(w_n) \) to be finite for \( \beta \) belonging to a fundamental system of semi-norms.

When \( w_n \) is a locally finite sequence, it follows of course that for every continuous semi-norm \( \beta \) the series \( \sum_n \beta(w_n) \) is convergent in \( \mathbb{R} \), that is to say

**Proposition III.1.4** A locally finite series \( \sum_n w_n \) is absolutely convergent.

We return to (III.1.2.f). Let \( \beta \) be a continuous semi–norm on \( w \). Then \( \beta \) allows a decomposition \( \beta = \beta \circ \pi_p \), for a certain index \( p \), and \( \beta \) a continuous semi–norm on \( w_p \) [34]. It follows that for the sequence \( l_n t \) in (III.1.2.f) one has \( \beta(l_n t) = \beta(\pi_p l_n t) = 0 \) for almost all \( n \). In other words, the sequence \( l_n t \) is locally finite.

Conversely, if \( w_n \) is an arbitrary locally finite sequence in \( w \), and \( v_n \) any sequence in a locally convex space \( V \), the sequence \( v_n \otimes w_n \) is again locally finite (in the projective tensor product \( V \otimes_\pi w \)). Indeed, when \( \alpha \) and \( \beta \) are continuous semi-norms in \( V \) and \( w \) respectively, \( (\alpha \otimes \beta)(v_n \otimes w_n) = \alpha(v_n)\beta(w_n) \) vanishes for almost all \( n \). The series \( \sum_n v_n \otimes w_n \) is therefore absolutely convergent. So, all series of the form (III.1.2.f) converge in \( c(\mathbb{N}) \otimes w \).

Finally, note that the decomposition is (III.1.2.f) is unique, because if one applies \( L_k \) to both sides one obtains indeed \( L_k(t) = \sum_n L_k(e_n \otimes f_n) = f_k \).

**Theorem III.1.5**

Let \( w \) be the strong dual of an \( \mathcal{L} \mathcal{F} \) – space.

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34 The semi-norms \( F \ni \xi \mapsto \sup_{x \in B}|\xi(x)\xi|, B \) bounded in \( F \), form a directed fundamental system of semi-norms for the strong topology on \( F' \). That \( B \) is bounded in \( F \) means \( B \) bounded in \( F_p \), for every \( p \). And so, as always in the case of projective limit topologies, a directed fundamental system of semi-norms is formed by the semi-norms \( \beta \circ \pi_p \) where for every \( p \) the \( \beta_p \) form a directed fundamental system of semi-norms for the topology on \( w_p \). Let \( \beta \) be an arbitrary continuous semi-norm, then there exist a constant \( M \) and a semi-norm \( \gamma \) on one of the \( w_p \) such that \( \beta \leq M \gamma \circ \pi_p \), so \( \beta \) vanishes on \( \ker \pi_p \), therefore \( \beta = \beta \circ \pi_p \) defines a semi-norm \( \beta^* \) on \( w_p \). Since \( \beta \circ \pi_p = \beta \leq M \gamma \circ \pi_p \), and \( \pi_p \) is onto, it follows that \( \beta \leq M \gamma \), so \( \beta^* \) is continuous.
Let \((e_n)_{n \in \mathbb{N}}\) be the canonical basis of \(\mathbb{C}^{(\mathbb{N})}\).

Then every element \(t \in \mathbb{C}^{(\mathbb{N})} \hat{\otimes} \mathcal{W}\) possesses a unique decomposition
\[
t = \sum_n e_n \otimes w_n
\]
with \((w_n)_{n \in \mathbb{N}}\) a locally finite sequence in \(\mathcal{W}\).

**Corollary III.1.6**

Let \(\mathfrak{g}\) be a complex Lie algebra, and \(\mathfrak{b}\) a subalgebra. Let \(\mathcal{W}\) be a locally convex \(\mathcal{O}(\mathfrak{b})\)-module that is the strong dual of an \(\mathcal{F}\)-space. Let \(\mathfrak{v}\) be a linear complement of \(\mathfrak{b}\) in \(\mathfrak{g}\), and \(x_1, x_2, \ldots, x_l\) a basis for \(\mathfrak{v}\). Then every element \(t\) in \(\mathcal{O}(\mathfrak{g}) \hat{\otimes} \mathcal{W}\) possesses a unique decomposition
\[
t = \sum_{\alpha \in \mathbb{N}^l} \hat{\alpha} x_1^{a_1} x_2^{a_2} \cdots x_l^{a_l} \otimes w_\alpha \quad \text{(locally finite series)}
\]
where the \(w_\alpha\) form a locally finite sequence in \(\mathcal{W}\).

It would be pleasant to have a more intrinsic explication of the topology on \(\mathcal{O}(\mathfrak{g}) \hat{\otimes} \mathcal{W}\), along the lines of Theorem III.1.1, but we do not see how this can be done. We restrict ourselves to a few comments.

Theorem III.1.5 describes the elements in \(\mathcal{O}(\mathfrak{g}) \hat{\otimes} \mathcal{W}\). The topology can be described as follows: a net \((t^\gamma)_{\gamma \in I}\) tends to zero when for every semi-norm \(\beta\) on \(\mathcal{W}\) all the corresponding nets \((\beta(w^\gamma))_{\gamma \in I}\) tend to zero in \(\mathbb{C}^{(\mathbb{N})}\).

A Fréchet space is a special type of \(\mathcal{F}\)-space. Corollary III.1.2.(ii) is consistent with Corollary III.1.6 because of the following

**Proposition III.1.7**

Let \(\mathcal{W} = \mathcal{F}'\) be the strong dual of an Fréchet space \(\mathcal{F}\). Then every locally finite sequence \((w_n)_{n \in \mathbb{N}}\) in \(\mathcal{W}\) is in fact finite.

**Proof** First note that for every \(f \in \mathcal{F}\) the sequence \(w_n(f)\) is finite. Indeed, the semi-norms \(w \mapsto \langle w, f \rangle\) define the weak topology on \(\mathcal{W} = \mathcal{F}'\), so they are a fortiori continuous semi-norms in the strong topology. Then define a bilinear form \(\mathcal{B}\) on \(\mathbb{C}^{\mathbb{N}} \times \mathcal{F}\) by the formula \(\mathcal{B}(\sum x_n e_n, f) = \sum x_n w_n(f)\).

\(\mathcal{B}\) is separately continuous. The continuity with respect to \(x\) is a consequence of the fact that the sequence \((w_n f)_{n \in \mathbb{N}}\) is finite. Then, for fixed \(x = \sum x_n e_n\) the form \(f \mapsto \sum x_n w_n(f)\) on \(\mathcal{F}\) is the
pointwise limit of the obviously continuous linear forms \( f \mapsto \sum_{n=1}^{m} x_n w_n(f) \), and is therefore continuous according to the uniform boundedness principle (valid in barrelled spaces, so in particular in Fréchet spaces).

\( \mathfrak{C}^N \) and \( F \) are Fréchet spaces, and so the separately continuous bilinear form \( \mathcal{A} \) is in fact continuous \([35]\). So, there exist a continuous semi-norm \( \beta \) on \( F \), and an integer \( N \), such that

\[
|\mathcal{A}(x,f)| \leq \sup_{n \in \mathbb{N}} |x_n| \beta(f) \quad \forall x \in \mathfrak{C}^N, \forall f \in F.
\]

This implies that \( w_n(f) = \beta(e_n, f) = 0 \) for \( n > N \), \( \forall f \in F \), in other words, \( w_n = 0 \), for all \( n > N \).

**Corollary III.1.8**

Let \( \mathcal{W} \) be the strong dual of an \( \mathcal{F} \)-space \( F = \lim_{p \to \infty} F_p \cdot (F_p)_{p \in \mathbb{N}} \) a sequence of Fréchet spaces. Then a sequence \( (w_n)_{n \in \mathbb{N}} \) in \( \mathcal{W} \) is locally finite in the sense of Definition III.1.3 if and only if for every \( p \in \mathbb{N} \) the sequence of restrictions \( (w_n|_{F_p})_{n \in \mathbb{N}} \) is finite.

So when \( \mathcal{W} \) is the space \( \mathcal{D}'(M) \) of distributions on a \( \mathcal{G}^\infty \)–manifold \( M \), the local finiteness indeed coincides with the usual notion of local finiteness, that is, a sequence of distributions is locally finite if and only if for every open and relatively compact set \( \mathcal{O} \) in \( M \) the sequence of restrictions to \( \mathcal{O} \) in \( M \) the sequence of restrictions to \( \mathcal{O} \) is finite.

**Proof of Corollary III.1.8** The restriction map \( \pi_p : E^\prime_p \longrightarrow E^\prime_p \) is continuous (for the strong topologies) and the image of a locally finite sequence under a continuous map is locally finite. But according to Proposition III.1.7 a locally finite sequence in \( E^\prime_p \) is in fact finite. So, local finiteness of \( (w_n)_{n \in \mathbb{N}} \) in \( \mathcal{W} \) implies finiteness for every restriction \( (w_n|_{F_p})_{n \in \mathbb{N}} \). The converse follows from the fact that (as mentioned before, in the paragraph following Proposition III.1.4) every continuous semi–norm \( \beta \) on \( \mathcal{W} \) allows a decomposition \( \beta = \beta\circ\pi_p \), for a certain index \( p \), and \( \beta \) a (continuous) semi–norm on \( E^\prime_p \).

