Convolution on homogeneous spaces
Capelle, Johan

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Document Version
Publisher's PDF, also known as Version of record

Publication date:
1996

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):
Chapter II

Topological Tensor Products over Algebras
Induced Locally Convex Modules

II.1 Topological Tensor Products over Algebras

The purpose of this chapter is roughly to equip the algebraic objects dealt with in Chapter I with suitable topologies. We will not attempt to do this in the most general setting of tensor products over rings, because in order to show the existence of certain topological tensor products we want to use some of the standard (i.e. Grothendieck’s) theory for topological tensor products of locally convex vector spaces. This means we have to assume that the rings involved contain, injectively, the field of complex numbers $\mathbb{C}$, or of the real numbers, in their centres. That is, the modules over these rings are locally convex vector spaces. The rings themselves become locally convex unital associative algebras.

**Definition II.1.1**  
*Locally Convex Associative Algebras*  
A complex locally convex unital associative algebra $A$ is a complex locally convex vector space $A$ equipped with an associative multiplication $A \times A \rightarrow A$, $(a,b) \mapsto ab$, containing a unit $1$, and such that the multiplication map $(a,b) \mapsto ab$ is a continuous bilinear map.

The field $\mathbb{C}$ is contained in $A$ through the imbedding $\mathbb{C} \ni a \mapsto a1$.

The essential point in the definition is the requirement of continuity of the bilinear multiplication map. A bilinear map that is continuous (in the usual sense of continuity) is said to be *jointly* continuous, to distinguish it from a separately continuous map, that is, one for which the maps obtained by fixing either of the variables are continuous. It is the distinction between jointly and separately continuous maps that plays an important part in this chapter (as always in the context of topological tensor products), and it turns out that for our purposes Definition II.1.1 is convenient. One should note that, for instance, the associative convolution algebra of compactly supported smooth
functions on a non-compact group $G$ is not a locally convex algebra in the sense of our definition, because the map $(\Phi, \Psi) \mapsto \Phi \ast \Psi$, $\Phi, \Psi \in \mathcal{D}(G)$, is not jointly continuous. On the other hand, the convolution algebra of compactly supported distributions on any Lie group is topological in the sense just defined (one can prove this fact along the lines of the proof of Lemma III.2.4 further down). Another example of a topological algebra in the sense defined here is the algebra $\mathcal{D}'_c(\mathbb{R})$ of distributions on $\mathbb{R}$ with supports in the closed interval $[0, \infty)$. We will stick to joint continuity.

**Definition II.1.2** A locally convex left module $V$ over a complex locally convex associative unital algebra $A$, is a left module over $A$, equipped with a topology, such that

i) over the field $\mathbb{C}$ (contained in $A$) the module $V$ is a locally convex vector space

ii) the external multiplication map $A \times V \ni (a, w) \mapsto aw \in V$ is a continuous bilinear map.

We assume all spaces introduced to be Hausdorff. This is only natural, but in the coming sections we will sometimes have to be more careful because some of the objects constructed are not automatically Hausdorff. Also, the spaces and algebras are usually assumed to be complex.

We have the following analogue of Theorem I.2.1.

**Theorem II.1.3** Let $A$ be a locally convex associative unital algebra. Let $V$ be a locally convex right module over $A$, $W$ a locally convex left module over $A$. Then there exist a complete locally convex Hausdorff space $X$, and an $A$–balanced continuous bi–linear map $\tau: V \times W \longrightarrow X$, such that every $A$–balanced continuous bi–linear map $\mathcal{B}: V \times W \longrightarrow Z$, into another complete locally convex Hausdorff space $Z$, decomposes as $\mathcal{B} = \alpha \tau$, where $\alpha$ is a uniquely determined continuous linear map $X \longrightarrow Z$.

$$
\begin{array}{ccc}
V \times W & \xrightarrow{\mathcal{B}} & Z \\
\tau \searrow & \swarrow_{\alpha} & \\
& X & 
\end{array}
$$

Moreover, the pair $X, \tau$ is unique in the sense that whenever the two pairs $X, \tau$ and $\tilde{X}, \tilde{\tau}$ both have the specified properties, there exists a linear topological isomorphism $X \longrightarrow \tilde{X}$ intertwining $\tau$ and $\tilde{\tau}$.
Definition II.1.4

X will be called the complete projective tensor product of V and W over A, and will be denoted $V \hat{\otimes}_A W$. The image $\tau(v,w)$ will be denoted $v \hat{\otimes}_A w$.

Proof of the theorem

Uniqueness is proved as usual. To show existence, first use the tensor–product–in–stages construction of Theorem I.5.1 (take $P = \mathbb{C}.1 \subseteq A$) to obtain the following commutative diagram of non-topological objects and maps (should the algebras and modules be real, read $\mathbb{R}$ for $\mathbb{C}$).

Here $J_\circ$ is the linear subspace of $V \hat{\otimes}_A W$ additively generated by elements of the form $va - v \hat{\otimes}_A aw$, $v \in V$, $w \in W$, $a \in A$.

Equip $V \hat{\otimes}_A W$ with the projective tensor product topology. Now $\theta$ being continuous, and $Z$ being complete, $l_\circ$ extends uniquely as a continuous map $l : V \hat{\otimes}_A W \rightarrow Z$, with $\theta = l \circ l_\circ$ [9]. Let $J$ be the closure of $J_\circ$ in $V \hat{\otimes}_A W$. Then $l_\circ$ vanishes on $J_\circ$, so the continuous map $l$ vanishes on $J$.

Therefore, there is a continuous map $\alpha_{\circ \circ} : V \hat{\otimes}_A W / J \rightarrow Z$ such that $\alpha_{\circ \circ} \pi = l$. This almost proves the theorem. However, in general the quotient of a complete topological vector space by a closed subspace may be incomplete in itself (unless the space is Fréchet, in which case the completeness of the quotient is guaranteed) [10]. One therefore has to extend $\alpha_{\circ \circ}$ to the completion $(V \hat{\otimes}_A W / J)^\sim$, with imbedding $\kappa$, of the Hausdorff space $V \hat{\otimes}_A W / J$. This leads to the following diagram:

To prove the theorem take $X = (V \hat{\otimes}_A W / J)^\sim$, and $\tau = \kappa \pi \tau \circ \circ$. Now $\theta$ determines $\alpha$ uniquely, because $\theta$ determines $l_\circ$ uniquely according to the general properties of tensor products, $l_\circ$ determines $l$ uniquely because $V \hat{\otimes}_A W$ is the completion of $V \hat{\otimes} W$ (i.e. $V \hat{\otimes} W$ equipped with the standard projective tensor product topology (as introduced by Grothendieck [11]), $l$ determines $\alpha_{\circ \circ}$ uniquely because $\pi$


is onto, and \( \alpha_{\infty} \) determines \( \alpha \) uniquely because \( (V \hat{\otimes} W / J) \) \( \hat{\sim} \) is the completion of \( V \hat{\otimes} W / J \). The continuity of \( \tau \) is simply a consequence of the continuity for the maps \( \tau_{\mathcal{C}}, \pi, \) and \( \kappa \).

When \( \mathbb{A} = \mathbb{C} \) we know that \( V \hat{\otimes} W \) is a dense subspace of \( V_{\mathbb{C}} \hat{\otimes} W \), more precisely, \( V \hat{\otimes} W \) is by definition the completion of \( V_{\mathbb{C}} \hat{\otimes} W \) when the latter is equipped with the projective tensor product topology. The proof given so far does by no means show that this is the case for more general \( \mathbb{A} \).

Nevertheless, there always exist a canonical map \( j : V \hat{\otimes} W / J \mathcal{O} = V_{\mathbb{C}} \hat{\otimes} W \to V_{\mathbb{C}} \hat{\otimes} W = (V \hat{\otimes} W / J) \).

Using the realization \( V_{\mathbb{A}} \hat{\otimes} W := (V \hat{\otimes} W / J) \) above, the map \( j \) is defined by \( j(v \hat{\otimes} w \mod J) = \kappa(v \hat{\otimes} w \mod J) \).

We have not shown that \( j \) is one-to-one (compare Corollary II.1.9).

Using the map \( j \) the behaviour of the tensor product \( V \hat{\otimes} W \) and the completed topological tensor product \( V_{\mathbb{A}} \hat{\otimes} W \) can be related as in the following diagram:

\[
\begin{array}{ccc}
V \otimes W & \overset{\varrho}{\longrightarrow} & Z \\
\downarrow \tau_{\mathcal{C}} & & \uparrow \alpha \\
V \hat{\otimes} W & \overset{j}{\longrightarrow} & V_{\mathbb{A}} \hat{\otimes} W
\end{array}
\]

From this diagram the fact that \( \tau \) is bilinear and \( \mathbb{A} \)-balanced is seen to be an immediate consequence of the fact that \( \tau_{\mathcal{C}} \) has such properties, \( j \) being linear.

Forgetting about the specific realization given, we can say the following. Consider the canonical map \( \pi : V \hat{\otimes} W \longrightarrow V_{\mathbb{A}} \hat{\otimes} W \), whose existence is ensured by the continuity and bilinearity of the map \( \tau : V \otimes W \longrightarrow V \hat{\otimes} W \). In the proof above, we have not shown that \( \pi \) is onto, and in general it is merely a homomorphism onto a dense subspace of \( V \hat{\otimes} W \). Its kernel is always \( J \), the closure of the additive hull of tensors \( v \mathfrak{a} \otimes w - v \otimes aw \).

\( \tau_{\mathcal{C}} \) spans \( V \hat{\otimes} W \) additively, so \( a fortiori \) linearly. We will see further on that the image of \( j \) is dense, and so that the image of \( \tau \) is dense.

The realization of the completed tensor product described above (as the completion of a quotient of a completion) is not quite satisfactory. We now proceed to describe \( V \hat{\otimes} W \) as the Hausdorff completion of \( V_{\mathbb{A}} \hat{\otimes} W \), with respect to a certain topology defined by semi-norms.

---

As usual, let $V \otimes \pi W$ denote the tensor product $V \otimes W$ equipped with the projective tensor product topology. Let $V \otimes \pi W$ denote the quotient $V \otimes W = V \otimes \pi W / J_\otimes$ equipped with the quotient topology. Let $\alpha$ and $\beta$ be semi-norms on $V$ and $W$. Then the seminorm $\alpha \otimes \beta$ on $V \otimes W$ will be defined by

$$
(\alpha \otimes \beta)(s) = \inf \left\{ \sum \alpha(x_i)\beta(y_i) \mid s = \sum x_i \otimes y_i \right\}.
$$

By $\alpha \otimes A \beta$ we will denote the quotient of this semi–norm with respect to the subspace $J_\otimes$. It follows that the expression for $\alpha \otimes A \beta$ becomes entirely similar:

$$
(\alpha \otimes A \beta)(s) = \inf \left\{ \sum \alpha(x_i)\beta(x_i) \mid s = \sum x_i \otimes A y_i \right\}.
$$

Let $(\alpha_i)_{i \in I}$ and $(\beta_j)_{j \in J}$ be directed fundamental systems of continuous semi-norms on $V$ and $W$ respectively. Then the family of seminorms $(\alpha_i \otimes \beta_j)_{i \in I, j \in J}$ is a directed fundamental system of continuous semi-norms for the projective tensor product topology on $V \otimes \pi W$ [12]. Consequently, the family of semi-norms $(\alpha_i \otimes \beta_j)_{i \in I, j \in J}$ is a directed system of continuous semi-norms for the topology on $V \otimes \pi W = V \otimes \pi W / J_\otimes$. As always, the topology on this space is Hausdorff if and only if $J_\otimes$ is closed in $V \otimes \pi W$.

