Gibbsian properties
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As has been seen in Chapter 3, even the physically simple transformation of heating produces non-Gibbsian behaviour. It would even be more interesting to say something about cooling dynamics. More generally one would like to study a Gibbs measure $\mu_0$ for an initial Hamiltonian $H$ which is subjected to a Glauber dynamics for another Hamiltonian $\bar{H}$, which gives rise to a trajectory $\mu_t$ where $t$ denotes time. Glauber dynamics at low temperatures describes fast cooling or "quenching". The question is to understand the behaviour of $\mu_t$, and in particular for which times it will be Gibbs. Since this is as yet too difficult on the lattice, we present our results for mean-field models.

While we give a formal definition of a mean-field model in the latter part, informally mean-field models can be viewed as the "zeroth-order" approximation to lattice systems. Physically, the idea of mean-field theory has a source in the idea that one replaces all interactions with a "mean field". Quite often, mean-field theory provides a convenient launching point to studying higher-order approximations (model on trees, e.g.) In general, dimensionality plays a strong role in determining whether a mean-field approach will work for any particular problem. In mean-field theory, many interactions are replaced by one effective interaction. Then it naturally follows that if the field or particle participates in many interactions in the original system, a mean-field model will be more accurate for such a system. This is true in cases of high dimensionality, or when the Hamiltonian includes long-range forces. The Ginzburg criterion is the formal expression of how fluctuations render mean-field theory a poor approximation, depending upon the number of spatial dimensions in the system of interest [1]. Investigations for mean-field models tend to reproduce the lattice results in many situations [37, 52] but often lead to an explicit knowledge of the parameter regions where Gibbsianness and non-Gibbsianness occur.

As we discussed in Section 2.4, the Gibbsian property strongly depends on continuity properties of single-site conditional probabilities. To understand discontinuous behaviour of conditional probabilities for the time-evolved model at fixed time $t$ one needs to look at the model resulting from the initial measure at time $s = 0$ under application of the dynamics in the space-time region for times $s$ between 0 and $t$. The hidden phase transitions responsible
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for the non-Gibbsian behaviour occur if there is a sensitive dependence of the model at time $s = 0$ obtained from constraining the space-time measure to certain configurations at time $s = t$. If a small variation of such a constraining configuration leads to a jump in the constrained initial measure it will (generically) be a bad configuration for the conditional probabilities of the system at time $t$. Small variation means in the lattice case a perturbation in an annulus far away from the origin. Small variation means in the mean-field case a small variation of the magnetization as a real number. In the independent spin-flip lattice example of [11] the chessboard configuration was a bad one, correspondingly in the independent spin-flip mean-field case of [36] the configurations with neutral magnetization equal to zero were bad ones for large enough times. Moreover, configurations with non-zero magnetization also appeared as points of discontinuity for the limiting conditional probabilities, in a particular bounded region of the parameter space of initial temperature and time. This phenomenon was called biased non-Gibbsianess in [36]. The complete analysis for the mean-field independent spin-flip situation was possible since the constrained system on the first layer could be understood on the level of the magnetization. The relevant quantities could be computed in terms of the rate-function for a standard quenched disordered model, namely the Curie-Weiss random-field Ising model with possibly non-symmetric random-field distribution of the quenched disorder.

To deal with the dependent-dynamics case a different route has to be taken since the dependence of the initial system on the conditioning is more intricate. As we will see, we will need to invoke the path large deviation principle for the dynamics with temperature $\beta^{-1}$ on the level of magnetizations. We will then have to minimize a cost functional of paths of magnetizations which is composed of the rate function along the path and an initial “punishment” term, which depends both on the initial Hamiltonian $H$ and the dynamical Hamiltonian $\bar{H}$, evaluated at the unknown initial point of the trajectory. The solution of the problem gives a surprising connection between path properties of the corresponding (integrable) dynamical system and Gibbs properties of a model of statistical mechanics. Solutions of the corresponding dynamical system will correspond to most probable ways for the system to evolve from an (unknown) initial state to a present. A phase transition in the “constrained” model will be shown to be connected with existence of several most probable histories of a current state of the system. We represent the aforementioned phase transition graphically in the form of the most probable histories for magnetization of the system in a space-time diagram. Such a diagram allows us to visualize all most probable curves for all types of conditioning. Moreover forbidden regions pop up, the history-curves do not enter these regions. These regions grow with time. In the case of independent spin-flip dynamics $\beta = 0$ starting configurations (positively magnetized) with magnetizations located
within an $\varepsilon$-neighbourhood of a certain positive value are only allowed. This
effect corresponds to a memory-losing. As a result we will provide a full
description of the regions of Gibbsian and non-Gibbsian behaviour as a function
of time, initial temperature, and dynamical temperature. As a special case the
previous results for infinite-temperature dynamics are reproduced. Furthermore we observe a new mechanism for the appearance of bad configurations
in the region of cooling from low temperatures with even lower temperatures.
These are related to periodic motion in the dynamical system. The correspond-
ning periodic curves are found numerically from the differential equations
governing both independent and dependent spin-flip dynamics.