We show, however, the following, by way of motivating our particular choice of tensor product topology:

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Proposition III.1.9
Whenever \( W \) is the dual of an \( \mathcal{L} \)–space that is not a Fréchet space, the inductive and projective tensor product topologies on \( C^{(\mathbb{N})} \otimes W \) do not coincide.

For example, \( W \) may be the space of distributions on a non-compact \( \Theta^{\infty} \)–manifold. The Proposition implies the inductive tensor product topology would have been the wrong choice. (This does not necessarily mean that our choice is the best, since there exist yet other tensor product topologies.)

Proof We construct a bilinear form \( \mathcal{B} \) on \( C^{(\mathbb{N})} \times W \) that is separately continuous, but not continuous. This implies that the linear form \( \mathcal{B} \) on \( C^{(\mathbb{N})} \otimes W \) that \( \mathcal{B} \) gives rise to is continuous with respect to the inductive tensor product topology, but discontinuous with respect to the projective tensor product topology, so that these topologies disagree (that is, the inductive tensor product topology is strictly finer than the projective tensor product topology).

Let \( W \) be the dual of an \( \mathcal{L} \)–space \( F \), with \( F \) not a Fréchet space. Let \( F_n \) be a defining sequence for \( F \). Take a sequence \( (f_n)_{n \in \mathbb{N}} \) in \( F \) that is not contained in any single \( F_m \). Let \( B \) be the linear map \( C^{(\mathbb{N})} \to F, B(\sum_n x_n e_n) = \sum_n x_n f_n \). Define \( \mathcal{B} : C^{(\mathbb{N})} \otimes W \to C \) by \( \mathcal{B}(x,w) = \langle Bx, w \rangle, w \in W, \langle ., . \rangle \) denoting the duality between \( F \) and \( F' = W \). The duality bracket being separately continuous, it follows that \( \mathcal{B} \) is separately continuous. For \( \mathcal{B} \) to be continuous it is necessary for \( B \) to map a suitable neighbourhood \( \mathcal{O} \) of 0 (e.g. the unit semi-ball belonging to a continuous semi-norm) onto an equicontinuous set \( B(\mathcal{O}) \) in \( F' \) (equicontinuous with respect to the strong dual \( F'_{\mathbb{b}} = W \)). An equicontinuous set in a locally convex space with respect to the strong dual is in fact the same as a bounded set [36], so \( B(\mathcal{O}) \) should be bounded. However, a basis of open neighbourhoods of 0 in \( C^{(\mathbb{N})} \) is formed by the sets \( \mathcal{O}_{\lambda} = \{ x \in C^{(\mathbb{N})} \mid \forall n \in \mathbb{N} \} \) where \( \lambda = (\lambda_n)_{n \in \mathbb{N}} \) is a series of strictly positive numbers. Clearly, \( B(\mathcal{O}_{\lambda}) = \{ \sum_{n=0}^{\infty} x_n f_n \mid x_n \leq \lambda_n, n \in \mathbb{N} \} \). In particular, since \( \mathcal{O}_{\lambda} \) contains \( \frac{1}{2} \lambda_n e_n \), for every \( n \), the set \( B(\mathcal{O}_{\lambda}) \) always contains a sequence of the form \( (\frac{1}{2} \lambda_n f_n)_{n \in \mathbb{N}} \), \( \lambda_n \) strictly positive. This sequence is unbounded, because it is not contained in any of the \( F_n \).

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[36] A basis of neighbourhoods of 0 in \( E_b \) is formed by the polars of bounded sets in \( E \), so the equicontinuous sets in \( E \) are the sets that are contained in the bipolars of bounded sets, that is, they are in fact the bounded sets.
III.2 Spaces of Distributions Concentrated on Submanifolds

A distribution on $\mathbb{R}^{l+n}$ that is concentrated on the closed submanifold $\{0\} \times \mathbb{R}^n$ (0 being the origin in $\mathbb{R}^l$, allows a unique decomposition as locally finite sum of derivatives of extensions to $\mathbb{R}^{l+n}$ of distributions on $\mathbb{R}^n$ (see also Section 0.3 in the introduction). The proof of this fact depends on the commutativeness of the transversal vectorfields $\frac{\partial}{\partial x_k}$, $1 \leq k \leq l$. Schwartz's theorem on this subject is well-known [37]. Topologically, spaces of distributions concentrated on closed regular submanifolds are closed subspaces of the distributions, and accordingly they are given the induced topology. We formulate a particular modification of Schwartz’s theorem which also describes the topological properties of the decomposition.

For this we first define the following. Let $M$ be a $\mathcal{C}^\infty$–manifold. Let $\mathcal{E}'(M)$ denote the space of compactly supported distributions on $M$. If $X$ is an operator $\mathcal{E}'(M) \rightarrow \mathcal{E}'(M)$, we will denote its transpose by $^\dagger X$. This is an operator in $\mathcal{E}(M)$, the space of smooth functions. We will call an operator $\mathcal{E}'(M) \rightarrow \mathcal{E}'(M)$ a differential operator if its transpose is so. If we introduce a differential operator, we think of it as operator defined in $\mathcal{E}'(M)$.

**Theorem III.2.1**

Let $M$ and $N$ be $\mathcal{C}^\infty$–manifolds, $N$ a closed regular submanifold of $M$. Let $\mathcal{E}'_N(M)$ denote the space of compactly supported distributions concentrated on $N$.

Let $X_1, X_2, \ldots, X_l$ be a set of commuting vectorfields defined in a neighbourhood of $N$, such that at every point of $N$ they span a subspace of the tangent space to $M$ that is supplementary to the tangent space to $N$. Let $\mathcal{S}$ be the abelian unital algebra of differential operators generated by $X_1, X_2, \ldots, X_l$. Let $\mathcal{S}'$ be the unital algebra of transpositions of these operators, an algebra acting on $\mathcal{E}'(M)$. Equip $\mathcal{S}'$ with the inductive limit topology with respect to its finite-dimensional subspaces.

Let $\imath$ be the imbedding $\imath: N \hookrightarrow M$. Let $\imath_*$ denote the push-forward $\mathcal{E}'(N) \rightarrow \mathcal{E}'(M)$, so

$$\langle \imath_* S, \Phi \rangle = \langle S, \Phi \\circ \imath \rangle, \Phi \in \mathcal{E}'(M).$$


[38] One must require that $N$ be closed. Otherwise the map $\imath$ defined in the theorem need no longer be onto.
Then the map \( \tilde{\iota}_*: \mathcal{G} \otimes \pi \mathcal{E}^l(N) \rightarrow \mathcal{E}^l(N) \), defined by
\[
\tilde{\iota}_*(Y \otimes S) := Y \iota S,
\]
is a linear topological isomorphism.

In particular, with respect to a basis \( (Y_\alpha)_{\alpha \in \mathbb{N}^l} \) of \( \mathcal{G} \), every \( T \in \mathcal{E}^l(N) \) possesses a unique decomposition as finite sum
\[
T = \sum_{\alpha \in \mathbb{N}^l} Y_\alpha \iota_\alpha S_\alpha, \quad S_\alpha \in \mathcal{E}^l(N).
\]

**Proof**  The proof we give is similar to the proof of Theorem II.5.6, using Lemma II.5.4 on homomorphisms between Fréchet–Montel spaces.

First some notation. When \( F \) is a vector space, \( F^{\mathbb{N}^l} \) will denote the space of maps \( \mathbb{N}^l \rightarrow F \), thought of as sequences \( (x_\alpha)_{\alpha \in \mathbb{N}^l} \) of vectors in \( F \), indexed over the multi–indices \( \alpha \). Let \( F^{(\mathbb{N}^l)} \) denote the subspace of finite sequences. Provide \( F^{\mathbb{N}^l} \) with a topology by identifying it with \( \prod_{\mathbb{N}^l} F \), countable product of copies of \( F \). Take for \( F \) a Fréchet-Montel space. Then \( F^{\mathbb{N}^l} \) is a Fréchet-Montel space as well.