To say more, we use a definition of the completion of a possibly non-Hausdorff space, again in terms of universal properties.

**Definition II.1.5**

*Let $E_\otimes$ be a topological vector space, not necessarily Hausdorff. Then a complete Hausdorff topological vector space $E$, together with a continuous linear map $\iota: E_\otimes \longrightarrow E$ will be called a Hausdorff completion of $E_\otimes$ iff every other continuous linear map $L: E_\otimes \longrightarrow F$, with $F$ a complete Hausdorff topological vector space, has a unique extension $L_\otimes: E_\otimes \longrightarrow F$ such that $L_\otimes = \iota L$.*

We use the term *extension* here in a broad sense, because $\iota$ may fail to be one-to-one.

---

12 See Alexandre Grothendieck, *Produits Tensoriels Topologiques et Espaces Nucléaires*, Chapitre I, §1, n° 1, Proposition 2, Part 2. **Fundamental** (with respect to a topology) means that the semi-norms generate the topology in question. **Directed** means that for every couple of members $p,q$ of the family there is another member $r$ that dominates both, in the sense that for some constant $M$ one has $p \leq Mr$, and $q \leq Mr$. The directedness of the system $\alpha_i \otimes \beta_j$ (not mentioned by Grothendieck) is evident, since if $\alpha \otimes A \beta$ and $\beta \otimes A \beta$, then it is obvious from the definition that $\alpha \otimes A \beta \leq A \alpha \otimes A \beta$. The directedness is crucial for the system of quotient semi-norms to be fundamental.
Proposition II.1.6

i) Every topological vector space has a unique Hausdorff completion

ii) If $\tau: E \longrightarrow \tilde{E}$ is a Hausdorff completion, then
   
a) $\tau$ is a homomorphism \[13\] onto a dense subspace of $E$
   
b) $\tau$ is one-to-one if and only if $E_{\circ}$ is Hausdorff.

The uniqueness in i) means that whenever $\tilde{\tau}: E_{\circ} \longrightarrow \tilde{E}$ is another Hausdorff completion of $E_{\circ}$, there exists a (necessarily unique) linear bicontinuous isomorphism $E \longrightarrow \tilde{E}$ intertwining $\tau$ and $\tilde{\tau}$.

Every topological vector space carries a canonical uniform structure (so that it makes sense to say that the space is complete). One proves Proposition II.1.6 by using the definition of completion of a possibly non-Hausdorff uniform space; such a completion is a Hausdorff space by definition \[14\] (without this requirement, one would not have uniqueness of completion).

Lemma II.1.7

Let $E_{\circ}$ be a topological vector space, not necessarily Hausdorff, with Hausdorff completion $\tau: E_{\circ} \longrightarrow E$. Let $J_{\circ}$ be a subspace of $E_{\circ}$, and let $J$ be the closure of $J_{\circ}$ in $E$. Let $\kappa: E/J \longrightarrow F$ be the Hausdorff completion of $E/J$.

Then $F$ is the Hausdorff completion of the four spaces $E_{\circ}/J_{\circ}, E/J_{\circ}, E_{\circ}/(J \cap E_{\circ}), E/J$, with maps into $F$ as indicated in the following diagram of canonical maps.

\[
\begin{array}{ccc}
E_{\circ}/J_{\circ} \overset{\sim}{\longrightarrow} E/J_{\circ} \longrightarrow E/J \longrightarrow \langle E/J_{\circ}/(J/J_{\circ}) \rangle \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\langle E_{\circ}/(J \cap E_{\circ})/J_{\circ} \rangle \longrightarrow E_{\circ}/(J \cap E_{\circ}) \longrightarrow E/J \longrightarrow \langle E/J_{\circ}/(J/J_{\circ}) \rangle \\
\end{array}
\]

(the bottom row maps are one-to-one homomorphisms between Hausdorff spaces).

13 That is, the associated linear isomorphism $E_{\circ}/\text{Ker}\tau \longrightarrow \text{Im}\tau$ is bicontinuous when these spaces are equipped respectively with the quotient topology and the topology induced by $E$. Equivalently, $\tau$ is continuous and open as a map onto $\text{Im}\tau$ when the latter is equipped with the topology induced by $E$.

In particular, let \( \theta \) be the canonical mapping \( \theta: E_\omega / J_\omega \rightarrow F / J \), and \( j = \xi \theta: E_\omega / J_\omega \rightarrow F \) the Hausdorff completion of the topological vector space \( E_\omega / J_\omega \). Then \( j \) is one-to-one if and only if \( J_\omega \) is closed in \( E_\omega \).

**Proof** The way things have been defined now, the proof is straightforward. We prove only that \( \xi \theta: E_\omega / J_\omega \rightarrow F \) is a Hausdorff completion. Let \( \pi_\omega \) be the canonical map \( E_\omega \rightarrow E_\omega / J_\omega \). Then \( \theta \) is defined by \( \theta \pi_\omega = \pi \). Let \( L_\omega \) be a continuous linear map \( E_\omega / J_\omega \rightarrow Z \), where \( Z \) is a complete Hausdorff space. One obtains by composition with \( \pi \) a continuous map \( E_\omega \rightarrow Z \), and then by extension a continuous map \( L_1: E \rightarrow Z \), such that \( L_1 \pi = L_\omega \pi_\omega \). \( L_1 \) vanishes on \( \nu(J_\omega) \), so, by virtue of its continuity and the fact that \( Z \) is Hausdorff, it also vanishes on \( J_\omega \). One obtains a continuous map \( E / J \rightarrow Z \), and then again by extension a continuous map \( L: F \rightarrow Z \), such that \( Lj = L \xi \theta = L_\omega \). Uniqueness of \( L \) is the result of a chain of uniquenesses (as in the proof of Theorem II.1.3).

The statement in the lemma concerning the injectivity of \( j \) is a consequence of Proposition II.1.6.(ii)(b).

The reason why we prove this lemma is the following. Though one could imagine various ways of defining a complete topological vector space \( V \#_A W \# \) (one could start with a topology \( V \#_A W \), then divide by a suitable ideal, then take a completion, or do these things in another order), all reasonable constructions lead to the same completed space. We do not work with the non-completed versions, which might possibly be non-Hausdorff anyway.

Lemma II.1.7, and the considerations in the proof of, and following, Theorem II.1.3, immediately lead to the following

**Theorem II.1.8**

Under the assumptions of Theorem II.1.3, let \( \#_A V \# A W \) denote the tensor product \( \#_A V \# A W \) equipped with the tensor product topology generated by the semi–norms

\[
(\alpha \#_A \beta)(s) = \inf \{ \sum \alpha(x_i)\beta(y_i) : s = \sum x_i \#_A y_i \}, \quad s \in \#_A V \# A W,
\]

for \( \alpha \) and \( \beta \) continuous semi–norms on \( V \) and \( W \).

Then \( \#_A V \# A W \) is the Hausdorff completion of \( \#_A V \# A W \).

The tensor map is \( \tau = j \tau_\omega \), where \( \tau_\omega \) is the canonical tensor map \( V \times W \rightarrow \#_A V \# A W \), and \( j: \#_A V \# A W \rightarrow \#_A V \# A W \) the Hausdorff completion of \( \#_A V \# A W \).
Corollary II.1.9

i) The map $j: V \hat{\otimes}_A W \longrightarrow V \hat{\otimes}_A W$ (see the final section of the proof of Theorem II.1.3) is one-to-one if and only if $V \hat{\otimes}_A W$ is Hausdorff when equipped with the topology indicated, which happens if and only if the subspace $J_\omega$ of $V \hat{\otimes}_C W$ generated by tensors $va \otimes w - v \otimes aw$ is closed in $V \hat{\otimes}_C W$.

ii) The image of the canonical map $\tau: V \hat{\otimes}_A W \longrightarrow V \hat{\otimes}_A W$ additively generates a dense subspace of $V \hat{\otimes}_A W$.

Proof of Corollary

Part i) is simply an application of Proposition II.1.6.(ii)(b), while ii) follows from Proposition II.1.6.(ii)(a), and the fact that the image of $\tau_\omega$ additively generates the whole of $V \hat{\otimes}_A W$.

Just as in the case of the ordinary projective abelian tensor product (of linear spaces, that is) the projective tensor product topology on $V \hat{\otimes}_A W$ is the finest for which the canonical $A$-balanced bilinear map $\tau$ is continuous. However, it is important to be aware of two important differences between this and the ordinary projective tensor product.

In the first place the semi-norms $\alpha \otimes \beta$ do not have the usual cross-norm property, that is, $(\alpha \otimes \beta)(v \otimes w)$ need by no means equal $\alpha(v)\beta(w)$. One decisive reason for this is that $v \otimes w$ may equal 0 without $v$ or $w$ being 0. For example, in the module $\bar{U}(g) \otimes W$ (see Section II.6) if $w \in W$ is a non-zero vector in $W$ killed by a particular $X \in \bar{U}(b)$, then

$$X \otimes w = 1 \otimes Xw = 0.$$ $\bar{U}(b)$ $\bar{U}(b)$

Choose continuous semi-norms $\alpha$ and $\beta$ with $\alpha(x)\neq 0$, $\beta(w)\neq 0$ to obtain an example where $(\alpha \otimes \beta)(v \otimes w) = 0 \neq \alpha(v)\beta(w)$.

Another matter is that the topology of $V \hat{\otimes}_A W$ might not be Hausdorff (though we have been unable to construct examples where it is not). In the example we will eventually be interested in, we prove that $V \hat{\otimes}_A W$ is a Hausdorff space, and $V \hat{\otimes}_A W$ its completion (see Chapter III).

Finally, we have chosen to deal with modules where the external multiplication is continuous, not merely separately continuous, and consistent with this we have chosen to use the projective tensor product topology on $V \otimes W$ and the quotient $V \hat{\otimes}_A W$. A theory involving separately continuous external multiplication is easily set up along the same lines, using Grothendieck's inductive tensor product topology, completions $V \hat{\otimes} W$, quotients, and completions. In Proposition 26
III.1.9 it will appear that in the case we are interested in this would lead to the wrong objects. There is still the possibility that other tensor product topologies would be even more suitable, however.