### 4.1 Probabilistic analysis

#### 4.1.1 Preliminaries

This section is devoted to a Gibbsian description of mean-field models and
one type of dynamics defined for them. We transfer the relevant definitions of
Gibbsianness and non-Gibbsianness from the lattice setup to the mean-field
setup. A very broad and a review type description of the connection between
these systems could be found in Le Ny [42].

**Curie-Weiss Ising model**

We first set the graph $G = (\mathcal{V}, \mathcal{E})$ of Chapter 2 to be a complete graph. Let
spin-variables sitting at vertices in $\mathcal{V}$ take values in $S = \{-1, +1\}$, as before.
Let $\Lambda$ be a sub-volume of $\mathcal{V}$. The graph structure suggests the isomorphism
$\Lambda \simeq [1,|\Lambda|]$, where $|\cdot|$ is the cardinality of the set, and the interval $[1,|\Lambda|]
contains only natural numbers. Hereafter we consider that the cardinality of
$\Lambda$ is $N$ and identify it with the interval $[1,N]$. We shall write $\sigma_{[1,N]}$ for $\sigma_{\Lambda}$,
the spin configuration $\sigma_{[1,N]}$ takes values in $S^N$. The a priori measure $\alpha(d\sigma_1)$
is chosen to be equidistribution. The finite-volume Gibbs measure at inverse
temperature $\beta$ and zero external field on $\sigma_{[1,N]}$ is given by

$$
\mu_{\beta,N}(d\sigma_{[1,N]}) = \frac{\exp \left\{ \frac{\beta}{2N} \sum_{(i,j) \in \mathcal{E}} \sigma_i \sigma_j \right\}}{Z_{\beta,N}} \alpha \otimes (d\sigma_{[1,N]}),
$$

where the normalization factor $Z_{\beta,N}$ is the standard partition function. The completeness of the graph allows a replacement of many-spin interactions by
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an effective interaction in the following way.

\[
\mu_{\beta,N}(d\sigma_{[1,N]}) = \frac{\exp \left\{ \frac{\beta}{2N} \left( \sum_{i \in V} \sigma_i \right)^2 \right\}}{Z_{\beta,N}} \alpha^{\otimes}(d\sigma_{[1,N]})
\]

\[
= \frac{\exp \left\{ -N \Phi \left( \frac{1}{N} \sum_{i \in V} \sigma_i \right) \right\}}{Z_{\beta,N}} \alpha^{\otimes}(d\sigma_{[1,N]}),
\]

(4.1)

where the function \( \Phi(x) = -\frac{\beta x^2}{2} \) is called a mean-field interaction, while the Hamiltonian of the system is \( H = N \Phi \left( \frac{1}{N} \sum_{i \in V} \sigma_i \right) \). We shall refer to this form of the finite-volume prescription as the mean-field finite-volume Gibbs measure in the sequel.

An infinite-volume Gibbs measure \( \mu \) is obtained by taking limits when \( N \to \infty \) in (4.1). A sequence \( \{\mu_{\beta,N}\} \) has a weak limit \( \mu \) according to the results of [16].

Generally, a mean-field model is defined as a sequence \( \{\mu_N\}_{N \in \mathbb{N}} \) of probability measures, such that each element \( \mu_N \) of this sequence is invariant under permutations of \( \sigma_{[1,N]} \). Moreover, this sequence is required to have a weak limit as \( N \to \infty \). As clear from the previous discussion, all these requirements are satisfied for \( \mu_N \) of form (4.1). Such a model is called Curie-Weiss Ising model in a vanishing external field. Having in mind that we will always work with this model, we shall refer to it also as a mean-field model.