As is clear from Corollary III.1.2.(i), its strong dual \( (F^{\mathbb{N}^l})' \) can be realized as \( (\mathcal{C}^{(\mathbb{N}^l)} \otimes \pi F_b) \), a complete space. The fact that \( \mathcal{C}^{(\mathbb{N}^l)} \) carries the inductive limit topology with respect to its finite-dimensional subspaces makes it possible to identify \( \mathcal{C}^{(\mathbb{N}^l)} \otimes \pi F_b \) by choosing a basis \( (Y_\alpha)_{\alpha \in \mathbb{N}^l} \) of \( \mathcal{G} \), \( Y_\alpha = Y_{\alpha_1}^{\alpha_2} \ldots Y_{\alpha_l}^{\alpha_l} \). The duality between \( \mathcal{G} \otimes \pi F_b \) and \( F^{\mathbb{N}^l} \) can then be described by
\[
<\sum Y_\alpha \otimes S_\alpha, (\varphi_\alpha)_{\alpha \in \mathbb{N}^l}> = \sum_{\alpha \in \mathbb{N}^l} <S_\alpha, \varphi_\alpha>, \quad \varphi_\alpha \in F, (S_\alpha)_{\alpha \in \mathbb{N}^l} \text{ a finite sequence in } F_b.
\]

Assume that the vectorfields given in the theorem are in fact defined globally (if not, replace \( M \) by a suitable open neighbourhood of \( N \)). Let \( \psi: \mathcal{E}(M) \rightarrow \mathcal{E}(N) \) denote the pull–back \( \psi^* = \Phi \circ \iota \) (i.e. \( \psi^* \) is restriction to \( N \)). This is a continuous linear map. Define the following map between Fréchet–Montel spaces:
\[
u: \mathcal{E}(M) \rightarrow \left[ \mathcal{E}(N) \right]^{\mathbb{N}^l},
\]
\[
u(\psi) = (\nu_\psi^* x_\alpha^{\alpha_1})_{\alpha \in \mathbb{N}^l}
\]
Every component \( \nu_\alpha(\psi) = \nu_\psi^* x_\alpha^{\alpha_1} \Phi \) is continuous, and so \( \nu \) is continuous. The transpose of \( \nu \) can now be calculated as:

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\[ t_u: \mathcal{G} \otimes \mathcal{D}'(N) \rightarrow \mathcal{D}'(M) \]

(III.2.1.a) \[ (t_u)(\mathcal{Y} \otimes S) = \mathcal{Y} \mid_{S}, \quad \mathcal{Y} \in \mathcal{G}, S \in \mathcal{D}'(N). \]

So \( t_u \) equals the map \( t_X \) defined in the theorem.

Locally there exist coordinates such that the vector fields look like \( x_i = \frac{\partial}{\partial x_i}, i=1,\ldots,l \), on \( U \times V, U \) and \( V \) open neighbourhoods of 0 in \( \mathbb{R}^l \) and \( \mathbb{R}^n \), and such that restricted to \( V \), the imbedding \( t \) looks like \( z \mapsto (0,z) \in U \times V \). Using polynomials in \( x_1,\ldots,x_l \) and suitable testfunctions, one shows that for every \( \psi \in \mathcal{D}(V) \) and every \( \beta \in \mathbb{N}^l \) there exists a testfunction \( \psi_\beta \in \mathcal{D}(\Omega) \) such that for every \( \alpha \in \mathbb{N}^l \) the restriction \( (x_1^\alpha x_2^\gamma \cdots x_l^\gamma) \psi_\beta \) equals \( \delta_{\alpha\beta} \psi \), where \( \delta_{\alpha\beta} = 1 \) when \( \alpha = \beta \), else 0. This means that if one views \( \psi_\beta \) as a testfunction on \( M \), and \( \psi \) as a testfunction on \( N \), one has \( u(\psi_\beta) = (\delta_{\alpha\beta} \psi)_{\alpha \in \mathbb{N}^l} \). Using a suitable \([39]\) partition of unity one shows that for every \( \psi \in \mathcal{D}(N) \) one can find \( \psi_\beta \in \mathcal{D}(\Omega) \) such that \( u(\psi_\beta) = (\delta_{\alpha\beta} \psi)_{\alpha \in \mathbb{N}^l} \). This yields that

(III.2.1.b) \[ \text{Im}\ u \text{ is dense}. \]

This implies that \( t_u \) is one–to–one.

In view of Lemma II.5.4 and (III.2.1.b) we get equivalence of the following four statements:

(III.2.1.c)

i) \( u \) is onto;

ii) \( u \) is a topological homomorphism;

iii) \( \text{Im}\ t_u \) is closed

iv) \( t_u \) is a topological isomorphism onto its range.

From Corollary III.1.2.(ii) and the description of \( t_u \) as given by (III.2.1.a) it follows that the image of \( t_u \) is the space formed by the finite sums \( \sum \mathcal{Y} \otimes v_S \). Now Schwartz’s theorem implies that the space of these sums equals exactly \( \mathcal{D}_N(M) \). A fortiori this means that \( \text{Im}\ t_u \) is closed, so that iii) is true. So the other 3 statements are true. It is (iv) that had to be proven for the theorem.

39 More precisely, choose a locally finite covering \( (\Omega_l)_{l \in \mathbb{N}} \) of \( \Omega \) by open and relatively compact sets in \( M \), of the form described. Let \( \Omega_c \) be the complement of \( \Omega \), which is open. Take a partition of unity \( (\chi_l)_{l \in \mathbb{N}} \) subordinate to the covering \( (\Omega_l)_{l \in \mathbb{N}} \), where \( \Omega_c = 1 \cup \{0\} \). Then the collection \( (\chi_l)_{l \in \mathbb{N}} \) belongs to \( \mathcal{D}(M) \) and has the property that \( \sum \chi_i \neq 1 \) in a neighbourhood of \( \Omega \). The collection of restrictions \( \chi_i \mid_{\Omega \cap \Omega_i} \) forms a partition of unity in \( \mathcal{D}(\Omega \cap \Omega_i) \), subordinate to the covering \( (\Omega_i \cap \Omega_i)_{i \in \mathbb{N}} \). Now if \( \psi \in \mathcal{D}(N) \), then \( \psi = \sum \chi_i \psi \). Since \( \chi_i \psi \) belongs to \( \mathcal{D}(\Omega_i \cap \Omega_i) \), one can find a collection \( (\psi_i,\beta \in \mathbb{N}^l) \), such that \( u(\psi_i,\beta) = (\delta_{\alpha\beta} \psi \chi_i)_{\alpha \in \mathbb{N}^l} \), and so \( u(\sum \psi_i,\beta) = (\delta_{\alpha\beta} \psi \chi_i)_{\alpha \in \mathbb{N}^l} \).
Corollaries III.2.2

Let $N$ be a closed regular submanifold of $M$, and let $X_1, X_2, \ldots, X_l$ be a set of commuting vector fields defined in a neighbourhood of $N$, and transversal to $N$, as in Theorem III.2.1. Then:

i) For every sequence of $C^\infty$ functions $(\psi_\alpha)_{\alpha \in \mathbb{N}}^l$ on $N$ there exists a $C^\infty$ function on $M$ such that for each $\alpha \in \mathbb{N}$ the restriction of $X_1^{\alpha} \ldots X_l^{\alpha} \Phi$ to $N$ is $\psi_\alpha$.

ii) Every $\mathcal{D}(\mathbb{R})$ with the property that $\Phi$ and all its derivatives vanish on $N$, can be approximated, in $\mathcal{D}(\mathbb{R})$, by a sequence $(\Phi_n)_{n \in \mathbb{N}}$ with $N \cap \text{supp}(\Phi_n) = \emptyset$, for all $n \in \mathbb{N}$.

iii) Let the manifold $M$ in Theorem III.2.1 be of the form $P \times N$, where $P$ is an $C^\infty$-manifold. Let $p$ be a point in $P$. Then $\mathcal{D}'(\{p\} \times N \times (P \times N))$ is linear–topologically isomorphic to $\mathcal{D}'(P) \otimes \pi \mathcal{D}(N)$. The canonical isomorphism is given by

$$\langle \omega \otimes S, \varphi \otimes \psi \rangle = \langle \omega, \varphi \rangle \langle S, \psi \rangle, \quad \omega \in \mathcal{D}'(P), S \in \mathcal{D}'(N), \varphi \in \mathcal{D}'(P), \psi \in \mathcal{D}(N).$$

i) extends the classical Theorem of Borel, see Corollary II.5.5.(i).

Proof of the Corollaries

i) is simply statement (III.2.1.c.i).

Statement ii) boils down to the fact that $C^\infty_0 := \{ \Phi \in C^\infty(\mathbb{R}) \mid N \cap \text{supp}(\Phi) = \emptyset \}$ is dense in $C^\infty = \{ \Phi \mid (D\Phi)|_N = 0 \text{ for all differential operators } D \}$. The point to note is that on the one hand $[C^\infty_0]^l = C^\infty_0(M)$ (by definition), and that on the other hand $C^\infty_0$ equals $\ker u$, so that $\text{Im} \, u$ is dense in $[C^\infty_0]^l$. The fact that $\text{Im} \, u$ equals $C^\infty_0(M)$ therefore implies that $[C^\infty_0]^l = [C^\infty_0]^l$, so $C^\infty_0$ is dense in $C^\infty_0$.