Conventions

From now on, \( E \otimes F \) will denote the tensor product over the complex numbers, \( E \otimes_F := E \otimes_F \mathbb{C} \). The projective tensor product topology will be the only tensor product topology considered.

II.2 Induced Locally Convex Modules over Algebras

We do not try to prove topological analogues of all the theorems in Chapter I. The following analogue of Theorem I.6.1, however, is important, and easy to show.

**Proposition II.2.1** Let \( A \) be a locally convex associative unital algebra, and let \( W \) be a complete Hausdorff locally convex module over \( A \). Then the algebraic isomorphism \( \gamma: A \otimes A W \rightarrow W, \gamma(a \otimes w) = aw, a \in A, w \in W, \) is a topological isomorphism when \( A \otimes A W \) is equipped with the projective tensor product topology. In particular, \( A \otimes A \pi W \) is Hausdorff and complete.

**Proof** Let \( M \) be the external multiplication map \( M(a, w) = aw, a \in A, w \in W \). Since \( M \) is continuous (by our definition of locally convex modules), and bilinear and \( A \)-balanced, it defines a unique continuous map \( \gamma: A \otimes A W \rightarrow W, \) with the property that \( \gamma(a \otimes w) = aw \). Moreover, the well-defined map \( \beta: W \rightarrow A \otimes A W, \beta(w) = 1 \otimes w, \) is continuous (because the tensor map \( \tau(a, w) = a \otimes w \) is continuous), and obviously \( \gamma \circ \beta \) is the identity map on \( W \). Moreover, \( \beta \gamma: A \otimes A W \rightarrow A \otimes A W \) is the identity on primitive tensors \( a \otimes w \), so it equals the identity on the dense subspace additively generated by these primitive tensors, and so, being continuous, it simply is the identity. So \( \gamma \) is a bicontinuous linear isomorphism.

Finally, let \( \gamma_0 \) be the map \( A \otimes A \pi W \rightarrow W, \gamma_0(a \otimes w) = aw \). This is a linear isomorphism, according to Theorem I.6.1. Apparently, \( \gamma \) is the unique continuous extension of \( \gamma_0 \) to \( A \otimes A W \) (see Theorem II.1.8). But \( \gamma_0 \) is already a linear isomorphism. This means that \( A \otimes A \pi W \) is a complete Hausdorff space.
The analogues of Theorem I.3.3, where \( V \) and \( W \) are locally convex bi–modules, and Proposition I.7.1 on associativity of tensor products, are easily proved, once the following essential point has been established.

**Proposition II.2.2**

Let \( B \) and \( A \) be topological associative unital algebras, and let \( V \) be a locally convex \( (B,A) \)-bi–module, \( W \) a locally convex left \( A \)-module.

Then \( V \hat{\otimes} W \) becomes a locally convex left \( B \)-module under the external multiplication defined by

\[
\hat{b}(v \hat{\otimes} A \hat{w}) = (bv) \hat{\otimes} A \hat{w}.
\]

This proposition implies that when forming tensor products one remains within the category of locally convex modules as defined in II.9.2, and that expressions like \( Z \hat{\otimes} A (V \hat{\otimes} A W) \) (compare Lemma I.7.1) are meaningful within this category.

For the proof we need the following two elementary lemmas on topological vector spaces.

**Lemma II.2.3**

Let \( E \) and \( F \) be topological vector spaces. Let \( M \) and \( N \) be subspaces of \( E \) and \( F \). Then the canonical linear isomorphism \( (E/M)\times(F/N) \rightarrow (E\times F)/(M\times N) \) is a topological isomorphism.

**Lemma II.2.4**

Let \( E, F \) and \( Z \) be locally convex spaces. Let \( \hat{\otimes} \) be a separately continuous bilinear map \( E \times F \rightarrow Z \), continuous on \( E_\circ \times F_\circ \), where \( E_\circ \) and \( F_\circ \) are dense in \( E \) and \( F \) respectively. Then \( \hat{\otimes} \) is continuous on \( E \times F \).

**Proofs of the lemmas** For the first lemma, note that the canonical map \( \Phi : E \times F \rightarrow (E/M) \times (F/N) \) has kernel \( M \times N \), and by factorization over the quotient \( E \times F / M \times N \) gives rise to the isomorphism \( \theta : E \times F / M \times N \rightarrow (E/M) \times (F/N) \), \( \Phi = \theta \pi \). Then \( \Phi \) is open, continuous, and onto, because it is the direct product of two maps that are open, continuous, and onto. The canonical projection \( \Pi \) is open, continuous and onto. It follows that \( \theta \) is bicontinuous.

For the second lemma, use the fact that every continuous semi–norm on a dense subspace \( E_\circ \) of a locally convex space \( E \) (with induced topology) is the restriction of a continuous semi-norm on
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E. (For example, a continuous semi-norm on a locally convex space \( E \)) is uniformly continuous, and so extends in a unique fashion to \( E \). Let \( r \) be a continuous semi–norm on \( E \). Since \( \mathcal{O} \) is continuous on \( E \), there exist continuous semi-norms \( p \) and \( q \) on \( E \) and \( F \) respectively such that

\[
r(\mathcal{O}(e, f)) \leq p(e)q(f), \quad e, f \in E, F.
\]

Extend \( p \) and \( q \) to \( E \) and \( F \). Fix \( e \). The left– and right–hand sides of this inequality are then continuous functions on \( F \). The inequality is valid on the dense subspace \( F \), and so on the whole of \( F \). So, the inequality is valid for \( e, f \in E, F \).

Proof of Proposition II.2.2 Fix \( b \in B \). Then the map \( V \times W \ni (v, w) \mapsto (bv) \hat{\otimes} w \in \hat{V} \otimes W \) is continuous, bilinear, \( A \)– balanced, and so there is a continuous linear map \( V \otimes W \ni (bv) \hat{\otimes} w \), and this by definition will be the multiplication by \( b \) in the space \( V \hat{\otimes} W \). This makes \( V \hat{\otimes} W \) into an algebraic left \( B \)–module. The real question is whether the map \( \tilde{\mathcal{M}}: B \times (V \hat{\otimes} W) \ni (b, t) \mapsto bt \in V \hat{\otimes} W \) is continuous, or, for example, merely separately continuous.

Use the associativity of projective topological tensor products over \( C \) [15]. The continuity of the map \( B \times V \ni (b, v) \mapsto b \in V \), ensures the continuity of the map \( B \times V \times W \ni (b, v, w) \mapsto (bv) \hat{\otimes} w \in \hat{V} \otimes W \). Moreover, this map is trilinear, so there is a continuous linear map \( \Lambda: \hat{V} \hat{\otimes} W \longrightarrow \hat{V} \otimes W \) such that \( \Lambda(bv \hat{\otimes} w) = (bv) \hat{\otimes} w \). Using the continuity of the map \( B \times (V \hat{\otimes} W) \ni (b, t) \mapsto bt \in \hat{B} \otimes (V \hat{\otimes} W) \), one finds that the map \( B \times (V \hat{\otimes} W) \ni (b, v) \hat{\otimes} w \mapsto (bv) \hat{\otimes} w \in \hat{V} \otimes W \) is continuous.

This is the essential point.

The latter map decomposes as two maps \( \Pi: B \times (V \hat{\otimes} W) \ni (b, v) \hat{\otimes} w \mapsto (b, v \hat{\otimes} w \text{ mod } J) \in B \times (V \hat{\otimes} W/J) \) (for the definition of \( J \) see the proof of Theorem II.1.3) and a map \( \mathcal{M}_o: B \times (V \hat{\otimes} W/J) \ni (b, v \hat{\otimes} w \text{ mod } J) \mapsto (b, v \hat{\otimes} w) \in V \hat{\otimes} W \). Lemma II.2.3 implies that \( B \times (V \hat{\otimes} W/J) \) has the quotient topology with respect to the map \( \Pi \), so that \( \mathcal{M}_o \) is continuous. The multiplication map \( \mathcal{M}: B \times (V \hat{\otimes} W) \longrightarrow \hat{V} \otimes W \) whose continuity we wish to demonstrate is a separately continuous extension of \( \mathcal{M}_o \) to \( B \times (V \hat{\otimes} W) \), so Lemma II.2.4 clinches the matter.

We are interested especially in the topological analogue of Theorem I.6.2 (concerning induced modules).

---

Let $B$ be a complete Hausdorff locally convex associative unital algebra, and $A$ a closed subalgebra containing 1.

Let $W$ be a complete Hausdorff locally convex left $A$–module.

Then there exist a complete Hausdorff locally convex left $B$–module $X$, and a continuous $A$–linear map $\kappa: W \rightarrow X$, such that every continuous $A$–linear map $\varphi$ mapping $W$ to a complete Hausdorff locally convex left $B$–module $Z$ has a unique continuous $B$–linear extension $\hat{\varphi}$, such that $\varphi = \hat{\varphi} \circ \kappa$.

Moreover, the pair $X, \kappa$ is unique in the sense that whenever the two pairs $X, \kappa$ and $\tilde{X}, \tilde{\kappa}$ both have the specified properties, there exists a continuous $B$–linear isomorphism $X \rightarrow \tilde{X}$ intertwining $\kappa$ and $\tilde{\kappa}$.

**Proof of Theorem II.1.5** Uniqueness is shown in the usual manner. To show existence take $X = B \otimes^A W$, $\kappa(w) = 1 \otimes^A w$. One can copy the proof of Theorem I.6.2, with hardly any modification. So really all one has to do is to remark that when $\varphi$ is a continuous $A$–module homomorphism $W \rightarrow Z$, where $Z$ is a complete Hausdorff locally convex $B$–module, the map $\mathcal{G}: B \otimes^A W \rightarrow Z$, defined by $\mathcal{G}(b, w) = b \varphi(w)$, is not only bi–additive and $A$–balanced, but actually continuous. One obtains a continuous map $\hat{\varphi}$ such that $\mathcal{G} = \hat{\varphi} \tau$. The fact that $B \otimes^A W$ is a locally convex left $B$–module is immediate from Proposition II.2.2. Moreover, $\hat{\varphi}$ is $B$–linear (a particular case of the topological analogue of Theorem I.3.3, which is easy to prove using Proposition II.2.2), and one gets the diagram:

$$
\begin{array}{c}
W \\
\kappa \downarrow \\
B \otimes^A W \\
\tau \downarrow \\
B \otimes^A W
\end{array}
\begin{array}{c}
\mathcal{G} \\
\hat{\varphi} \\
\end{array}
\begin{array}{c}
Z
\end{array}
$$

**Corollary II.2.6** Let $B$, $A$ and $W$ be as in Theorem II.2.5. Then $B \otimes^A W$ is characterized by the universal property described there. The map $\hat{\varphi}$ is the extension to the Hausdorff completion $j: B \otimes^A W \rightarrow B \otimes^A W$ of the map $\hat{\varphi}$ defined in Theorem I.6.2.
**Definition II.2.7** \( B \hat{\otimes} \mathcal{W}_A \) will be called the locally convex \( B \)-module induced from the \( A \)-module \( \mathcal{W} \).