Gibbsianness for mean-field models

We aim to transfer the notion of non-Gibbsianness to mean-field models. We will make use (again) of the underlying graph structure. Let us remind the reader what the tokens intrinsically identifying a Gibbs nature of a measure are. An infinite-volume measure \( \mu \) is Gibbs if and only if its finite-volume restrictions are uniformly non-null w.r.t. conditioning on exterior and quasi-local as functions of the conditioning. This definition does not exploit the graph’s structure and refers to a general graph. We note that this definition cannot be modified anyhow in the case when spins of a statistical mechanics model interact in a local fashion. This was the case of trees (see Chapter 3) and the case for general lattices \( \mathbb{Z}^d \).

There are two crucial differences between the lattice and the mean-field case:

(i) each site is a neighbour of any other. This fact implies that there are no “far-away” regions and an effect of conditioning on “outside” of any finite region is immediately transmitted to the concerned region;

(ii) in the lattice setup the continuity properties of a family of finite-volume conditioned restrictions are studied. The exact state where this family
comes from was identified by a boundary law. In the mean-field situation there is necessity of actual taking limits as volumes grow, not only considering the infinite-volume measure.

This requires to adjust the idea of conditioning. We start from noticing that the random variables \( \sigma_i, i \in [1, N] \) are exchangeable. This means that their joint distribution is independent of the order in which \( \sigma_i \)'s are observed, for instance for \( \sigma_1 \) and \( \sigma_2 \) vectors \( (\sigma_1, \sigma_2) \) and \( (\sigma_2, \sigma_1) \) have the same distribution. Exchangeability comes from the fact that all spin-variables are i.i.d.

Thus the number of pluses and minuses plays a role. The product of a priori measures \( \alpha \otimes (\cdot) \) for large \( N \) behaves as follows \([22]\)(or, in English and in greater generality, \([30]\))

\[
\alpha \otimes (d\sigma_{[1,N]}) = \int_{-1}^{1} \theta^t_N (1 - \theta)^{N - t_N} dG(\theta),
\]

where \( t_N \) is the number of pluses. It further holds that \( G \) is the distribution function of the limiting frequency \( m_N(\sigma) = \lim_{N \to \infty} m_N^N = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \sigma_i \). The empirical measure \( m \) in the language of statistics is sufficient for the unknown parameter \( \theta \). Some estimates on the speed of the convergence of \( m_N^N \) to \( m \) could be found in \([8]\).

The exchangeability of \( \sigma_{[1,N]} \) implies that the conditionings which are permutations of one another will give rise to the same result, effectively the empirical measure only matters. Generally, the empirical measure \( L \) of a configuration \( \sigma_{[1,N]} \) is defined as \( L(\sigma_{[1,N]})(\cdot) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\sigma_i}(\cdot), \) as in e.g. \([48, \text{ see Chapter 4}]\). For binary single-site state space \( S \), a magnetization or empirical mean \( m \) is treated.

We stress that the non-nullness is always granted for mean-field models as follows from the finite-volume representation of \( \mu \).

When a system’s size is finite and equal to \( N \), the “rest of the world” for a single spin is a configuration of size \( N - 1 \). By convention we will refer to a single spin as \( \sigma_1 \) and to the “rest of the world” as \( \sigma_{[2,N]} \). A magnetization value \( m_2^N \) for a system of a configuration of size \( N - 1 \) belongs to \( \{-1, -1 + \frac{2}{N-1}, \ldots, 1 - \frac{2}{N-1}, 1\} \).

**Definition 4.1.1.** A single-site finite-volume conditional probability \( \gamma_{\beta,1,N}(d\sigma_1|m_2^N) \) is defined as follows:

\[
\gamma_{\beta,1,N}(d\sigma_1|m_2^N) := \mu_{\beta,N}(d\sigma_1|\sigma_{[2,N]}),
\]

where \( \sigma_{[2,N]} \) is any spin-configuration such that \( m_2^N = \frac{1}{N-1} \sum_{j=2}^{N} \sigma_j \).