For iii) choose coordinates $x_1, \ldots, x_l$ in a neighbourhood of $p$, and choose for $\mathcal{G}$ the algebra of constant coefficient differential operators in $x_1, \ldots, x_l$, operating in a neighbourhood of $N$ as subspace of $P \times N$. Let $\delta_p$ be the Dirac measure at $p$. Then the map $\mathcal{G} \longrightarrow \mathcal{D}'(\{p\} \times (P \times N)), \theta \longmapsto \delta_p \otimes \theta$, is a linear–topological isomorphism, since both $\mathcal{G}$ and $\mathcal{D}'(\{p\} \times (P \times N))$ carry the inductive limit topologies with respect to their finite-dimensional subspaces (cf. Theorem II.5.6). Let $\theta$ be the (canonical) map

$$\mathcal{D}'(P) \otimes \mathcal{D}'(N) \longrightarrow \mathcal{D}'(P \times N), \langle \theta(S \otimes T), \varphi \otimes \psi \rangle = \langle S \varphi \rangle \langle T, \psi \rangle.$$ Then

$$\theta((Y \delta_p \otimes S)) = Y \delta_p S = \widetilde{\iota}_\theta(Y \otimes S)$$

where $\widetilde{\iota}_\theta$ is the isomorphism defined in Theorem III.2.1. It follows that $\theta$ restricted to $\mathcal{D}'(\{p\} \times (P \times N))$ is an isomorphism. \[58\]
Theorem III.2.3
Let $N$ and $M$ be as in Theorem III.2.1. Let $\iota_\#$ denote the push-forward $\mathcal{D}'(N) \longrightarrow \mathcal{D}'(M)$, $\langle \iota_\# S, \phi \rangle = \langle S, \phi \circ \iota \rangle$, $\phi \in \mathcal{D}(M)$. Then the identity

$$\iota_\#(\psi S) = 
\psi \iota_\# S,$$

defines a linear topological isomorphism $\iota_\#: \mathcal{G} \otimes \mathcal{D}'(N) \longrightarrow \mathcal{D}'_N(M)$.

In particular, with respect to a basis $(\psi_\alpha)_{\alpha \in \mathbb{N}}$ of $\mathcal{G}$, every $T \in \mathcal{D}'_N(M)$ possesses a unique decomposition as locally finite series

$$T = \sum_{\alpha \in \mathbb{N}} \psi_\alpha \iota_\# S_\alpha,$$

$S_\alpha \in \mathcal{D}'(N)$.

For the proof we need the following lemma.

Lemma III.2.4
Let $\mathcal{G}$ be a complex unital algebra of differential operators on a $\mathcal{C}^\infty$–manifold $M$. Let $\mathcal{G}$ possess a countable basis over $\mathcal{C}$, and equip $\mathcal{G}$ with the inductive limit topology with respect to its finite-dimensional subspaces. Then $\mathcal{D}'(M)$ and $\mathcal{G}'(M)$ are locally convex $\mathcal{G}$–modules.

The kind of algebra we think of is an algebra generated by a finite number of vector fields, such as the universal enveloping algebra of a Lie group realized as right (or left) invariant differential operators on the group.

Proof of Lemma III.2.4. The only thing that really needs to be shown is the continuity of the bilinear map $\mathcal{D} : \mathcal{G} \times \mathcal{D}'(M) \longrightarrow \mathcal{D}'(M)$, $\mathcal{D}(Y,T) = YT$.

For $K$ a compact set in $M$, let $\mathcal{D}_K(M)$ denote $\{ \varphi \in \mathcal{D}(M) | \text{ supp } \varphi \subset K \}$, a closed subspace of $\mathcal{D}(M)$. The space $\mathcal{D}(M)$ has the inductive limit topology with respect to the spaces $\mathcal{D}_K(M)$. Let $[\mathcal{D}_K(M)]'$ denote the strong dual of $\mathcal{D}_K(M)$, and let $\pi_K$ denote the projection $\mathcal{D}'(M) \longrightarrow [\mathcal{D}_K(M)]'$, that is, for $T \in \mathcal{D}'(M)$, $\pi_K T$ is the restriction of $T$ to $\mathcal{D}_K(M)$. Let $\Pi_K$ be the map $\mathcal{G} \times \mathcal{D}'(M) \longrightarrow \mathcal{G} \times [\mathcal{D}_K(M)]'$, $\Pi_K(Y,S) = (Y, \pi_K S)$. For $Y \in \mathcal{G}$ the transpose $^t Y$, a differential operator, is local, that is to say that it maps $\mathcal{D}_K(M)$ into itself. Consequently, there exists a bilinear map $\mathcal{D}_K$ such that the following diagram commutes:
Explicitly $\mathcal{B}_K$ is given by

\begin{equation}
(\text{III.2.4.a}) \quad \mathcal{B}_K(Y,S) = S^t Y
\end{equation}

where $t_Y$ denotes the transpose of $Y$, seen as operator in $\mathcal{D}_K(M)$. The bilinear map $\mathcal{B}_K$ is separately continuous. Its continuity with respect to $Y$ is a consequence of the fact that every linear map defined on $\mathcal{E}$ is continuous (Proposition II.5.7), and $\mathcal{B}_K$ is continuous with respect to $S$ because in view of (III.2.4.a) for fixed $Y$ the map $S \mapsto \mathcal{B}_K(Y,S)$ is the transpose of the map $t_Y$.

The space $\mathcal{E}$ is of the type $\mathcal{C}^\mathbb{N}$, and $\mathcal{D}_K(M)$ is a closed subspace of the reflexive Fréchet space $\mathcal{E}(M)$, so both $\mathcal{E}$ and $[\mathcal{D}_K(M)]'$ are strong duals of reflexive Fréchet spaces. Under these circumstances the Dieudonné-Schwartz Theorem guarantees the continuity of all separately continuous bilinear maps defined on the product, into any third locally convex space [40]. But then $\pi_K \mathcal{B}$ is continuous for all $K$. Since $\mathcal{D}'(M)$ carries the projective topology with respect to the maps $\pi_K$ this means that $\mathcal{B}$ is continuous.

**Proof of Theorem III.2.3**

The proof cannot be conducted along the same lines as the proof of Theorem III.2.1. In fact, if in imitation of that proof one defines a map $u: \mathcal{D}'(M) \rightarrow [\mathcal{D}(N)]^\mathbb{N}$, its transpose $t_u$ will fail to be a topological homomorphism. Instead, we reduce to Theorem III.2.1.

First of all, the final statement in the theorem is contained in Schwartz's theorem. Now if one compares Theorem III.1.5 it is clear that algebraically the map $\tilde{\mathcal{B}}_*$ is indeed a linear isomorphism. The remaining problem is to show that it is actually bicontinuous, a fact that is by no means obvious.

Assume again that the vectorfields are defined on all of $M$. Then $\mathcal{D}'(M)$ is an $\mathcal{E}$–module, according to Lemma III.2.4. So the bilinear map $\mathcal{E} \times \mathcal{D}'(M) \ni (Y,T) \mapsto YT \in \mathcal{D}'(M)$ is continuous. Since

[40] J. Dieudonné and L. Schwartz, "La dualité dans les espaces ($\mathcal{F}$) et ($\mathcal{F}'$)," *Ann. Inst. Fourier Grenoble* I (1949), 61-101, Theorem 9. This theorem actually says that a bilinear map $B: E \times F \rightarrow G$ which is hypocontinuous is automatically continuous when $E$ and $F$ are strong duals of Fréchet spaces at least one of which is reflexive ($G$ can be any locally convex space). In our case, both $E$ and $F$ are strong duals of reflexive spaces, and therefore barrelled, and so every separately continuous bilinear map into a locally convex space is hypocontinuous, that is to say, the set of linear maps $x \mapsto B(x,y)$ is equicontinuous when $y$ is restricted to a bounded set in $F$ (and *mutatis mutandis*). So, in this case separate continuity implies continuity.
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is continuous, this also settles the continuity of the bilinear map \( \mathcal{E} \times \mathcal{D}'(\mathbb{R}) \ni (Y, T) \mapsto Y \circ T \in \mathcal{D}'(M) \). But that is equivalent to the continuity of \( \tilde{\varphi} \). So this is no problem.

As to the converse, let \( p \) be a point in \( N \). Choose a neighbourhood \( \Omega \) of \( p \), with \( \Omega \) open in \( M \), such that there are coordinates in which the vectorfields look like \( X_i = \frac{\partial}{\partial x_i} \), for \( i = 1, \ldots, \ell \), on \( U \times V \), where \( U \) is an open neighbourhood of 0 in \( \mathbb{R}^\ell \), and \( V \) a neighbourhood of \( p \) in \( N \), and \( U \) and \( V \) are such that restricted to \( V \) the imbedding \( \varphi \) takes the shape \( V \ni z \mapsto (0, z) \in U \times V \). Let \( K_0 \) be an open neighbourhood of \( p \) in \( V \), relatively compact in \( V \). Let \( \xi \in \mathcal{D}(U) \) be equal to 1 in a neighbourhood of 0, and let \( \zeta \in \mathcal{D}(V) \) be equal to 1 in a neighbourhood of \( K \). Define \( \gamma(u, v) = \xi(u) \zeta(v) \), and extend \( \gamma \) as testfunction on \( M \). One then has \( \text{supp} \gamma \subset \Omega \), \( \gamma \equiv 1 \) in a neighbourhood of \( \varphi(K) \), and \( (\gamma \varphi) \circ \varphi = \zeta \varphi(X \varphi) \circ \varphi \), \( \varphi \in \mathcal{C}, \varphi \in \mathcal{D}(M) \). And so

\[
(III.2.4.b) \quad \gamma Y \circ S = Y \circ \zeta S \quad \forall \varphi \in \mathcal{C}, S \in \mathcal{D}'(\mathbb{R}).
\]