### II.3 Induced Locally Convex Modules over Complemented Algebras

The purpose in this section is to describe what happens when one induces a module from \( A \) to \( B \), when \( B \) is something like a free module over \( A \). Such modules are relatively easy to manage.

**Definition II.3.1**

Let \( A \) be a closed subalgebra (containing 1) of the complete Hausdorff locally convex associative unital algebra \( B \). Then a **multiplicative complement** of \( A \) in \( B \) is a closed linear subspace \( L \) (containing 1) of \( B \) such that the map \( L \hat{\otimes} A \ni \ell \otimes a \mapsto \ell a \in B \), induced by the multiplication map restricted to \( L \times A \), is a topological isomorphism.

This implies that the finite sums \( \sum_i \ell_i a_i \), \( \ell_i \in L \), \( a_i \in A \), are dense in \( B \). It is not difficult to find examples of algebras \( A \) and \( B \), \( A \subset B \), where these complements exist. To take one example: let \( G \) be a Lie group which is the direct product of two subgroups \( H \) and \( K \). Let \( B = \mathcal{S}'(G) \), the convolution algebra of compactly supported distributions on \( G \), which is indeed a (complete, Hausdorff) locally convex associative unital algebra (the bilinear convolution map being continuous, as can be proved in the manner of the proof of Lemma III.2.4). Let \( A = \mathcal{S}'(H) \). Then \( L = \mathcal{S}'(K) \) is a multiplicative complement of \( \mathcal{S}'(H) \) (which happens to be an algebra as well), because of the well-known isomorphism \( \mathcal{S}'(K \times H) \cong \mathcal{S}'(K) \hat{\otimes} \mathcal{S}'(H) \).

Roughly speaking, whenever \( A \) has a topological multiplicative complement in \( B \), modules induced from \( A \) to \( B \) are relatively easy to manage.

**Proposition II.3.2**

Let \( A \) be a closed subalgebra (containing 1) of the complete Hausdorff locally convex associative unital algebra \( B \), and let \( \mathcal{W} \) be a complete Hausdorff locally convex module over \( A \). Assume that there exists a multiplicative complement \( L \) of \( A \) in \( B \).

*Then the induced locally convex \( B \)-module \( B \hat{\otimes} \mathcal{W}_A \) has the following properties:*
a) The canonical map \( \kappa : \mathcal{W} \to B \hat{\otimes} \mathcal{W}, \kappa(w) = 1 \hat{\otimes} w \) is a one-to-one homomorphism onto a topological direct factor. In short, \( \mathcal{W} \) is a direct factor in \( B \hat{\otimes} \mathcal{W} \).

b) Let \( B_o \) be the dense subspace of \( B \) spanned by finite sums \( \sum_i f_i a_i, f_i \in \mathcal{L}, a_i \in A \); in short, \( B_o = L \hat{\otimes} A \). Then equipped with the projective tensor product topology (as defined in Theorem II.1.8), the space \( B_o \hat{\otimes} \mathcal{W} \) is a Hausdorff space with completion

\[
B_o \hat{\otimes} \mathcal{W} \hookrightarrow B \hat{\otimes} \mathcal{W} \xrightarrow{j} B \hat{\otimes} \mathcal{W}
\]

For any multiplicative complement \( L \) of \( A \) in \( B \) the composition of linear maps

\[
L \hat{\otimes} \mathcal{W} \xrightarrow{i} B \hat{\otimes} \mathcal{W} \xrightarrow{j} B \hat{\otimes} \mathcal{W}
\]

\[
l \hat{\otimes} w \mapsto l \hat{\otimes} w, \quad b \hat{\otimes} w \mapsto b \hat{\otimes} w
\]
is a linear topological isomorphism. In particular, \( B \hat{\otimes} \mathcal{W} \) is the direct sum of \( L \hat{\otimes} \mathcal{W} \) and the closed additive hull \( J \) of the set of tensors \( l a \hat{\otimes} w - l \hat{\otimes} aw, l \in \mathcal{L}, a \in A, w \in \mathcal{W} \). Moreover, \( J \) is the kernel of the well-defined continuous linear map \( B \hat{\otimes} \mathcal{W} \ni l a \hat{\otimes} w \mapsto l \hat{\otimes} aw \in L \hat{\otimes} \mathcal{W}, l \in \mathcal{L}, a \in A, w \in \mathcal{W} \). Finally, \( J \) is closed in \( B \hat{\otimes} \mathcal{W} \).

d) Restriction of the isomorphism under c) to \( L \hat{\otimes} \mathcal{W} \) yields an isomorphism

\[
L \hat{\otimes} \mathcal{W} \xrightarrow{\alpha} B_o \hat{\otimes} \mathcal{W} \xrightarrow{\beta} B_o \hat{\otimes} \mathcal{W}
\]

which is bicontinuous for the projective tensor product topologies. In particular, \( B_o \hat{\otimes} \mathcal{W} \) is the direct sum of \( L \hat{\otimes} \mathcal{W} \) and the additive hull \( J_o \) of the set of tensors \( l a \hat{\otimes} w - l \hat{\otimes} aw, l \in \mathcal{L}, a \in A, w \in \mathcal{W} \). Moreover, \( J_o \) is the kernel of the well-defined continuous linear map \( B \hat{\otimes} \mathcal{W} \ni l a \hat{\otimes} w \mapsto l \hat{\otimes} aw \in L \hat{\otimes} \mathcal{W}, l \in \mathcal{L}, a \in A, w \in \mathcal{W} \). Finally, \( J_o \) is closed in \( B_o \hat{\otimes} \mathcal{W} \).

In part b) of the proposition no claim is made that \( B \hat{\otimes} \mathcal{W} \) is Hausdorff, i.e., that \( j \) is one-to-one. The fact that this remains uncertain is the main technical reason why the proof should be as long-winded as it is.

**Proof of Proposition II.3.2** Compare the proof of Proposition I.7.5, and consider the following commutative diagram of maps

\[
\begin{array}{ccc}
L \hat{\otimes} \mathcal{W} & \xrightarrow{\iota_o} & B_o \hat{\otimes} \mathcal{W} & \xrightarrow{\pi_o} & B_o \hat{\otimes} \mathcal{W} \\
\downarrow \alpha & & \downarrow \beta & & \downarrow Y \\
L \hat{\otimes} \mathcal{W} & \xrightarrow{\iota} & B \hat{\otimes} \mathcal{W} & \xrightarrow{\pi} & B \hat{\otimes} \mathcal{W}
\end{array}
\]

32
The maps $\alpha : L \otimes_w \to L \hat{\otimes} W$ and $\beta : B \otimes_w \to B \hat{\otimes} W$ are the completions of $L \otimes_w W$ and $B \otimes_w W$, when these are equipped with the projective tensor product topologies. The map $\gamma$ is the one defined in part b) of the proposition. The bottom row maps are the ones defined in the proposition. The map $\nu_\sigma$ is a bi–restriction of $\nu$, and $\pi_\sigma$ can only be said to be the map that makes the diagram commutative. Note that $\pi$ is the canonical map discussed in the remark following the proof of Theorem II.1.3. We first look at the bottom row maps.

One shows in the usual way that $\pi$ and $\nu$ are well–defined, and continuous. We construct a continuous inverse for $\pi \nu$, making explicit that there exists an isomorphism $B \hat{\otimes} W = (L \hat{\otimes} \Lambda) \hat{\otimes} W = L \hat{\otimes} (A \hat{\otimes} W) = L \hat{\otimes} W \to B \hat{\otimes} W$. The first of these is the canonical tensor map $(s, w) \mapsto s \otimes_w$, the last map sends $l \otimes_a w$ to $l \otimes (aw)$. Furthermore the associativity of the projective tensor product comes into play in the third map, the isomorphism $B = L \hat{\otimes} A$ in the second. All these maps are continuous. The result is a continuous bilinear map $\Pi : B \hat{\otimes} W \to L \hat{\otimes} W$, into a complete Hausdorff space, with the property that $\Pi(la, w) = l \otimes aw$. But this map is also $\Lambda–$balanced. Indeed, $\Phi(b, \alpha w)$ equals $\Phi(b, \alpha w)$ for $b = la$, $l \in \Lambda$, $a, \alpha \in \Lambda$, $w \in W$. But the fact that sums $\sum \lambda_i a_i, l_i, a_i \in \Lambda, \lambda_i \in \Lambda$, are dense in $B$, and the bilinearity and continuity of $\Phi$, ensure that $\Phi$ is $\Lambda–$balanced on all of $B \times W$. By the universal property of completed projective tensor products over $\Lambda$, there exists a continuous map $\Psi : B \hat{\otimes} W \to L \hat{\otimes} W$, characterized by $\Psi(la \otimes w) = l \otimes aw, l \in \Lambda, a \in \Lambda, w \in W$. This map does the trick, that is, $\Psi \nu \nu$ is seen to equal the identity on tensors $l \otimes w$, and therefore on all of $L \hat{\otimes} W$, and likewise $\nu \nu \Psi$ equals the identity on tensors $l \otimes w$, and so on all of $B \hat{\otimes} W$.

Part a) of the proposition is immediate now, since the isomorphism $\nu \nu$ intertwines the canonical map $\kappa : W \to B \hat{\otimes} W$ with the map $\overline{\kappa} : W \to L \hat{\otimes} W, \overline{\kappa}(w) = l \otimes w$. It is well known that the latter map is a one-to-one homomorphism onto a direct factor (for this conclusion to be valid $W$ must indeed be complete) [16]. It follows that $\nu$ is one-to-one, and that $\pi$ is onto.

Since $\Psi$ inverts $\nu \nu$, the map $\Psi \pi$ is a continuous projection $B \hat{\otimes} W \to B \hat{\otimes} W$. Explicitly this map sends $l \otimes r \otimes w$ to $l \otimes lerw$. Then $B \hat{\otimes} W$ becomes the topological direct sum of the image and the kernel of this map, that is, $B \hat{\otimes} W = \text{Im}(\Psi \pi) \oplus \text{Ker}(\Psi \pi) = \text{Im}(\nu \nu) \oplus \text{Ker}(\nu \nu) = (L \hat{\otimes} W) \oplus J$. Furthermore, the kernel of $\nu \nu \pi$ equals the range of the complementary projection $I - \nu \nu \pi$, the map that sends $la \otimes w$ to

16 See Alexandre Grothendieck, *Produits Tensoriels Topologiques et Espaces Nucléaires*, Chapitre I, §1, n° 3, Lemme 3. The idea of the proof is obvious: one decomposes $L$ as the topological sum of the complex line $\mathbb{C}1$ and a hyperplane.
The image under $I-\psi\pi$ of the dense subspace $B_0 \otimes W = (L \otimes A) \otimes W$ yields a dense subspace of $J$. Since $B_0 \otimes W$ is the additive hull of tensors $la \otimes w$, $l \in L$, $a \in A$, $w \in W$, it follows that $J$ is the closure of the additive hull of tensors $la \otimes w$, $l \in L$, $a \in A$, $w \in W$.

b) and d) remain to be shown.