The infinite-volume conditioning is obtained by taking a formal limit when the \( N \) grows. A discrete-valued magnetization \( m_2^N \in \{-1, -1 + \frac{2}{N-1}, \ldots, 1 - \frac{2}{N-1}, 1\} \)
\( \frac{2}{N-1}, 1 \) converges to a real-valued magnetization \( m \in [-1, 1] \) under such a limiting procedure.

\[
\lim_{N \to \infty} \gamma_{\beta,1,N}(d\sigma_1|m_N) =: \gamma_{\beta,1}(d\sigma_1|m) \quad (4.3)
\]

The former definition reflects the main idea of mean-field theory of replacing many-bodies interaction with an average interaction. Equivalently, the whole mean-field system could be seen as a system containing only two spins \( -\sigma_1 \) and \( \sigma_{\text{aver}} \), where \( \sigma_1 \in S \) and \( \sigma_{\text{aver}} \) lives in a continuous space \([-1,1]\).

For a lattice or a tree a configuration is said to be bad if it broadcasts the information of “far-away” regions. Oppositely, a configuration is said to be good if it stops the influence of far regions. In the mean-field setup this reads:

**Definition 4.1.2.** A point \( \hat{m} \in [-1,1] \) is said to be good for a mean-field model if and only if:

1. There exists a neighborhood of \( m_N^2 \) such that, for all \( \alpha \) in this neighborhood the following holds. For all sequences \( \alpha_N \in \{-1, -1 + \frac{2}{N-1}, \ldots, 1 - \frac{2}{N-1}, 1\} \) with the property \( \lim_{N \to \infty} \alpha_N = \alpha \) the limit

\[
\gamma_{\beta,1}(|m_1\alpha) = \lim_{N \to \infty} \gamma_{\beta,1,N}(|m_1\alpha_N)
\]

exists and is independent of the choice of the sequence \( \alpha_N \).

2. The function \( \alpha \mapsto \gamma_{\beta,1}(|m_1\alpha) \) is continuous at \( \alpha = \hat{m} \).

A point \( \hat{m} \) is bad, if it is not good. Here we are in the position to give a rigorous definition of Gibbsianness for a mean-field model.

**Definition 4.1.3.** A mean-field model at inverse temperature \( \beta \) is called Gibbs if and only if it has no bad points.

The Definition 4.1.1 could be extended to any finite volume and, moreover, finite-volume conditional probabilities may be expressed in terms of single-site conditional probabilities. This representation is proven in [48, see Chapter 4, Proposition 4.2.2] for a general mean-field interaction. From the present standpoint the most important statement of the aforementioned proposition is

\[
\gamma_{\beta,n}(d\sigma_n|m) := \lim_{N \to \infty} \mu_{\beta,N}(d\sigma_n|\sigma_{[n+1,N]}) = \prod_{i=1}^{n} \gamma_{\beta,1}(d\sigma_i|m), \quad (4.5)
\]

where \( m = \lim_{N \to \infty} \frac{1}{N-n} \sum \sigma_i \).
4.1.2 Spin-flip transform of mean-field model

In the present setup the finite-volume measure (4.1) may be rewritten in such a way that it will explicitly contain the effective parameter $m_1^N$, namely

$$
\mu_{\beta,N}(d\sigma_{[1,N]}) = \exp \left\{ \frac{N \beta (m_1^N)^2}{2} \right\} \alpha^\otimes(d\sigma_{[1,N]}),
$$

(4.6)

where $m_1^N$ is an empirical mean of the configuration $\sigma_{[1,N]}$. This representation is equivalent to (4.1) and both will be treated as a definition of the finite-volume Gibbs measure for Curie-Weiss Ising model in a vanishing external field.

Given an initial Gibbs mean-field model with measures $\mu_{\beta,N}$ (4.6), our aim is to investigate the Gibbs properties of the transformed model under site-wise independent spin-flip evolution. Hence, rewriting (2.7), we get

$$
\mu'_{\beta',\beta,N}(d\eta_{[1,N]}) = \sum_{\sigma_{[1,N]}} \tilde{\mu}_{\beta',\beta,N}(d\sigma_{[1,N]}) = \sum_{\sigma_{[1,N]}} \mu_{\beta,N}(d\sigma_{[1,N]}) \prod_{i=1}^N k(\sigma_i, \eta_i)
$$

(4.7)

The corresponding joint model is given by $\tilde{\mu}_{\beta',\beta,N}(d\sigma_{[1,N]})$.