Let \( \Psi \) be the inverse of \( \tilde{\varphi} \). Take \( K \subset N \) compact. Let \( \pi_K \) be the restriction operator \( \mathcal{D}'(N) \rightarrow [\mathcal{D}'(\mathbb{R})]^* \). Let \( \Phi_K \) be the composite map \((1 \otimes \pi_K) \circ \Psi \), so

\[
\Phi_K : \mathcal{D}'(M) \rightarrow \mathcal{E} \otimes \pi \left[ \mathcal{D}'(\mathbb{R}) \right],
\]

\[
\Phi_K(Y \circ S) = Y \otimes (\pi_K S) \quad \forall \varphi \in \mathcal{C}, S \in \mathcal{D}'(\mathbb{R}).
\]

We claim that \( \Phi_K \) is continuous. To show this, decompose \( \Phi_K \) according to the following diagram:

\[
\begin{array}{ccc}
\mathcal{D}'(M) & \xrightarrow{\Phi_K} & \mathcal{E} \otimes \pi \left[ \mathcal{D}'(\mathbb{R}) \right] \\
\downarrow \Phi & & \downarrow \Phi \\
\mathcal{E}'(M) & \xrightarrow{\Phi_K} & \mathcal{E} \otimes \pi \mathcal{C}'(\mathbb{R})
\end{array}
\]

Here \( \Phi \) is the (inverse of) the isomorphism described in Theorem III.2.1, and \( \mathcal{E}'(M) \) signifies multiplication by \( \gamma \). The diagram commutes because \((1 \otimes \pi_K) \circ (1 \otimes \pi_K) = (1 \otimes \pi_K) \circ (1 \otimes \pi_K) = (1 \otimes \pi_K) \circ (1 \otimes \pi_K) = (1 \otimes \pi_K) \circ (1 \otimes \pi_K) = \Phi_K(Y \circ S)\). Here equation (III.2.4.b) is the important ingredient. The maps \( \mathcal{E}'(M) \) and \( 1 \otimes \pi_K \) are continuous for well-known reasons, \( \Phi \) is continuous according to Theorem III.2.1.

Recapitulating, we have that every point \( p \) in \( N \) possesses a compact neighbourhood \( K \) (in \( N \)) such that the map \( \Phi_K = (1 \otimes \pi_K) \circ \Psi \) is continuous. But \( \mathcal{D}'(N) \) has the initial topology with respect to the maps \( \pi_K \). This is easy to see, it simply means that a net \( (S_\alpha)_{\alpha \in \Lambda} \) in \( \mathcal{D}'(N) \) tends to 0 iff for every \( K \)
the net $(\pi K_\alpha)_{\alpha \in A}$ tends to 0 in $\mathcal{D}_K'(\mathbb{N})$. As noted in the paragraphs following Corollaries III.1.2, this means that $\mathcal{E}\hat{\mathcal{D}}'(\mathbb{N})$ has the initial topology with respect to the maps $1 \otimes \pi K : \mathcal{E}\hat{\mathcal{D}}'(\mathbb{N}) \longrightarrow \mathcal{E}\hat{\mathcal{D}}_K'(\mathbb{N})$. But that means that since all maps $(1 \otimes \pi K) \circ \Psi$ are continuous, $\Psi$ itself (the inverse of $\tilde{\nu}_*$) is continuous.

**Corollary III.2.5**

Let the manifold $\mathbb{M}$ in Theorem III.2.1 be of the form $\mathbb{P} \times \mathbb{N}$, with $\mathbb{P}$ a $C^\infty$–manifold. Let $p$ be a point in $\mathbb{P}$. Then $\mathcal{D}'(\mathbb{P} \times \mathbb{N})$ is linear–topologically isomorphic to $\mathcal{D}'(\mathbb{P}) \hat{\otimes} \mathcal{D}'(\mathbb{N})$.

The canonical isomorphism is given by

$$<w \otimes s, \varphi \otimes \psi> = <w, \varphi><s, \psi>, \quad w \in \mathcal{D}'(\mathbb{P}), s \in \mathcal{D}'(\mathbb{N}), \varphi \in \mathcal{D}(\mathbb{P}), \psi \in \mathcal{D}(\mathbb{N}).$$

**Proof** See proof of Corollary III.2.2.(iii)

Finally, the notation $\tilde{\nu}_*$ in Theorems III.2.1 and III.2.3 has been chosen for the following reasons. The pushforward $\nu_*$ is a continuous linear map $\mathcal{D}'(\mathbb{N}) \longrightarrow \mathcal{D}'(\mathbb{M})$. Equip $\mathcal{E}$ with the inductive limit topology with respect to its finite-dimensional subspaces. According to Lemma III.2.4, the space $\mathcal{D}'(\mathbb{M})$ is an $\mathcal{E}$–module. And so, in accordance with Theorem II.2.5 (with $w = \mathcal{D}'(\mathbb{N}), \varphi = \nu_*, \chi = \mathcal{E}\hat{\otimes} \mathcal{D}'(\mathbb{N}), A = \mathcal{E}, \kappa(\mathcal{S}) = 1 \otimes \mathcal{S}$), one obtains a map $\tilde{\nu}_*$ as in the following diagram.

$$\begin{array}{ccc}
\mathcal{D}'(\mathbb{N}) & \xrightarrow{\nu_*} & \mathcal{D}'(\mathbb{M}) \\
\kappa \downarrow & & \uparrow \tilde{\nu}_* \\
\mathcal{E}\hat{\otimes} \mathcal{D}'(\mathbb{N}) & & \\
\end{array}$$

Theorem III.2.3 could be reformulated by saying that the induced map $\tilde{\nu}_*$ is a topological isomorphism with image $\mathcal{D}'_N(\mathbb{M})$. Thanks to this isomorphism we can view $\mathcal{D}'_N(\mathbb{M})$ as an induced $\mathcal{E}$–module, the module induced from being simply the $\mathcal{E}$–module (that is, the linear space) $\mathcal{D}'(\mathbb{N})$.

It turns out that, in the context of Lie groups, this remains true even when there is not a complete set of transversal commuting vectorfields available. The next section shows this.
III.3  Distributions Concentrated on Subgroups

We now come to the main theorem of the first three chapters, concerning spaces of distributions carried by subgroups.

Let $G$ be a real Lie group, and let $B$ be a closed subgroup. Let $\mathfrak{g}$ and $\mathfrak{b}$ denote the Lie algebras of $G$ and $B$ respectively, and let $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{b})$ denote their respective complex universal enveloping algebras, equipped with the inductive limit topologies with respect to their finite-dimensional subspaces.

View $\mathcal{U}(\mathfrak{g})$ as the algebra generated by the right invariant vectorfields on $G$. Lemma III.2.4 shows that the space $\mathcal{D}(G)$ of distributions on $G$ is a locally convex left $\mathcal{U}(\mathfrak{g})$–module. Likewise $\mathcal{D}(B)$ is a locally convex left $\mathcal{U}(\mathfrak{b})$–module. By $\mathcal{D}_B(G)$ we denote the space of distributions on $G$ that are concentrated on $B$, which is a $\mathcal{U}(\mathfrak{g})$–submodule of $\mathcal{D}(G)$ because $\mathcal{D}_B(G)$ is invariant for all differential operators on $G$.

Let $^\star$ be the formal imbedding $B \hookrightarrow G$. Let $^\star*$ be the push-forward $\mathcal{D}(B) \longrightarrow \mathcal{D}(G)$, defined by

$$<^\star* S, \varphi> = <S, \varphi \circ \iota>$$

Then $^\star*$ is a continuous $\mathcal{U}(\mathfrak{b})$–linear map $\mathcal{D}(B) \longrightarrow \mathcal{D}(G)$. Formally, the pull-back $^\star*:\mathcal{D}(G) \longrightarrow \mathcal{D}(B), ^\star* \varphi = \varphi \circ \iota$, intertwines the left regular action of $B$ on itself and the restriction to $B$ of the left regular action of $G$ on itself. Passing over to the infinitesimal action one obtains that

$$u^\star* \varphi = ^\star* u \varphi,$$

Transposition yields that $^\star u = u^\star, u \in \mathcal{U}(\mathfrak{b})$, that is, $^\star$ is indeed $\mathcal{U}(\mathfrak{b})$–linear.

Since $^\star$ is a left $\mathcal{U}(\mathfrak{b})$–linear continuous map from the left $\mathcal{U}(\mathfrak{b})$–module $\mathcal{D}(B)$ into the left $\mathcal{U}(\mathfrak{g})$–module $\mathcal{D}(G)$, it follows from Theorem II.2.5 (on the universal property of induced locally convex modules) that there exists a continuous left $\mathcal{U}(\mathfrak{g})$–linear map $^\star : \mathcal{U}(\mathfrak{g}) \otimes \mathcal{D}(B) \longrightarrow \mathcal{D}(G)$, defined by $^\star(u \otimes S) = S^\star u$

$$\mathcal{D}(B) \xrightarrow{\iota^\star} \mathcal{D}(G)$$

(III.3.0.a)
**Theorem III.3.1  Distributions Concentrated On Subgroups**

Let $B$ be a closed subgroup of a real Lie group $G$.

Then the space $\mathcal{D}_B(G)$ of distributions concentrated on $B$ can be seen as the induced locally convex $\mathcal{U}(\mathfrak{g})$-module $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{D}'(B)$. More precisely, the map $\tilde{\omega}$ defined by (III.0.3.a) is a $\mathcal{U}(\mathfrak{g})$-linear topological isomorphism onto $\mathcal{D}_B(G)$.