The composite map $\pi_0$ is inverted by the map $\Psi_0$, $\Psi_0(la \otimes w) = la \otimes w$, $l \in L$, $a \in A$, $w \in W$ (compare the proof of Proposition I.7.5). The top and bottom rows being isomorphisms, it follows that $\gamma$ is one-to-one, with dense image.

Once we know that $\gamma$ is one-to-one, it becomes clear that $\iota_0$, $\pi_0$, and $\Psi_0$ are merely bi–restrictions of $\iota$, $\pi$, and $\Psi$. Equip $B_0 \otimes A$ with the topology induced from $B \otimes A$, making $\gamma$ a completion. Then $\iota_0$, $\pi_0$, and $\Psi_0$ are continuous (since $L \otimes W$ and $B_0 \otimes W$ are equipped with the projective tensor product topologies). What remains to be shown, is that the topology thus introduced on $B_0 \otimes A$ coincides with the projective tensor product topology, that is, with the quotient topology with respect to $\pi_0$. We know that $\pi_0$ is continuous and onto. That $\pi_0$ is open is easily seen to be a consequence of the fact that $\pi$ is open. This shows b).

d) is now clear □

Note that $J = \ker \pi$ is the closure of $J_0 = \ker \pi_0$, the latter equalling the additive hull of tensors $la \otimes w$, $l \in L$, $a \in A$, $w \in W$, and that, moreover, $J_0$ is closed in $B_0 \otimes W$. As the proof makes clear, the essential reason for this is that under the projection $P = \psi \pi B \otimes W \longrightarrow B \otimes W$, $P(la \otimes w) = la \otimes w$, the subspace $B_0 \otimes W$ is invariant (that is, it is mapped into itself).

Some basic problems in the construction of topological tensor product over algebras are now out of the way. From now on it will be tacitly assumed that

**Convention**  All locally convex modules and algebras introduced are complete and Hausdorff.

### II.4 Some Examples

The purpose is eventually to show that the space of distributions concentrated on a subgroup of a Lie-group can in a meaningful way be interpreted as an induced locally convex module (see Section III.3). This requires more preparation. In the present short section we give some minor examples of induced locally convex modules.
Example II.4.1
Let $G$ be a Lie-group, and $K$ a compact subgroup. We denote by $\mathcal{E}_K^\prime(G)$ the space of distributions concentrated on $K$. This is a locally convex convolution algebra. Moreover, the space $\mathcal{D}(G)$ of distributions on $G$ is a topological right module under $\mathcal{E}_K^\prime(G)$, when the external multiplication $\mathcal{D}(G) \times \mathcal{E}_K^\prime(G) \to \mathcal{D}(G)$ is again defined to be convolution. (This is based on a lemma similar to Lemma III.2.4 below, that is, the external multiplication can be shown to be continuous). Let $W$ be any locally convex left $\mathcal{E}_K^\prime(G)$ module (one might take $\mathcal{D}(G)$ itself, as a left module). Then $\mathcal{D}(G) \hat{\otimes} W$ is a well-defined locally convex vector space.

Example II.4.2
Let $\mathcal{E}(G)$ be the space of compactly supported distributions on the Lie group $G$. This is itself a convolution algebra, because the convolution map $\mathcal{E}(G) \times \mathcal{E}(G) \to \mathcal{E}(G)$ is continuous. So $\mathcal{E}_K^\prime(G)$ is a subalgebra, and $\mathcal{E}(G) \hat{\otimes} W$ is a well-defined induced locally convex left $\mathcal{E}(G)$–module.

Example II.4.3
In the preceding example, replace $\mathcal{E}_K^\prime(G)$ by $\mathcal{E}(K)$, identified with the space of those distributions on $G$ concentrated on $K$ which have transversal order 0 (to see how $\mathcal{E}(K)$ fits into $\mathcal{E}_K^\prime(G)$ see Section III.3). One then obtains an $\mathcal{E}(G)$–module from an $\mathcal{E}(K)$–module. This can be seen as a form of induced representation.

II.5 Spaces of the type $\mathcal{C}^{(\mathbb{N})}$
We equip the universal enveloping algebra $\mathcal{U}(g)$ of a Lie group $g$ with a 'natural' topology, and we study properties of this topology. For this we first define a topology on the space $\mathcal{C}^{(\mathbb{N})}$ of finite sequences of complex numbers.

Let $\mathcal{C}^{(\mathbb{N})}$ be the space of finite sequences of complex numbers, finite meaning that for every single sequence $(t_n)_{n \in \mathbb{N}} \in \mathcal{C}^{(\mathbb{N})}$ there is a number $N$ such that $t_n = 0$ for $n > N$. Equipped this space with the strict inductive limit topology with respect to the subspaces $\mathcal{C}^{(m)}$ of sequences $(\sigma_i)_{i \in \mathbb{N}}$ with $\sigma_i = 0$, $i > m$. It is well-known that $\mathcal{C}^{(\mathbb{N})}$ is the strong dual of the Fréchet space $\mathcal{C}^{\mathbb{N}}$ of all sequences of complex numbers, when the latter is equipped with the topology of pointwise convergence. Moreover, $\mathcal{C}^{(\mathbb{N})}$ is a reflexive space [17].
**Definition II.5.1** A space of the type $C(\hat{F})$ will be a locally convex vector space linear-topologically isomorphic to $C(\hat{F})$.

Typical examples of spaces of which can be made into spaces of the type $C(\hat{F})$ are spaces of polynomials, or symmetric algebras over finite dimensional spaces.

**Lemma II.5.2** Let the infinite dimensional vector space $E$ be the union of a countable number of finite dimensional subspaces; equivalently, let $E$ possess a countable base. Then $E$ admits a topology which makes $E$ into a space of the type $C(\hat{F})$. This topology is the finest among all possible locally convex topologies on $E$.

Moreover, when $E$ is a space of the type $C(\hat{F})$, any increasing sequence of finite dimensional subspaces $E_i$, with union $E$, is a defining sequence for $E$ as inductive limit.

Finally, when $F_i, i=1,2,...$, are finite dimensional subspaces of $E$, such that $E = \bigoplus_{i=1}^{\infty} F_i$, then $E$ carries the direct sum topology.

**Proof** Let the vector space $E$ be the union of a countable number of finite dimensional subspaces. Construct a countable base $f_i, i=1,2,...$ for $E$. Then $E$ consists of all vectors $\sum \lambda_i f_i$, with $\lambda_i = 0$ for all except a finite number of indices $i$. So, $E$ is isomorphic to $C(\hat{F})$ (no topology involved yet). Equip $C(\hat{F})$ with the topology described in the introduction to Definition II.5.1. Then every linear map defined on $C(\hat{F})$ into any locally convex space is continuous, because its restrictions to the finite dimensional subspaces $C_{\hat{F}}$ are continuous. This means for one thing that the identity map from $C_{\hat{F}}$ with the current topology into $C(\hat{F})$ equipped with any other locally convex topology, will be continuous, so the current topology is the finest locally convex topology on $C(\hat{F})$, and so on $E$.

Let $E_i$ be any increasing series of finite dimensional subspaces, with union $E$, and let $E$ be the space $E$ equipped with the inductive limit topology with respect to this sequence. Then the identity map $E \longrightarrow E$ is bicontinuous.

---

17 For more detail, see, e.g., François Trèves, *Topological Vector Spaces, Distributions and Kernels*, Chapter 22. Reflexivity can be checked immediately. It is perhaps easier to note that $C(\hat{F})$ is the inductive limit of Montel spaces, and therefore a Montel space (that is, locally convex, Hausdorff, barrelled, and with closed bounded sets being compact). Montel spaces are well-known to be reflexive (N.Bourbaki, *Éléments de Mathématique; Première Partie: Les Structures Fondamentale de l’Analyse; Livre V: Espaces Vectoriels Topologiques; Fascicule de Résultats §7, n° 9*).
For the final statement, use the fact that the direct sum topology is defined as the finest locally convex topology for which all imbeddings \( \varphi_i: F_i \subset E \) are continuous. This implies that with respect to this direct sum topology, any map \( L \) defined on \( E \), into any topological space, is continuous iff its restrictions \( L \circ \varphi_i \) are continuous.

We prove that a space of distributions concentrated at a single point is a space of the type \( \mathfrak{C}^{(\mathbb{N})} \). We need the following technical lemma.

**Lemma II.5.4** *Homomorphisms between Fréchet-Montel spaces*

*Let \( E \) and \( F \) be Fréchet-Montel spaces [18].

Let \( u \) be a continuous linear map \( E \rightarrow F \). Equip \( E' \) and \( F' \) with the strong topologies. Then the following are equivalent:

i) \( u \) has closed range 

ii) \( u \) is a topological homomorphism

iii) \( u' \) has closed range

iv) \( u' \) is a topological homomorphism

The equivalence also holds when \( E \) and \( F \) are Banach spaces.*

**Proof** This lemma supplements a well-known theorem on homomorphisms between Fréchet spaces. Grothendieck, for example, shows the equivalence of i), ii), and iii), and, moreover, he shows that iv) implies i), ii), and iii) [19]. He shows that iv) is actually equivalent to i), ii), and iii) when \( E \) and \( F \) are Banach spaces. So what we have to show ourselves is that i), ii), and iii) imply iv) when \( E \) and \( F \) are Fréchet-Montel spaces. Note that Fréchet-Montel spaces and Banach spaces are quite distinct subcategories of the category of Fréchet spaces; only finite dimensional spaces belong to both.

So, assume that i), ii), and iii) are true. Since \( \text{Im} u \) is a closed subspace of a Fréchet-Montel space it is a Fréchet-Montel space in its own right. The map \( \hat{u}: E/\text{Ker} u \rightarrow \text{Im} u \) induced by \( u \) is a linear-topological isomorphism with respect to the quotient and subspace topologies on these spaces. This means for one thing that \( E/\text{Ker} u \) is also a Fréchet-Montel space (this needed a proof, because in

---

18 So, complete locally convex spaces with metrizable topologies (therefore, also Hausdorff and barrelled), in which closed bounded sets are compact. A typical Fréchet-Montel space is the space of \( \mathcal{G}^{\infty} \)-functions on a \( \mathcal{G}^{\infty} \)-manifold, with the usual topology of locally uniform convergence of all derivatives.

19 A. Grothendieck, *Topological Vector Spaces* (New York: Gordon & Breach, 1973), Chapter IV, Part 2, Section 4, Theorem 3. He shows that the following are equivalent as well: v) \( u \) is a weak homomorphism; vi) \( u' \) is a weak homomorphism; vi) \( u' \) has weakly closed range. We leave these out to have a neatly symmetric lemma.
Chapter II: Topological Tensor Products over Algebras; Induced Locally Convex Modules

A quotient of Fréchet-Montel space by a closed subspace need not be Montel. The transpose \( t: (Im \cdot)' \to (\mathcal{E}/\ker u)' \) is then an isomorphism as well (in our case, for the strong topologies). What we have to show instead, however, is that the map \( (t\cdot)': F'/\ker t \to \text{im} t \) is an isomorphism (for the quotient and subspace topology induced by the strong duals \( F' \) and \( \text{E}' \)).