The study of the Gibbs properties is based on the investigation of continuity properties of the single-site conditional distributions $\gamma'_{\beta',\beta,t,1}$ of the corresponding transformed model. As before,

$$
\gamma'_{\beta',\beta,t,1}(d\eta_1|m') := \lim_{N \to \infty} \frac{1}{N-1} \sum_{i=2}^N k(\sigma_i, \eta_i)
$$

Fix a transformed configuration $\eta$, equivalently, fix the corresponding magnetization $m'$. The completeness of the graph relaxes requirements for non-Gibbsian behaviour of the transformed measure. An existence of a phase transition for the two-layered model with fixed second layer becomes a sufficient condition for non-Gibbsianness. Suppose that a finite-volume joint model with the fixed particular $m'$ admits several infinite-volume measures corresponding to different magnetizations (say) $m_{i1}'(m')$ and $m_{i2}'(m')$. The distributions $\gamma'_{\beta',\beta,t,1}(\cdot|m')$ turn out to be functions of the magnetization $m^*$. Therefore varying the conditioning in a neighbourhood of $m'$, the single-site conditional distributions will obtain discontinuities of a jump type. We are left with a problem of identifying such a magnetization(s) $m^*$ of the initial spins $\sigma$, so the model started at $\sigma$ and evolved ends up in configuration $\eta$, equivalently, with magnetization $m'$. The study of evolution of the system from the state $m^*$ to the state $m'$ involves large deviations theory for trajectories of stochastic processes. Such an approach allows to keep track, in addition, of a most probable evolution of the system.
4.1.3 Another view on spin-flip evolution

The spin-flip evolution on the level of individual spins was explained in Section 2.5. To be more specific, let \( c(\pm, m_N) \) be a single-site rate for \( \sigma_1 \) to flip from plus- to minus-state, when the magnetization of the rest of the system (consisting \( N - 1 \) components) is \( m_N \). Analogously, \( c(-, m_N) \) is defined. We are going to apply a \emph{temperature-dependent} spin-flip dynamics. This is reflected in the fact that the spin-flip rates depend implicitly (for now) on the temperature \( \beta' \) of the dynamics. Moreover, their values depend on the initial inverse temperature \( \beta \) and time. The time-dependence of the rates occurs via the time-dependence of the magnetization \( m \). The explicit formula for the rates will be given later. The dynamics is called \emph{constrained} or \emph{interacting} if the inverse temperature \( \beta' \) of the dynamics is non-zero. Unless the opposite is stated a dynamics is considered with both \( \beta \) and \( \beta' \) not equal to zero.

We require the dynamical measure \( \mu_{\beta'} \) to be \emph{time-reversible}. Equivalently, this requirement for the single-site spin-flip rates may be re-expressed as follows:

\[
\frac{c(-, m)}{c(+, m)} = \exp \{2\beta' m\},
\]

where \( c(\pm, m) \) are the rates depending on the magnetization of infinite-volume. See Appendix B.1 for an explanation.

In other words,

\[
c(\sigma_1, m) = R(m) \exp \{-\sigma_1 \beta' m\},
\]

with a function \( R(m) \) giving a time-rescaling.

Adjusting the form of the linear generator \( L \) of the configurations spin-flip dynamics from the Section 2.5, we define a generator \( L_{\beta',N} \) for local functions \( \bar{F} : \Omega_{[1,N]} \rightarrow \mathbb{R} \)

\[
L_{\beta',N} \bar{F}(\sigma_{[1,N]}) = \sum_{i=1}^{N} c \left( \sigma_i, \frac{1}{N-1} \sum_{j:j\neq i} \sigma_j \right) \left( \bar{F}(\sigma_{[1,N]}) - \bar{F}(\sigma_{[1,N]}^i) \right),
\]

where \( \sigma_{[1,N]}^i \) is the configuration that is flipped at the site \( i \).

In the light of (4.6) it is wise to trace how spin-flip dynamics acts on the magnetization. We remark that by permutation invariance the continuous time process, induced on the empirical average, is a Markov chain. Namely, suppose that \( \bar{F} : \{-1, 1 - \frac{2}{N}, \ldots, 1 - \frac{2}{N}, 1\} \rightarrow \mathbb{R} \) is a function on the possible magnetization values at size \( N \), then the linear generator \( \hat{L}_{\beta',N} \) acting
on such functions $F$ defined as $(L_{\beta',N}F) \circ m_1^N = L_{\beta',N}(F \circ \sigma_{[1,N]}^s)$ with

$$
\hat{L}_{\beta',N}F(m_1^N) = N \frac{1 + m_1^N}{2} c \left( +, l(N) \left( m_1^N - \frac{1}{N} \right) \right) \left( F \left( m_1^N - \frac{2}{N} \right) - F \left( m_1^N \right) \right) + N \frac{1 - m_1^N}{2} c \left( -, l(N) \left( m_1^N + \frac{1}{N} \right) \right) \left( F \left( m_1^N + \frac{2}{N} \right) - F \left( m_1^N \right) \right)
$$

(4.11)

where $l(N) = \frac{N}{N-1}$. Clearly, $l(N)$ may be omitted when $N$ is sufficiently large.