Similarly, as a $\mathcal{U}(\mathfrak{g})$-module $\mathcal{L}_B(G)$ equals $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{L}'(B)$.

For the proof we first recall a few well-known facts.

**Lemma III.3.2**

Let $G$ be a Lie group, and $B$ a closed subgroup. Let $G$ and $B$ have Lie algebras $\mathfrak{g}$ and $\mathfrak{b}$. Let $l$ be a linear complement of $\mathfrak{b}$ in $\mathfrak{g}$.

Then there exists an open neighbourhood $U$ of 0 in $l$ such that the map $\Phi: U \times B \rightarrow G, \Phi(X,b) = \exp X.b$, is a diffeomorphism onto the open set $[\exp(U)].B$.

**Proof**  Let $\pi$ be the canonical projection $G \rightarrow G/B$, and $p$ the image of the identity. Then $G/B$ is a homogeneous $G$-space, and $B$ is the stability group of $p$. Choose a neighbourhood $U$ of 0 in $l$ such that the composite map $\alpha: U \rightarrow G/B, \alpha(X) = \pi(\exp X)$, is a diffeomorphism onto an open neighbourhood of $p$ [41]. We also assume $U$ is so small that the differential of the restriction of the exponential map to $l$ is one-to-one on all of $U$. Then $\Phi: U \times B \rightarrow G, \Phi(X,b) = \exp X.b$, is one-to-one. Indeed, when $\Phi(X,b) = \Phi(\tilde{X}, \tilde{b})$ it follows that $\alpha(X) = \pi(\Phi(X,b)) = \Phi(\mathfrak{b})$, $\Phi(\tilde{X}, \tilde{b}) = \alpha(\tilde{X})$, so $X = \tilde{X}$, whence $b = \tilde{b}$. Now it is easily seen that the differential of $\Phi$ is one-to-one on all of $U \times B$. Comparing dimensions one sees that $\Phi: U \times B \rightarrow G$ is an imbedding into a manifold of equal dimension, which means that it is a diffeomorphism onto an open subset of $G$.

Let $G$ be a real Lie group with Lie algebra $\mathfrak{g}$. Let $\mathcal{D}_0(\mathfrak{g})$ denote the space of distributions on $\mathfrak{g}$ concentrated at 0, and let $S(\mathfrak{g})$ denote the complex symmetric algebra over $\mathfrak{g}$. Then there is a canonical isomorphism $S(\mathfrak{g}) \rightarrow \mathcal{D}_0(\mathfrak{g})$ that associates to a monomial $X^{q_1} \ldots X^{q_k} \in S(\mathfrak{g})$ the...

---

[41] This is always possible, $B$ being closed. Compare theorems about local sections, as in V.S. Varadarajan *Lie Groups, Lie Algebras, and Their Representations* (Berlin: Springer Verlag, 1974, ed. 1984), Lemma 2.9.3 and the proof of Theorem 2.9.4.
distribution

\[ \varphi \mapsto \frac{\partial|\alpha|}{\partial t_1^{\alpha_1} \cdots \partial t_k^{\alpha_k}}|_{t_1=\cdots = t_k=0} \varphi(t_1 X_1 + \cdots + t_k X_k). \]

When \( \mu \) is a linear form on \( \mathfrak{g} \), let \( \chi_{\mu} \) denote the associated character on \( \mathfrak{g} \) defined by \( \chi_{\mu}(X) = e^{\mu(X)} \), \( X \in \mathfrak{g} \), and for \( P \in S(\mathfrak{g}) \) let \( P(\mu) \) denote the value of \( P \) at the point \( \mu \) (here \( P \) is in the usual way viewed as a polynomial on the dual of \( \mathfrak{g} \)). With these notations the isomorphism \( S(\mathfrak{g}) \to \mathcal{D}'(\mathfrak{g}) \) so defined can be defined more intrinsically by \( S(\mathfrak{g}) \ni P \mapsto P(\delta_0) \in \mathcal{D}'(\mathfrak{g}) \), \( \langle P(\delta_0), \chi_{\mu} \rangle := P(\mu) \). Since \( \mathfrak{g} \) is a linear space, \( \mathcal{D}'(\mathfrak{g}) \) has a natural structure of abelian convolution algebra, and for this convolution structure the isomorphism \( S(\mathfrak{g}) \to \mathcal{D}'(\mathfrak{g}) \) thus defined is even algebraic.

On the other hand, let \( \mathcal{D}'_e(G) \) denote the space of distributions on \( G \) that are concentrated at the neutral element \( e \). This is a locally convex convolution algebra. Let \( U(\mathfrak{g}) \) be realized as the right invariant differential operators on \( G \), that is, let \( U(\mathfrak{g}) \) be identified with its image under the infinitesimal left regular representation in \( \mathcal{D}'(G) \). Then the map \( \mathcal{U}(\mathfrak{g}) \to \mathcal{D}'_e(G) \), \( u \mapsto u \delta_e \), is an isomorphism of locally convex algebras. The isomorphism is topological in view of Theorem II.5.6, and algebraic in view of the more general identity \( u(T*S) = (uT)*S \), \( T \in \mathcal{D}'(G) \), \( S \in \mathcal{E}'(G) \), for \( u \) any right invariant differential operator.

Finally, the exponential map is a diffeomorphism on a suitable neighbourhood of \( 0 \in \mathfrak{g} \), so that the push-forward \( \exp_\ast \) is a topological isomorphism \( \mathcal{D}'_0(\mathfrak{g}) \to \mathcal{D}'_e(G) \).

**Lemma III.3.3  The Symmetrizer Map as Push-forward**

Under the (topological and algebraic) isomorphisms \( S(\mathfrak{g}) \cong \mathcal{D}'_0(\mathfrak{g}) \) and \( \mathcal{U}(\mathfrak{g}) \cong \mathcal{D}'_e(G) \) the symmetrizer map \( \lambda : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \) corresponds to the push-forward \( \exp_\ast : \mathcal{D}'_0(\mathfrak{g}) \to \mathcal{D}'_e(G) \), that is,

\[ \exp_\ast(P(\delta_0)) = \lambda(P)\delta_e \quad \text{for } P \in S(\mathfrak{g}). \]

**Proof** For \( \varphi \in \mathcal{D}(G), X \in \mathfrak{g} \), consider the expression (\#) \( \frac{d^m}{dt^m} \bigg|_{t=0} \varphi(\exp tX) \). On the one hand this can be written as \( \langle X^m(\delta_0) \varphi, \exp \rangle = \langle \exp_\ast(X^m(\delta_0)), \varphi \rangle \). On the other hand, when \( L \) denotes the left regular representation of \( G \) in \( \mathcal{D}'(G) \), (\#) can be written \( \frac{d^m}{dt^m} \bigg|_{t=0} \langle L \exp tX \delta_e, \varphi \rangle \), that is, \( \langle L X^m \delta_e \varphi, \varphi \rangle \). In other words, \( \exp_\ast(X^m(\delta_0)) \) equals \( L X^m \delta_e \), usually denoted \( X^m \delta_e \). Let \( \gamma : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \) be the map corresponding to \( \exp_\ast \) under the identifications indicated. Then \( \gamma(\varphi) = X^m \varphi, X \in \mathfrak{g} \). Since the property \( \lambda(X^m) = X^m, X \in \mathfrak{g} \), fully characterizes \( \lambda \), it follows that \( \gamma \) coincides with \( \lambda \). \[ \square \]
Proof of Theorem III.3.1 Let \( l, U, \) and \( \Phi \) be as in Lemma III.3.2. The push-forward \( \Phi_*, \) restricted to \( \mathcal{D}'(U) \otimes \mathcal{D}'(B), \) and seen as a map into \( \mathcal{D}'(G), \) can be expressed as follows, using the convolution product on \( G: \)

\[
\Phi_*(R \otimes S) = (\exp_\ast(R)) \ast (\ast_s S), \quad R \in \mathcal{D}'(U), \ S \in \mathcal{D}'(B).
\]

Indeed, let \( \mathcal{M}: G \times G \longrightarrow G, \) denote multiplication, \( \mathcal{M}(g_1, g_2) = g_1 g_2. \) Let \( \Gamma: B \times B \longrightarrow G \times G \) denote the map \( \Gamma(X, b) = (\exp X, \ast_s b). \) Then \( \Phi = \mathcal{M} \circ \Gamma. \) So for \( \kappa \in \partial \mathcal{D}(G), \ R \in \mathcal{D}'(U), \ S \in \mathcal{D}'(B), \)

\[
<\Phi_*(R \otimes S), \kappa> = <R \otimes S, (\kappa \circ \mathcal{M})> = <(\exp_\ast R) \ast (\ast_s S), \kappa>.
\]

The distribution \( \exp_\ast R \) has compact support, because \( \exp \) restricted to \( U \) is proper. In (III.3.3.a) take \( R = \mathcal{P}(\partial \mathcal{D}(g)), \ P \in \mathcal{S}(l) \) (for the notation \( \mathcal{P}(\partial \mathcal{D}(g)) \) see introduction to Lemma III.3.3). According to Lemma III.3.3, \( \mathcal{P}(\partial \mathcal{D}(g)) = \lambda(P) \delta_g, \) so that

\[
\Phi_*(P(\partial \mathcal{D}(g)) \otimes S) = \lambda(P) \ast_s S, \quad P \in \mathcal{S}(l), \ S \in \mathcal{D}'(B).
\]