For a moment take \( \text{E} \) and \( \text{F} \) to be merely locally convex, \( u \) continuous. As always in the case of duals of subspaces and quotients, one identifies canonically \( (\text{Im} \cdot)' \) with \( F'/\text{im} t \), and \( \text{im} t \) with \( (\ker u)' \). Furthermore, \( \text{im} t \) equals \( (\ker u)' \). In view of the bipolar theorem \( \text{im} u' \) is in general dense in \( (\ker u)' \).

The maps \( t\cdot \) and \( (t\cdot)' \) correspond under these identifications, as follows:

\[
\begin{align*}
(\text{Im} \cdot)' & \xrightarrow{t\cdot} (\text{E}/\ker u)' \\
F'/\text{im} t & \xrightarrow{(t\cdot)'} \text{im} t \subset (\ker u)' \subset \text{E}'
\end{align*}
\]

We have that \( t\cdot \) is an isomorphism, and that \( \text{im} t \) equals \( (\ker u)' \) because it is closed. So, in order to prove that \( (t\cdot)' \) is an isomorphism it will suffice to show that the canonical identifications \( (\text{Im} \cdot)' = F'/\text{im} t \), and \( (\text{E}/\ker u)' = (\ker u)' \), are in the present circumstances topological isomorphisms for the strong duals involved.

In general, let \( \text{N} \) be a closed subspace of a semi-reflexive space \( \text{E} \). Then the strong dual \( \text{N}' \) is linear-topologically isomorphic to the quotient \( \text{E}'/\text{N}^\circ \) of the strong dual of \( \text{E}' \) by the annihilator of \( \text{N} \). In our case, \( \text{F} \) is a Montel space, and therefore reflexive, so the isomorphism \( (\text{Im} \cdot)' = \text{E}'/\text{N}^\circ \) follows.

The proof is as follows. Let \( \text{E}^\# \) denote the strong dual of \( \text{E} \). The dual of \( \text{E}^\#/\text{N}^\circ \) can be identified with \( \text{N}^\circ \), as subspace of \( (\text{E}^\#)' = \text{E} \). By the bipolar theorem \( \text{N}^\circ \) equals \( \text{N} \), since \( \text{N} \) is a closed subspace of \( \text{E} \). On the other hand, the dual of \( \text{N}^\circ \) also equals \( \text{N} \), because a closed subspace of a reflexive space is reflexive (for example, because reflexivity is equivalent to bounded sets being weakly relatively compact).

Since a locally convex topology on a linear space is always equal to the topology of uniform convergence on equicontinuous sets in the dual, one can show that \( \text{E}'/\text{N}^\circ \) and \( \text{N}' \) are topologically equal by showing that under the identification of their duals the equicontinuous sets correspond. Now in general, under the identification of the dual \( (\text{F}/\text{M})' \) of a quotient \( \text{F}/\text{M} \) with the orthogonal \( \text{M}^\circ \) as subspace of \( \text{F}' \), the equicontinuous sets in \( (\text{F}/\text{M})' \) are easily seen to correspond to the equicontinuous sets in \( \text{F}' \) that are contained in \( \text{M}^\circ \). In the present case, therefore, the equicontinuous sets in the dual of \( \text{E}^\#/\text{N}^\circ \) can be identified as those equicontinuous sets in \( \text{E} \) (as dual to \( \text{E}^\# \)) that are contained in \( \text{N} \). What one has to show, then, is that the latter equal the equicontinuous sets in \( \text{N} \) (as dual to \( \text{N}^\circ \)). This, however, becomes obvious once it is seen that the equicontinuous sets in a locally convex space \( \text{F} \), (equicontinuous with respect to its strong dual \( \text{F}^\# \)), are no other than the bounded sets. Indeed, a basis of neighbourhoods of \( 0 \) in \( \text{F}^\# \) is formed by the polars of bounded.
F'/\text{Im}u^\circ$ is indeed topological.

Secondly, it is known that every compact set in the quotient space $E/N$ of a Fréchet space $E$ by a closed subspace $N$ is the image under the canonical map $E \longrightarrow E/N$ of a compact set in $E$ [22]. That implies that the canonical isomorphism $(E/N)' = N^\circ \subset E$ is topological for the compact dual topologies on $E/N$ and $E$. Now assume that both $E$ and $E/N$ are Montel spaces as well. Then the compact dual topologies in question coincide with the strong topologies. This proves that the isomorphism $(E/	ext{Ker}u)' = (\text{Ker}u)^\circ$ is topological, and the proof is complete.

For $m \in \mathbb{N}$ let $\mathbb{N}^m$ be the set of multi-indices $(\alpha_1, \ldots, \alpha_m)$, and let $\mathbb{C}^{\mathbb{N}^m}$ be the complex-linear space of maps $\mathbb{N}^m \longrightarrow \mathbb{C}$. Elements in $\mathbb{C}^{\mathbb{N}^m}$ are denoted $(x_\alpha)_{\alpha \in \mathbb{N}^m}$. Equip $\mathbb{C}^{\mathbb{N}^m}$ with the topology of pointwise convergence (making it obviously isomorphic to $\mathbb{C}^\mathbb{N}$). Its dual $(\mathbb{C}^{\mathbb{N}^m})'$ is the space of complex linear maps $\mathbb{N}^m \longrightarrow \mathbb{C}$ that vanish almost everywhere on $\mathbb{N}^m$, and is denoted $\mathbb{C}^{(\mathbb{N}^m)}$.

Equipped with the strong dual topology this is a space of the type $\mathbb{C}^{(\mathbb{N})}$. The duality is defined in the obvious fashion, so by sets in $E$, so the equicontinuous sets in $E$ are the sets that are contained in the bipolars of bounded sets, that is, they are in fact the bounded sets.

This is a consequence of the following (Grothendieck, Chapter I, Section 14, Exercise 2): let $M$ a compact topological space, $E$ a Fréchet space. Let $\mathcal{C}(M;E)$ denote the Fréchet space of continuous functions with values in $E$. Let $F$ a closed subspace of $E$. The canonical projection $\pi$ induces a map $\Pi: \mathcal{C}(M;E) \longrightarrow \mathcal{C}(M;E/F)$, $f \longmapsto \pi f$. Not only is this map is (obviously) continuous, but it is also open, so by the open mapping theorem it is in fact onto. The result we need then follows by taking for $M$ a compact subset of $E/F$.

That $\Pi$ is indeed a surjective homomorphism, can be seen as follows. A basis of convex neighbourhoods of $0$ in $\mathcal{C}(M;E)$ is formed by sets $\mathcal{C}(M;U) = \{ f \in \mathcal{C}(M;E) \mid f(U) \subset U \}$, where $U$ varies over a basis of convex neighbourhood of $0$ in $E$. Let $\mathcal{C}(M;U)_0$ be subset of $\mathcal{C}(M;U)$ consisting of finite sums $\sum f_i x_i$, where the $f_i$ are complex-valued continuous functions on $M$ with $0 \leq f_i \leq 1$, $\sum f_i \equiv 1$, and with the $x_i$ belonging to $U$. Then $\mathcal{C}(M;U)_0$ is actually dense in $\mathcal{C}(M;U)$. Indeed, if $f \in \mathcal{C}(M;U)$, and $p$ a continuous semi-norm on $E$, one constructs a finite open covering $(V_i)_{i=1}^n$ of $M$ with every $V_i$ containing a point $t_i$ such that $p(f(t_i) - f(t)) \leq 1$ for all $t \in V_i$. Set $x_i = f(t_i)$, and $\Psi = \sum f_i x_i$, where the $f_i$ form a partition of unity subordinate to the covering $(V_i)_{i=1}^n$. Then $\Psi$ belongs to $\mathcal{C}(M;U)_0$, while $\Phi(t) - \Psi(t) = \sum f_i(t)(\Phi(t) - f(t))$, so that $p(\Phi(t) - \Psi(t)) \leq 1$ for all $t \in M$, $p$ being arbitrary, this proves the density claimed.

The image $\prod(\mathcal{C}(M;U)_0)$ equals the (convex) set of sums $\sum f_i \pi(x_i)$, $0 \leq f_i \leq 1$, $\sum f_i \equiv 1$, $x_i \in U$, so $\prod(\mathcal{C}(M;U)_0)$ actually equals $\mathcal{C}(M;\pi(U))_0 = \{ \sum f_i z_i, 0 \leq f_i \leq 1, \sum f_i \equiv 1, z_i \in \pi(U) \}$. Now $\pi(U)$ is a convex neighbourhood of $0$ in $E/F$, $\pi$ being open. And so, as above, $\prod(\mathcal{C}(M;U)_0) = \mathcal{C}(M;\pi(U))_0$ is dense in $\mathcal{C}(M;\pi(U))$. And so $\prod$ maps a neighbourhood of $0$ to a set whose closure is a neighbourhood of $0$. If $\prod$ being a continuous linear map between Fréchet spaces, the conclusion is that $\Pi$ is indeed a surjective homomorphism.

22 A. Grothendieck, Topological Vector Spaces (New York: Gordon & Breach, 1973), Chapter IV, Part 2, Section 1, Proposition 1.
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\[
\langle (x_\alpha)_{\alpha \in \mathbb{N}^m}, (\lambda_\alpha)_{\alpha \in \mathbb{N}^m} \rangle := \sum_{\alpha \in \mathbb{N}^m} x_\alpha \lambda_\alpha, \\
(x_\alpha) \in \mathbb{C}[\mathbb{N}^m], (\lambda_\alpha) \in \mathbb{C}[\mathbb{N}^m].
\]

Let \( \mathcal{E}'(\mathbb{R}^m) \) denote the space of smooth functions on \( \mathbb{R}^m \), and let \( u \) be the map \( \mathcal{E}'(\mathbb{R}^m) \to \mathcal{C}[\mathbb{N}^m], u(\varphi) = (\frac{\partial^{\lvert \alpha \rvert} \varphi}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}(0))_{\alpha \in \mathbb{N}^m} \). Then \( u \) is a continuous linear map with dense range, and its transpose is the one–to–one map \( t_u \), defined by \( t_u((\lambda_\alpha)_{\alpha \in \mathbb{N}^m}) = \sum_{\alpha \in \mathbb{N}^m} (-1)^{\lvert \alpha \rvert} \lambda_\alpha \delta_{0}^{(\alpha)} \), where \( \delta_{0}^{(\alpha)} \) denotes the derivative of order \( \alpha \) of the Dirac measure at \( 0 \). Moreover, \( u \) vanishes on the (non-closed) subspace \( \mathcal{E}'_{oo}(\mathbb{R}^m) = \{ \varphi \in \mathcal{E}'(\mathbb{R}^m) | 0 \notin \text{supp} \varphi \} \), and one shows (by approximation by sequences) [23] that this space is dense in \( \text{Ker} u \). And so, \( \text{Im} t_u \) is dense in the annihilator of \( \mathcal{E}'_{oo}(\mathbb{R}^m) \), that is, in the space \( \mathcal{E}'(\mathbb{R}^m) \) of distributions concentrated at \( 0 \). With these facts in mind one can apply Lemma II.5.4 to obtain the following:

**Corollary II.5.5** Let \( u \) be as above.

The following statements are equivalent:

i) \( u \) is onto 
(a classical theorem of Borel [24])

ii) \( u \) is a topological homomorphism

iii) \( t_u \) has closed range 
(Schwartz's theorem on distributions concentrated at a single point)

iv) \( t_u \) is a topological isomorphism onto its range.