The asymptotic rates in (4.11) with $N \to \infty$, for the magnetization are

$$
c_{\pm}(m) = \frac{1 \pm m}{2} c(\pm, m) = R(m) \frac{1 \pm m}{2} \exp\{ \mp \beta' m \} \quad (4.12)
$$

We treat the prefactor $N$ in (4.11) as of a scaling for test functions $F$.

Summarizingly, we would like to study a stochastic spin-flip evolution of infinite spin-configurations $\sigma$. The dynamics of the process $\{\sigma_s, s \geq 0\}$ is described by a semi-group $S(s) = e^{sL_{\beta',N}}$ on the level of spins. The exchangeability allows to study equivalent real-valued process $\{m_s := m(\sigma_s), s \geq 0\}$. The corresponding semi-group of the dynamics is $S_m(s) = e^{sL_{\beta',N}}$.

### 4.1.4 Deterministic behaviour

As a warm-up we ask ourselves how the typical paths for the infinite-temperature starting measure look like for large $N$, when the former measure is subjected to the unconstrained dynamics. To answer this question we look how a magnetization of the system behaves as a function of time. Mathematically, we consider a process $\{m_s, s \geq 0\}$ taking values in $[-1, 1]$ and depending on time $s$ started at the initial value $m_0$. For fixed $N$ we write

$$
\frac{d}{ds} E_N^{m_0} m_s = E_N^{m_0} \frac{d}{ds} m_s = E_N^{m_0} \frac{d}{ds} (S_m(s)m_0) = E_N^{m_0} \hat{L}_{\beta',N} m_s
$$

(4.13)

Interchanging expectation and derivative causes no problem because $N$ is finite. Choosing as a test function in (4.11) a simple identity, we get

$$
\frac{d}{ds} E_N^{m_0} m_s = E_N^{m_0} \hat{L}_{\beta',N} m_s =
$$

$$
E_N^{m_0} \left[ (1 - m_s) c \left( -, l(N)(m_s + \frac{1}{N}) \right) - (1 + m_s) c \left( -, l(N)(m_s - \frac{1}{N}) \right) \right]
$$

(4.14)

Taking the limit $N \to \infty$ at both sides, the stochastic process for the magnetization $m_s$ concentrates on a certain deterministic path $m(s)$, see e.g. [23,
Chapter 2. The expectations on both sides may be omitted, because \( m(s) \) is not random anymore. As a shortcut we write \( m \) every time the function \( m(s) \) is meant. Its time derivative is denoted by a dot, i.e. \( \dot{m} \). Choosing \( F \) to be an identity function, we get a deterministic differential equation.

\[
\frac{d}{ds} m = (1 - m)c(-, m) - (1 + m)c(+, m) \\
= (1 - m)R(m)e^{\beta m} - (1 + m)R(m)e^{-\beta m} \\
= 2R(m)(\sinh(\beta m) - m \cosh(\beta m))
\]

The total rate is

\[
c(m) := c_+(m) + c_-(m)
\]

Employing the expressions for the asymptotic rates (4.12), the total rate is given by

\[
c(m) = R(m)(\cosh(\beta m) - m \sinh(\beta m)) \\
= R(m) \cosh(\beta m)(1 - m \tanh(\beta m))
\]

To simplify the problem, we choose the time rescaling to be

\[
R(m) = \frac{1}{\cosh(\beta m) - m \sinh(\beta m)},
\]

making the total rate constant. Taking into account the chosen time rescaling (4.17), the deterministic time evolution results in the ODE

\[
\dot{m} = 2 \frac{\sinh(\beta m) - m \cosh(\beta m)}{\cosh(\beta m) - m \sinh(\beta m)}
\]

In the case \( \beta' = 0 \) the equation reduces to the linear equation \( \dot{m} = -2m \, ds \) which describes the relaxation of the magnetization to zero under the unconstrained infinite-temperature dynamics.