Let \( J \) be the map \( \mathcal{S}(l) \otimes \mathcal{D}'(B) \longrightarrow \mathcal{D}'(U \times B), \ J(P \otimes S) = P(\partial \mathcal{D}(g)) \otimes S, \) which is, according to Corollary III.2.5, a topological isomorphism onto its image \( \mathcal{D}'(U \times B) \). On general grounds, the map \( \Phi_*, \) when seen as a map \( \mathcal{D}'(U \times B) \longrightarrow \mathcal{D}'(G), \) is a topological isomorphism onto its image \( \mathcal{D}'(G) \) [42]. It follows that \( \Phi_*, J \) is a topological isomorphism \( \mathcal{S}(l) \otimes \mathcal{D}'(B) \longrightarrow \mathcal{D}'(G), \) with image \( \mathcal{D}'(G), \) and, moreover, from (III.3.3.b) it follows that \( \Phi_*, \mathcal{A} = \kappa_\mathcal{A}, \) where \( \Lambda: \mathcal{S}(l) \otimes \mathcal{D}'(B) \longrightarrow \mathcal{U}(g) \otimes \mathcal{D}'(B) \) is the map \( \Lambda(P \otimes S) = \lambda(P) \cdot \mathcal{A} \) \( S \). One obtains the following commutative diagram, where \( \kappa \) is the canonical imbedding \( S \longrightarrow \mathcal{U}(g) \otimes \mathcal{D}'(B): \)

\[
\begin{array}{ccc}
\mathcal{D}'(B) & \xrightarrow{\kappa_*} & \mathcal{D}'(G) \\
\mathcal{U}(g) \otimes \mathcal{D}'(B) & \xrightarrow{\Lambda} & \mathcal{S}(l) \otimes \mathcal{D}'(B) \\
\end{array}
\]

The purpose is to show that \( \kappa_* \) is a topological isomorphism with image \( \mathcal{D}'(G). \) This now follows from the fact that \( \Lambda \) is a topological isomorphism (according to Proposition II.6.1 and the discussion

42 In general, if \( U \subseteq V, \) \( U \) and \( V \) open subsets of a manifold, then \( \mathcal{D}'(U) \) is not contained in \( \mathcal{D}'(V). \) However, when \( F \) is a closed subset of \( V, \) and \( U \) an open neighbourhood of \( F, \) contained in \( V, \) the space \( \mathcal{D}'(U) \) is contained not only in \( \mathcal{D}'(U), \) but also in \( \mathcal{D}'(V), \) and the topologies of \( \mathcal{D}'(U) \) and \( \mathcal{D}'(V) \) coincide on \( \mathcal{D}'(U). \) Now, being a diffeomorphism, \( \Phi_*, \) is an isomorphism \( \mathcal{D}'(U \times B) \longrightarrow \mathcal{D}'(\exp U \cdot B), \) and so \( \mathcal{D}'(\exp U \cdot B) \) induces the same topology onto \( \mathcal{D}'(G) \) as does \( \mathcal{D}'(G). \)
III.4 Distributions Concentrated on Certain Orbits in Homogeneous Spaces

In this section we use Theorem III.3.1 to obtain a similar theorem for spaces of distributions concentrated on certain orbits of homogeneous spaces.

In general, let \( \mathcal{X} \) be a \( \mathcal{C}^\infty \)–homogeneous space under the action of the real Lie group \( G \), and let \( B \) be a closed subgroup of \( G \) with closed orbit \( \mathcal{Y} := \mathcal{Y} \) for a certain \( p \in \mathcal{X} \). Let \( B_p \) be the stability group of \( p \) in \( B \), so \( B_p = \{ b \in B | bp = p \} \). Consider \( \mathcal{Y} \) in its own right as a homogeneous space for the action of \( B \), with manifold structure \( B/B_p \). Then \( \mathcal{Y} \) is a regular submanifold of \( \mathcal{X} \), in the sense that the formal inclusion map \( \mathcal{Y} \) is a regular imbedding [43].

Let \( \mathcal{D}'(\mathcal{Y}) \) be the space of distributions on \( \mathcal{Y} \), a \( \mathcal{U}(\mathfrak{b}) \)–module in the usual manner. Likewise \( \mathcal{D}'(\mathcal{X}) \) is a \( \mathcal{U}(\mathfrak{g}) \)–module. The push-forward \( \mathcal{D}'(\mathcal{X}) \rightarrow \mathcal{D}'(\mathcal{Y}) \) is evidently \( B \)–equivariant, that is, it intertwines the natural representations of \( B \) in \( \mathcal{D}'(\mathcal{X}) \) and \( \mathcal{D}'(\mathcal{Y}) \). Therefore \( \mathcal{D}'(\mathcal{X}) \) is \( \mathcal{U}(\mathfrak{b}) \)–linear as well.

In accordance with Theorem II.2.5 one obtains the usual diagram:

\[
\begin{align*}
\mathcal{D}'(\mathcal{Y}) & \xrightarrow{\mathcal{D}'(\mathcal{X})} \mathcal{D}'(\mathcal{Y}) \\
\mathcal{U}(\mathfrak{g}) \otimes \mathcal{D}'(\mathcal{Y}) & \xrightarrow{\mathcal{U}(\mathfrak{b}) \otimes \mathcal{D}'(\mathcal{Y})} \mathcal{U}(\mathfrak{b}) \otimes \mathcal{D}'(\mathcal{Y})
\end{align*}
\]

Here \( \tilde{\mathcal{D}}(\mathfrak{b}) \) is the continuous \( \mathcal{U}(\mathfrak{g}) \)–linear map determined by \( \tilde{\mathcal{D}}(\mathfrak{b}) u \otimes S = u \otimes S \), \( S \in \mathcal{D}'(\mathcal{Y}) \). Its image is \( \mathcal{D}'(\mathcal{Y}) \), that is, the space of distributions on \( \mathcal{X} \) concentrated on the submanifold \( \mathcal{Y} \).

In general, \( \tilde{\mathcal{D}}(\mathfrak{b}) \) has no reason to be one-to-one. One would like, in general, to show that \( \tilde{\mathcal{D}}(\mathfrak{b}) \) is a topological homomorphism, so that one can view the \( \mathcal{U}(\mathfrak{g}) \)–module \( \mathcal{D}'(\mathcal{Y}) \) as a quotient of \( \mathcal{U}(\mathfrak{g}) \otimes \mathcal{D}'(\mathcal{Y}) \) by a closed \( \mathcal{U}(\mathfrak{g}) \)–submodule. We deal, however, with the simpler case where \( \tilde{\mathcal{D}}(\mathfrak{b}) \) is an isomorphism. This happens when the stability group \( H = G_p \) of \( p \) in \( G \) is contained in \( B \). One obtains a generalization of Theorem III.3.1.

\[\text{II.4.0.a} \]
Theorem III.4.1  **Distributions Concentrated on Certain Orbits**

Let \( \mathcal{X} \) be a homogeneous space under the action of the real Lie group \( G \). Let \( p \) be a fixed point in \( \mathcal{X} \), with stabilizer \( H \). Let \( B \) be a closed subgroup of \( G \) which contains \( H \). Let \( \mathcal{Y} \) denote the orbit of \( p \) under the action of \( B \). Then \( \mathcal{Y} \) is a closed regular submanifold of \( \mathcal{X} \).

Let \( g \) and \( b \) denote the Lie algebras of \( G \) and \( B \) respectively.

Let \( \mathcal{D}'(\mathcal{Y}) \) denote the \( \mathcal{U}(b) \)-module of distributions on the homogeneous \( B \)-space \( \mathcal{Y} \), and let \( \mathcal{D}'(\mathcal{X}) \) denote the \( \mathcal{U}(g) \)-module of distributions on \( \mathcal{X} \) that are concentrated on \( \mathcal{Y} \).

Then the map \( \tilde{\tau}_\alpha \) in (III.4.0.a) is a topological \( \mathcal{U}(g) \)-linear isomorphism onto \( \mathcal{D}'(\mathcal{Y}) \), in other words, as a \( \mathcal{U}(g) \)-module \( \mathcal{D}'(\mathcal{Y}) \) equals \( \mathcal{U}(g) \otimes \mathcal{D}'(\mathcal{Y}) \).

Similar statements are valid for distributions with compact support.

**Comment** This theorem makes it possible to handle the infinitesimal action of the group on spaces of distributions concentrated on certain submanifolds in a way that takes account of the various commutation relations. This is useful in determining certain spaces of invariant distributions. In Bruhat theory the Schwartz form is used, and certain estimates are obtained on the dimensions of spaces of invariant distributions concentrated on submanifolds \([44]\). In principle the form used in Theorem III.4.1 allows a more detailed analysis. Applications can be found in our Chapters V, VI, and VII. Theorem V.3.1 cannot be proved without Theorem III.4.1 (so we believe).