Since both i) and iii) are familiar theorems, one obtains

**Theorem II.5.6**

Let \( M \) be a \( \mathcal{C}^\infty \)–manifold. Let \( p \) be a point in \( M \). Let \( \mathcal{E}'_p(M) \) denote the space of distributions concentrated at \( p \), a closed subspace of \( \mathcal{E}'(M) \). Then the topology induced by \( \mathcal{E}'(M) \) onto \( \mathcal{E}'_p(M) \) is the inductive limit topology with respect to finite dimensional subspaces, so it is a space of type \( \mathcal{C}(\mathbb{N}) \). Furthermore, \( \mathcal{D}'(M) \) also induces this topology onto \( \mathcal{E}'_p(M) \).

---

23 Not entirely trivial, if one wants to avoid using the results in the theorem. Explicitly: let \( \psi \) be a real-valued \( \mathcal{C}^\infty \)– function on \( \mathbb{R} \) that is 1 for \( x \leq 1 \), and 0 for \( x \geq 1 \). If \( \varphi \in \mathcal{E}'(\mathbb{R}^n) \) is such that all its derivatives vanish at 0, the sequence \( (\varphi_n)_{n \in \mathbb{N}}, \varphi_n(x) = \psi(n|x|)\varphi(x) \) tends to \( \varphi \) in \( \mathcal{E}'(\mathbb{R}^n) \), while \( \forall n \in \mathbb{N}, 0 \notin \text{supp} \varphi_n \). The convergence holds because for every \( k \in \mathbb{N} \), and every multi-index \( \alpha \), \( |x|^k \varphi^{(\alpha)}(x) \) is bounded on \( |x| \leq 1 \). A similar statement is valid for general compact sets instead of \{0\}, see Laurent Schwartz, *Théorie des Distributions* (Paris: Hermann, 1966), Chapter III, §7, Theorem XXVIII.

As to the final statement, it is true in general that for $K$ any compact subset of $M$, $\mathcal{D}'(M)$ and $\mathcal{E}'(M)$ induce the same topology onto $\mathcal{D}_K(M)$ [25].

**Conjecture** I suspect that on $\mathcal{C}(\hat{\alpha})$ the inductive limit topology with respect to finite dimensional subspaces is in fact the only locally convex topology with respect to which $\mathcal{C}(\hat{\alpha})$ is complete. Of course, if this is true, Theorem II.5.6 is an immediate consequence.

Spaces of the type $\mathcal{C}(\hat{\alpha})$ in many ways they behave like finite dimensional spaces.

**Proposition II.5.7** Let $E$ be a space of the type $\mathcal{C}(\hat{\alpha})$. Then $E$ has the following properties:

i) $E$ is a complete locally convex Hausdorff space

ii) Every linear map defined on $E$, into any locally convex space, is continuous

iii) Every subspace $Z$ of $E$ is a topological direct factor (that is, there is another subspace $W$ such that $E=Z \oplus W$, topological direct sum of (closed) subspaces)

iv) Every subspace $Z$ of $E$ is either finite dimensional, or of the type $\mathcal{C}(\hat{\alpha})$ (when equipped with the induced topology)

Let $E$ and $F$ both be two spaces of the type $\mathcal{C}(\hat{\alpha})$. Then

v) Every bilinear map $\mathcal{B}$ defined on $E \times F$, into any locally convex space, is continuous

vi) On $E \otimes F$ the projective and the inductive tensor product topologies coincide, and make $E \otimes F$ into a space of the type $\mathcal{C}(\hat{\alpha})$.

In particular, in this topology $E \otimes F$ is complete.

In other words, the category of finite dimensional and $\mathcal{C}(\hat{\alpha})$ spaces is closed under all kinds of familiar operations. Spaces of the type $\mathcal{C}(\hat{\alpha})$ are special cases of spaces discussed by Grothendieck under the heading Sommes directes de droites [26]. Further on we look into the more interesting case of tensor products $E \otimes F$ where $F$ is no longer of the type $\mathcal{C}(\hat{\alpha})$.

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25 The essential point about this is that in spite of the fact that when $M$ is non-compact the topology of $\mathcal{E}'(M)$ is strictly stronger than the topology induced by $\mathcal{D}'(M)$ onto $\mathcal{E}'(M)$, a net $(T_\alpha)_{\alpha \in I}$ tending to 0 in $\mathcal{D}'(M)$, with the property that the supports of the $T_\alpha$ remain within a fixed compact set $K$, is also a convergent net in $\mathcal{E}'(M)$. Indeed, let $\chi \in \mathcal{D}(M)$ be equal to 1 in a neighbourhood of $K$. Let $B$ be a bounded set in $\mathcal{E}$. Then $\chi \cdot B$ is a bounded set in $\mathcal{D}(M)$. Since $\langle T_\alpha, \varphi \rangle = \langle T_\alpha, \chi \varphi \rangle$, and $(T_\alpha)_{\alpha \in I}$ tends to 0 uniformly on $\chi \cdot B$, $(T_\alpha)_{\alpha \in I}$ tends to 0 uniformly on $B$.

26 A. Grothendieck, *Topological Vector Spaces*. Chapter IV, Part 1, Section 6, especially Propositions 8-10.
Proof Throughout the proof, let $E_i$ be a defining sequence for $E$, $E_i$ finite dimensional.

As to i), use the fact that every direct sum of complete spaces is complete, and compare Lemma II.5.2.

Every linear map defined on $E$, into any topological vector space, is continuous because all its restrictions to the finite dimensional spaces $E_i$ are continuous. Hence (ii).

For the third statement, let $Z$ be any subspace of $E$. Let $Z_i$ be defined by $Z_i = Z \cap E_i$. It is easy to construct inductively a sequence of projections $\Pi_i : E_i \rightarrow E_i$ with the property that $Z_i = \ker(\Pi_i)$, and such that restricted to $E_i$, $\Pi_{i+1}$ coincides with $\Pi_i$. The sequence $\Pi_i$ then defines a projection $\Pi : E \rightarrow E$, with $\ker \Pi = Z$. Like all linear maps on $E$, $\Pi$ is continuous, so (iii) results.

In general, a closed subspace of an inductive limit space need not itself be an inductive limit. Here, however, $Z$ is the strict inductive limit of the subspaces $Z_i$. In the first place, the sequence $Z_i$ is a sequence of finite dimensional Fréchet spaces with the topology on $Z_i$ coinciding with the topology induced from $Z_{i+1}$. Let $\tilde{Z}$ be the union $Z$, equipped with the inductive limit topology, so a space of the type $C(\mathbb{N})$. Let $j : \tilde{Z} \rightarrow E$ be the imbedding $j(z) = z$. Since $\tilde{Z}$ is of type $C(\mathbb{N})$, $j$ is continuous. Moreover, $j$ is one-to-one, with image $Z$. Moreover, there exists an inverse of $j$ defined on the whole of $E$. Since $E$ is of the type $C(\mathbb{N})$, this inverse is also continuous. The result is that $j$ is a homomorphism, that is, a linear isomorphism onto $Z$.

To prove (v), first note that since all linear maps on $E$ and $F$ are continuous, every bilinear map on $E \times F$ is separately continuous. Furthermore, as a strong dual of a reflexive space, $E$ is barreled, which in turn implies that every separately continuous bilinear map into a locally convex space is hypocontinuous, that is to say, the set of linear maps $a \mapsto \mathcal{B}(a, b)$ is equicontinuous when $b$ is restricted to a bounded set in $E$ (and mutatis mutandis). The continuity of $\mathcal{B}$ is then for example a result of the Dieudonné–Schwartz Theorem, according to which a hypocontinuous bilinear map $E \times F \rightarrow Z$ is continuous when $E$ and $F$ are strong duals of Fréchet spaces at least one of which is reflexive, and $Z$ any locally convex space [27] (compare the proof of Lemma III.2.4).

Now consider the tensor product $E \otimes F$, with canonical tensor map $\tau : E \times F \rightarrow E \otimes F$.

Evidently, $E \otimes F$ is the countable union of finite dimensional subspaces. The projective tensor product

---

topology on $E \otimes F$ can be described as the finest locally convex topology for which $\tau$ is continuous. But according to (v) the tensor map $\tau$ will be continuous regardless of the choice of locally convex topology on $E \otimes F$. So the projective tensor product topology is simply the finest locally convex topology on $E \otimes F$, making this space into a space of the type $c^{(\infty)}$ (according to Lemma II.5.2). Moreover, the inductive tensor product topology on $E \otimes F$ is in general finer than the projective tensor product topology [28], so in this case identical to it. This proves vi) $\Box$

**Corollary II.5.8**

i) *Equipped with the inductive limit topology with respect to finite dimensional subspaces, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a complete Hausdorff locally convex associative unital algebra.*

ii) *Let $\mathfrak{b}$ be a Lie subalgebra, then $\mathcal{U}(\mathfrak{b})$ is a closed subspace of $\mathcal{U}(\mathfrak{g})$. More precisely, the canonical imbedding of $\mathcal{U}(\mathfrak{b})$ into $\mathcal{U}(\mathfrak{g})$ is a topological homomorphism with closed range.*

iii) *Let $L$ be any linear subspace of $\mathcal{U}(\mathfrak{g})$. Then $L$ is a closed subspace of $\mathcal{U}(\mathfrak{g})$, and itself of the type $c^{(\infty)}$.*

iv) *Let $\mathfrak{b}$ be a Lie subalgebra, and let $L$ be linear subspace of $\mathcal{U}(\mathfrak{g})$ containing 1, such that $\mathcal{U}(\mathfrak{g}) = L \otimes \mathcal{U}(\mathfrak{b})$, in the usual sense that the map $L \otimes \mathcal{U}(\mathfrak{b}) \ni l \otimes b \mapsto lb \in \mathcal{U}(\mathfrak{g})$ is a linear isomorphism. Then the map $L \otimes \mathcal{U}(\mathfrak{b}) \ni l \otimes b \mapsto lb \in \mathcal{U}(\mathfrak{g})$ is a topological isomorphism, when $L \otimes \mathcal{U}(\mathfrak{b})$ is equipped with any of the standard tensor product topologies.*

**Comment** The essential point about i) is that the multiplication in $\mathcal{U}(\mathfrak{g})$ is continuous, because bilinear maps on $\mathcal{U}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g})$ are continuous (Proposition II.5.7.v). Another proof of this fact can be given by showing that $\mathcal{U}(\mathfrak{g})$ is isomorphic (as locally convex algebra) to the convolution algebra of distributions at the identity of a Lie group, and by using the fact that the convolution algebra of distributions with compact support on a Lie group is a locally convex algebra in our sense, that is, convolution in $C^0(G)$ is continuous. To show this last fact, however, one also uses (in some form or other) the Dieudonné-Schwartz Theorem (as in the proof of Lemma III.2.4), so that this amounts more or less to the same thing.