### 4.1.5 Large deviations for stochastic processes

Consider a finite-volume configuration \( \sigma_N \). Let, as usual, \( s \) be a time variable. The induced Markov process \( \{m_s, s \geq 0\} \) on a finite volume with \( N \) spins performs jumps upwards or downwards of size \( \frac{2}{N} \). We remind that \( m_s \) concentrates on a deterministic path \( m(s) \) when \( N \) is large. Let \( p_\beta(m) = \frac{c_-(m(s))}{c_-(m(s)) + c_+(m(s))} \) be the temperature- and \( m(s) \)-dependent probability for a magnetization to go down for large \( N \). Clearly, the asymptotic probability to go up is then given by \( 1 - p_\beta(m) \). Whenever it is clear from the context, we, as before, write \( m \) instead of \( m(s) \) and, additionally, \( p_\beta \) for \( p_\beta(m) \).
Let \( z_{\sigma_{[1,N]}}(s) \in \{-1, -1 + \frac{2}{N}, \ldots, 1 - \frac{2}{N}, 1\} \) be the path of the magnetization for the Markov chain (with the generator as in (4.11)) evolved at inverse temperature \( \beta' \) with the initial condition to be distributed according to the Curie-Weiss measure \( \mu_{\beta,N} \). Denote by \( \mathbb{P}_{\beta,\beta',N} \) the law of the paths \( \{ z_{\sigma_{[1,N]}}(s) \}_{s \in [0,t]} \).

**Theorem 4.1.4.** Suppose \( \{ z_{\sigma_{[1,N]}}(0) \} \) satisfies a large deviation principle (LDP) with rate function \( I_0 \) and rate \( N \), then the measure \( \mathbb{P}_{\beta,\beta',N} \) satisfies a large deviation principle in \( L^2[-1,1] \) with rate \( N \) and rate function

\[
I(m) = I_0(m(0)) + \int_0^t \mathcal{L}_{\beta'}(m(s), \dot{m}(s)) ds
\]

(4.19)

with Lagrange density \( \mathcal{L}_{\beta'}(m(s), \dot{m}(s)) \) given by

\[
\mathcal{L}_{\beta'}(m, \dot{m}) = \frac{\dot{m}}{2} \ln \left( \frac{\dot{m} + \sqrt{16p_{\beta'}(1 - p_{\beta'}) + \dot{m}^2}}{4p_{\beta'}} \right) - \frac{1}{2} \sqrt{16p_{\beta'}(1 - p_{\beta'}) + \dot{m}^2} + 1,
\]

(4.20)

where \( p_{\beta'} = p_{\beta'}(m) \).

**Proof of Theorem.** There are two ingredients needed to prove the statement of the theorem. The first ingredient we need is the static large deviation principle for the magnetization in the initial measure, the Curie-Weiss measure with inverse temperature \( \beta \). Secondly, the formalism of Feng and Kurtz [17] allows us to obtain the form of the Lagrangian density, then the correctness of the present statement is a mere application of [17, Theorem 13.7].

We start with the proposition about static LDP. It reads as follows.

**Proposition 4.1.5.** The distribution of magnetization \( m_1^N = \frac{1}{N} \sum_{i=1}^N \sigma_i \) w.r.t. the Curie-Weiss measure at inverse temperature \( \beta \) obeys a large deviation principle with rate \( N \) and rate function \( I_0 \):

\[
I_0(m) = \Phi(m_1^N) + I(m_1^N),
\]

(4.21)

where

\[
I(m_1^N) = \frac{1 + m_1^N}{2} \log(1 + m_1^N) + \frac{1 - m_1^N}{2} \log(1 - m_1^N)
\]

(4.22)

is the rate function for the symmetric Bernoulli distribution and \( \Phi(m_1^N) = \frac{-\beta(m_1^N)^2}{2} \) is a mean-field interaction.
Proof of Proposition. Proposition 4.1.5 follows from Varadhan’s Lemma. Suppose a probability distribution satisfies an LDP principle with a known rate function and rate $N$ and suppose we consider the probability distribution with density $Ce^{-N\Phi(x)}$ relative to the first density. Under suitable conditions of the function $\Phi(x)$ (boundedness and continuity will suffice) this probability distribution will satisfy an LDP with the same rate $N$ and rate function obtained by adding $\Phi(x)$ to the first rate function and subtracting a constant.