**Proof** Let \( I, U, \) and \( \Phi \) be as in Lemma III.3.2. Consider the map \( \Psi: U \times \mathcal{Y} \longrightarrow \mathcal{X} \), defined by \( \Psi(x,y) = \exp_x \cdot y \). Compare the diagram

\[
\begin{array}{ccc}
U \times B & \xrightarrow{\Phi} & G \\
\downarrow & & \downarrow \\
U \times \mathcal{Y} & \xrightarrow{\Psi} & \mathcal{X}
\end{array}
\]

The vertical maps are canonical submersions, and \( \Phi \) is a diffeomorphism onto its image. It follows that \( \Psi \) is a one-to-one submersion and therefore also a diffeomorphism onto its image. This also shows that \( \mathcal{Y} \) is a closed, regular submanifold of \( \mathcal{X} \).

For the push-forward \( \Psi_* \) there is no immediate equivalent of (III.3.3.a), because the analogous convolution product need not be defined. However, take \( R \in \mathcal{E}'(U), S \in \mathcal{D}'(\mathcal{Y}), \phi \in \mathcal{D}(\mathcal{X}) \). Then

using the Fubini Theorem for distributions one shows that \( \langle \Psi_\ast (R \otimes S), \varphi \rangle \) equals \( \langle R, \psi \rangle \), where \( \psi \) is the \( G' \)-function on \( U \) defined by \( X \mapsto \langle \tau \exp X \cdot \ast S, \varphi \rangle \), and where \( \tau \) denotes the quasi-regular representation of \( G \) in \( \mathcal{D}'(\mathcal{F}) \). Consequently, for an arbitrary monomial \( X_{\alpha_1} \ldots X_{\alpha_k} \in S(I) \) one obtains

\[
\Psi_\ast (X_{\alpha_1} \ldots X_{\alpha_k}) = \frac{\partial |h|}{\partial t_{\alpha_1} \ldots t_{\alpha_k}} \bigg|_{t_1 = \ldots = t_k = 0} \tau \exp(t_1 X_{\alpha_1} + \ldots + t_k X_{\alpha_k}) \cdot \ast S
\]

And so

\[
\Psi_\ast (P(X \cdot \ast S)) = \lambda(P) \cdot \ast S \quad P \in S(I), S \in \mathcal{D}'(\mathcal{F}).
\]

The proof is then completed by further imitating the proof of Theorem III.3.1.

### Examples III.4.2

i) With \( H = \{e\} \) one retrieves, of course, Theorem III.3.1.

ii) Take \( B = H \), that is, let \( \mathcal{F} \) be a homogeneous space under the action of the Lie group \( G \), let \( p \) be a point in \( \mathcal{F} \), and let \( H \) be the fixed point group of \( p \). The distributions concentrated at \( p \) form the \( \mathcal{O}(g) \)-module:

\[
\mathcal{O}_p(\mathcal{F}) = \mathcal{O}(g) \cdot \mathcal{O}(h) \subseteq \mathcal{O}(g) \cdot \mathcal{O}(h)
\]

where \( \mathcal{C} \) acts as trivial \( \mathcal{O}(h) \)-module. This expression can be reworked into the shape

\[
[\mathcal{O}(g)]/[\mathcal{O}(h)],
\]

a well-known expression \([45]\). When \( H \) is connected, the subspace of \( h \)-invariant elements in \( \mathcal{O}_p(\mathcal{F}) \) corresponds to the space of \( G \)-invariant differential operators (see the paragraph following Proposition IV.9.3). When \( \mathcal{F} \) is reductive in the sense that there exists an \( \text{ad} h \)-invariant complement \( \mathcal{I} \) of \( h \) in \( b \), it follows from Theorem I.7.6 that under the isomorphism \( S(I) = S(I) \otimes \mathcal{C} \xrightarrow{\Lambda} \mathcal{O}(g) \otimes \mathcal{O}(h) \cdot \mathcal{C} \) the action of an element \( \gamma e \cdot h \) on \( \mathcal{O}(g) \otimes \mathcal{O}(h) \cdot \mathcal{C} \) corresponds to the action of \( \text{ad} X \) on \( S(I) \). With this in mind one re-establishes the familiar result that the \( G \)-invariant differential operators on \( \mathcal{F} \) correspond to the \( \text{ad} h \)-invariant elements in \( S(I) \) (still assuming \( H \) is connected).

In Chapters IV and V we look at larger classes of \( G \)-invariant linear operators in \( \mathcal{D}'(\mathcal{F}) \), associated to distributions that are no longer concentrated at one point. Theorem III.4.1 makes it possible to calculate the infinitesimal action of the group on these.

---

As to global actions, unless \( B = G \) the induced module is not invariant under the action of the \( G \).
However, for the smaller group \( B \) the global action can be expressed in terms of the module.

**Definition III.4.3**

*Let \( B \) be a closed subgroup of the Lie group \( G \). Let \( V \) be a \( \mathcal{U}(g) \)-module that is also a \( \mathcal{O}(g) \)-module. Then \( V \) is called a \( \mathcal{U}(g), B \)-module when the two module structures are compatible, in the sense that*

\[
\begin{align*}
\text{i)} & \quad X v = \left. \frac{d}{dt} \right|_{t=0} (\exp tX).v & \quad \forall x \in B, v \in V \\
\text{ii)} & \quad b \cdot v = (Ad_b \cdot \cdot \cdot )b v & \quad \forall b \in B, \forall v \in \mathcal{U}(g), v \in V.
\end{align*}
\]

This is not quite the standard definition by Lepowski of a \( \mathcal{O}(g), M \)-module, \( M \) a compact subgroup of \( G \). That is to say, apart from the compatibility conditions in Definition III.4.3 in the Lepowski definition it is required that every vector be \( M \)-finite \cite{46}. We will not need that property. Our purpose is merely to make explicit how the natural \( \mathcal{O}(g), B \)-module structure of \( \mathcal{D}'(\mathcal{O}(g)) \) is reflected in its form as induced module. This is expressed in the following easy proposition, which will be useful later on.

**Proposition III.4.4.** \( \mathcal{D}'(\mathcal{O}(g)) \) as \( \mathcal{O}(g), B \)-module

\[
\begin{align*}
\text{i)} & \quad \mathcal{D}'(\mathcal{O}(g)) \text{ is a } B \text{-invariant subspace of } \mathcal{D}'(\mathcal{O}(g)), \text{ and is a } \mathcal{O}(g), B \text{-module.} \\
\text{ii)} & \quad \text{Let } \tau \text{ and } \tilde{\tau} \text{ denote the quasi-regular representations of } B \text{ in } \mathcal{D}'(\mathcal{O}(g)) \text{ and } \mathcal{D}'(\mathcal{O}(g)) \text{ respectively. Then under the isomorphism } \tilde{\tau}_*: \mathcal{U}(g), \mathcal{O}(g) \mathcal{D}'(\mathcal{O}(g)) \rightarrow \mathcal{D}'(\mathcal{O}(g)) \text{ the representation } \tau \text{ of } B \text{ in } \mathcal{D}'(\mathcal{O}(g)) \text{ corresponds to the representation } b \mapsto Ad(b)\mathcal{O}(g)\tilde{\tau}_b \\
\text{of } B \text{ in } \mathcal{U}(g), \mathcal{O}(g) \mathcal{D}'(\mathcal{O}(g)). \text{The latter is defined by}
\end{align*}
\]

\[
(\text{III.4.4.a}) \quad (Ad(b)\mathcal{O}(g)\tilde{\tau}_b)(u, \Phi, S) = (Ad(b)(u), \mathcal{O}(g)\tilde{\tau}_b S) & \quad \forall b \in B, u \in \mathcal{U}(g), S \in \mathcal{D}'(\mathcal{O}(g)).
\]

\[
\text{iii)} & \quad \text{Similar statements are valid for } \mathcal{D}'(\mathcal{O}(g)) \text{ instead of } \mathcal{D}'(\mathcal{O}(g)).
\]

**Proof** For i) one merely has to note that \( \mathcal{D}'(\mathcal{O}(g)) \) itself is a \( \mathcal{O}(g), G \)-module, and that, since \( \text{supp}(\tau_g T) = g.\text{supp} T \), \( g \in G, T \in \mathcal{D}'(\mathcal{O}(g)) \), it follows that \( \mathcal{D}'(\mathcal{O}(g)) \) is a \( B \)-invariant subspace.

ii) is simply a matter of computation: take $b \in B, X \in \mathfrak{g}, S \in \mathcal{O}^\prime(\mathcal{Y})$. Since $\tau_b \circ X = (\text{Ad} X) \circ \tau_b$ (as operators in $\mathcal{O}^\prime(\mathcal{Y})$), and $\tau_b \circ_\ast S = \circ_\ast \tau_b S$, it follows that $\tau_b (X \circ_\ast S) = (\text{Ad} X) \circ_\ast \tau_b S$. This identity is easily extended to all of $\mathcal{O}(\mathfrak{g})$, so

\begin{equation}
\tau_b (u \circ_\ast S) = (\text{Ad} X) \circ_\ast \tau_b S \tag{III.4.4.b}
\end{equation}

which can be rewritten $\tau_b (u \circ_\ast S) = \circ_\ast (\text{Ad} X) \circ_\ast \tau_b S$. 

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