28 The projective tensor product topology is the topology of uniform convergence on equicontinuous sets of bilinear forms on $E \times F$, the inductive tensor product topology is the topology of uniform convergence on separately equicontinuous bilinear forms on $E \times F$, and so the latter is finer than the former.
II.6 Induced Locally Convex Modules over Universal Enveloping Algebras

The (complex) universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a real Lie group $\mathcal{G}$ can be identified with the convolution algebra of distributions concentrated at the origin of the group $\mathcal{G}$. In view of the discussion in Section II.5, we equip $\mathcal{U}(\mathfrak{g})$ with the inductive limit topology with respect to its finite-dimensional subspaces, making it into a space of the type $C(\mathbb{N})$.

As in Section I.7, let $\mathfrak{b}$ be a subalgebra of $\mathfrak{g}$. Assume that $W$ is a locally convex $\mathcal{U}(\mathfrak{b})$–module. This is quite a reasonable assumption. For example, let $\mathcal{B}$ act smoothly and transitively on a $C^\infty$–manifold $\mathcal{Y}$. Then the space of distributions on $\mathcal{Y}$ is a (complete, Hausdorff) locally convex $\mathcal{U}(\mathfrak{b})$–module, $\mathcal{U}(\mathfrak{b})$ acting through the infinitesimal representation (this is shown in Chapter III).

With this assumption on $W$, according to Theorem II.2.5 the vector space $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} W$ becomes a locally convex left $\mathcal{U}(\mathfrak{g})$–module characterized by the universal property that every continuous $\mathcal{U}(\mathfrak{b})$–linear map $\varphi$ into a complete locally convex $\mathcal{U}(\mathfrak{g})$–module $Z$ has a unique continuous $\mathcal{U}(\mathfrak{g})$–linear extension $\tilde{\varphi}: \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} W \longrightarrow Z$, that is to say, $\tilde{\varphi}(1 \otimes_{\mathcal{U}(\mathfrak{b})} w) = \varphi(w)$.

Combine Corollary II.5.8 and Proposition II.3.2 to obtain the following:

**Proposition II.6.1**

Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{b}$ a subalgebra.

Let $W$ be a locally convex $\mathcal{U}(\mathfrak{b})$–module.

Then the induced locally convex module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} W$ has the following properties.

i) The canonical map $\kappa: W \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} W, \kappa(w) = 1 \otimes_{\mathcal{U}(\mathfrak{b})} w$, is a one–to–one homomorphism onto a topological direct factor.

ii) Equipped with the projective tensor product topology over $\mathcal{U}(\mathfrak{b})$ the space $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} W$ is a Hausdorff space, with completion $\mathcal{U}(\mathfrak{g}) \hat{\otimes}_{\mathcal{U}(\mathfrak{b})} W$.

iii) Let $L$ be any linear subspace of $\mathcal{U}(\mathfrak{g})$, containing $1$, such that $\mathcal{U}(\mathfrak{g}) = L \mathcal{U}(\mathfrak{b})$, in the sense that the map $L \otimes \mathcal{U}(\mathfrak{b}) \ni l \otimes \mathfrak{b} \longmapsto l \mathfrak{b} \in \mathcal{U}(\mathfrak{g})$ is a linear isomorphism. Then the composition of linear maps
is a linear topological isomorphism. In particular, \( \mathcal{U}(g) \otimes W \) is the direct sum of \( L \otimes W \) and the closed additive hull of the tensors \( l \otimes b \otimes w, l \in L, b \in \mathcal{U}(b), w \in W \). Moreover, \( J \) is the kernel of the well-defined continuous linear map \( \mathcal{U}(g) \otimes W \ni l \otimes b \otimes w \mapsto l \otimes b \otimes w \in L \otimes W, l \in L, b \in \mathcal{U}(b), w \in W \).

iv) Restriction of the map in iii) yields a linear isomorphism

\[
L \otimes W \longrightarrow \mathcal{U}(g) \otimes W \longrightarrow \mathcal{U}(g) \otimes \mathcal{U}(b)
\]

which is bicontinuous for the projective tensor product topologies on these spaces. In particular, \( \mathcal{U}(g) \otimes W \) is the direct sum of \( L \otimes W \) and the subspace \( J \) generated additively by tensors \( l \otimes b \otimes w, l \in L, b \in \mathcal{U}(b), w \in W \). Moreover, \( J \) is the kernel of the well-defined continuous linear map \( \mathcal{U}(g) \otimes W \ni l \otimes b \otimes w \mapsto l \otimes b \otimes w \in L \otimes W, l \in L, b \in \mathcal{U}(b), w \in W \).

This means also that \( J \) is closed in \( \mathcal{U}(g) \otimes \pi W \).

The proposition is a topological extension of Proposition I.7.5. To obtain a proof, in Proposition II.3.2 take \( B=\mathcal{U}(g), A=\mathcal{U}(b) \). There is in this case no distinction between \( B \) and \( A \), since \( L \otimes \mathcal{U}(b) \) is complete (according to Corollary II.5.8).

The isomorphism \( L \otimes W = \mathcal{U}(g) \otimes \mathcal{U}(b) \) is very useful, and one can use it to prove certain intrinsic statements about \( \mathcal{U}(g) \otimes \mathcal{U}(b) \). We first look at the concept of transversal order.

First note that each subspace \( \mathcal{U}(g)(m) \otimes \mathcal{U}(b) \) (see Section I.7) of \( \mathcal{U}(g) \) is closed, and is isomorphic to \( S(\mathcal{I})(m) \otimes \mathcal{U}(b) \) under the map \( \Lambda_m(P \otimes b) = \lambda(P)b \), according to Proposition II.5.7 (compare the paragraphs following the proof of Theorem I.7.6).

The space \( S(\mathcal{I})(m) \) is finite dimensional, so that for any complete locally convex space \( W \) the tensor product \( S(\mathcal{I})(m) \otimes W \) equipped with, for example, the projective tensor product topology, is a complete locally convex space. Moreover, \( S(\mathcal{I})(m) \otimes W \) can be seen as the finite direct sum of a number of copies of \( W \). It is a subspace (with induced topology) of \( S(\mathcal{I}) \otimes W \), and so a subspace with induced topology of \( S(\mathcal{I}) \otimes W \). Being complete, it is a closed subspace.

Let \( W \) be a (complete, Hausdorff) locally convex \( \mathcal{U}(b) \)-module. We know from Proposition I.7.9 that if \( I \) is a linear complement of \( b \) in \( g \), then under either of the isomorphisms
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\[
\begin{align*}
\mathcal{S}(l) \otimes \mathcal{W} & \longrightarrow \mathcal{O}(g) \otimes \mathcal{W} \longrightarrow \mathcal{O}(g)_{\mathcal{O}(b)} \\
\text{or } P \otimes \mathcal{W} & \longmapsto \chi(P) \otimes \mathcal{W}, \quad u \otimes \mathcal{W} \longmapsto u \otimes \mathcal{W}
\end{align*}
\]

(II.6.1.a)

the subspaces \(\mathcal{S}(l)^{(m)} \otimes \mathcal{W}\) correspond to the spaces \([\mathcal{O}(g)_{\mathcal{O}(b)}] \otimes \mathcal{W}\) of tensors of transversal order not exceeding \(m\). Moreover, we know from Proposition II.6.1 (slightly adjusted for the occasion) that these isomorphisms are topological for the projective tensor product topologies. Using the facts concerning \(\mathcal{S}(l)^{(m)} \otimes \mathcal{W}\) that we just mentioned, we see that

\[
[\mathcal{O}(g)_{\mathcal{O}(b)}] \otimes \mathcal{W}
\]

is a complete Hausdorff space, and therefore equals \([\mathcal{O}(g)_{\mathcal{O}(b)}] \otimes \mathcal{W}\). One thus obtains a topological variant of Proposition I.7.9.

**Proposition II.6.2 Transversal order, topological**

Let \(\mathcal{W}\) be a locally convex \(\mathcal{O}(b)-\)module.

i) Equipped with its own projective tensor product topology, the space of tensors of transversal order at most \(m\):

\[
[\mathcal{O}(g)_{\mathcal{O}(b)}] \otimes \mathcal{W} = [\mathcal{O}(g)_{\mathcal{O}(b)}] \otimes \mathcal{W}
\]

is a complete Hausdorff space

ii) The canonical map \([\mathcal{O}(g)_{\mathcal{O}(b)}] \otimes \mathcal{W} \longrightarrow \mathcal{O}(g)_{\mathcal{O}(b)} \otimes \mathcal{W}\) is a one-to-one linear-topological homomorphism onto a direct factor;

in short, \([\mathcal{O}(g)_{\mathcal{O}(b)}] \otimes \mathcal{W} \) is a direct factor in \(\mathcal{O}(g)_{\mathcal{O}(b)} \otimes \mathcal{W}\).

iii) \([\mathcal{O}(g)_{\mathcal{O}(b)}] \otimes \mathcal{W} \) is a \(\mathcal{O}(b)-\)submodule in \(\mathcal{O}(g)_{\mathcal{O}(b)} \otimes \mathcal{W}\).

**Proof** Only ii) still needs some comment. One can use Lemma II.3.3 to prove that \(\mathcal{S}(l)^{(m)}\) has a topological complement \(\mathcal{K}\) in \(\mathcal{S}(l)\), which gives rise to a topological direct sum \(\mathcal{S}(l) \oplus \mathcal{K} \oplus \mathcal{W} = (\mathcal{M} \oplus \mathcal{W}) \oplus (\mathcal{S}(l)^{(m)} \otimes \mathcal{W})\). The isomorphism \(\Lambda:\)

\[
\begin{align*}
\mathcal{S}(l) \otimes \mathcal{W} & \longrightarrow \mathcal{O}(g) \otimes \mathcal{W} \longrightarrow \mathcal{O}(g)_{\mathcal{O}(b)} \\
\text{or } P \otimes \mathcal{W} & \longmapsto \chi(P) \otimes \mathcal{W}, \quad u \otimes \mathcal{W} \longmapsto u \otimes \mathcal{W}
\end{align*}
\]

yields ii) \(\square\)