Let $k$ be the number of pluses in a configuration $\sigma_{[1,N]}$ with mean $m_N$. The number of pluses may be expressed as $1+m_N/2N$. Applying the apparatus of [35] to our framework, the heuristics described before, slightly more formally, reads:

$$-\frac{1}{N} \ln \alpha (d\sigma_{[1,N]}) = -\frac{1}{N} \ln \left( \frac{N}{2N} \right) 2^{-N} \to I(m_N^1),$$

where $\sigma_{[1,N]}$ is any configuration such that $\sum_{i=1}^N \sigma_i = m_N^1$ and

$$-\frac{1}{N} \ln \int C e^{-N\Phi(m_N^1)} \alpha (d\sigma_{[1,N]}) \to \inf (-\Phi(m_N^1) - I(m_N^1)) = \sup (\Phi(m_N^1) + I(m_N^1))$$

□

When a magnetization path is considered, it is clear that $m_N^1$ is nothing but $z_{\sigma_{[1,N]}}(0)$ and for large $N$ the value $m_N^1$ concentrates on $m(0)$.

The Lagrangian density is computed employing the notion of non-linear generator introduced in [17]. The non-linear generator $\hat{\mathcal{H}}_{\beta',N}$ acts on test functions $F : \{-1, -1 + \frac{2}{N}, \ldots, 1 - \frac{2}{N}, 1\} \to \mathbb{R}$ and is defined as

$$\left( \hat{\mathcal{H}}_{\beta',N}F \right)(x) = \lim_{N \to \infty} \frac{1}{N} e^{-NF(x)} \left( \hat{\mathcal{L}}_{\beta',N}e^{NF} \right)(x) \quad (4.23)$$

The integrand in (4.19) is given by the Legendre transform of $\left( \hat{\mathcal{H}}_{\beta',N}F \right)(m)$ with $F$ being an identity function.

As explained in Appendix B.2, computing the right-hand side for a finite $N$ and taking limits with respect to the volume size yields

$$\mathcal{H}(m, F'(m)) := \left( \hat{\mathcal{H}}_{\beta',N}F \right)(m) = c_-(m)(e^{2F'(m)} - 1) + c_+(m)(e^{-2F'(m)} - 1) \quad (4.24)$$

Due to the chosen rescaling $R(m)$ (4.17) involved into the definition of the rates we deduce that $c_- = p_{\beta'}(m)$ and $c_+ = 1 - p_{\beta'}(m)$. Thence our non-linear generator reads

$$\mathcal{H}(m, F'(m)) = p_{\beta'}(m)(e^{2F'(m)} - 1) + (1 - p_{\beta'}(m))(e^{-2F'(m)} - 1) \quad (4.25)$$
Performing a Legendre transform with \( F(m) = m \)

\[
\mathcal{L}_{\beta'}(m, \dot{m}) = \sup_{\lambda} (\lambda \dot{m} - \mathcal{H}(m, \lambda)),
\]

we arrive at the Lagrangian density which governs the path large deviations as stated in [17, Theorem 13.7 or, in particular situations, Examples 1.5 or 1.12]. This concludes our treatment of the proof.

4.1.6 Minimal cost problem

The joint model connects a layer of initial spins \( \sigma \) and a layer of the evolved (during time \( s = t \)) spins \( \eta \). We would like to identify whether the two-layer model exhibits a phase transition for initial spins when the configuration of the second layer is fixed. Analogously, we say that the second layer has a certain magnetization \( m' \). Conditioning on the second layer induces a conditional distribution on the magnetization values \( m_0 \) at time \( s = 0 \). This is reflected in the following corollary.

**Corollary 4.1.6.** The conditional distribution of the initial magnetization \( m_0 \) taken according to the law of the paths \( \mathbb{P}_{\beta', \beta, N} \), conditioned to end in the final condition \( m' \) at time \( s = t \), satisfies a large deviation principle with rate \( N \) and rate function given by

\[
E_{m'}(m_0, \beta, \beta') = \Phi(m_0) + I(m_0) + \inf_{m(s), \varphi(t) = m'} \int_0^t \mathcal{L}_{\beta'}(\varphi, \dot{\varphi}) ds - \text{Const}(m')
\]

(4.26)

This rate function allows to compute the large-\( N \) asymptotics of the probability to find the system in a final magnetization \( m' \) at time \( s = t \) by computing the value of the rate function in the minimizing path to \( m' \).

The existence of a phase transition for the joint model in our setup corresponds to the fact that a large deviations functional (4.20) weighs several functions \( \varphi_i(s), i \in \mathbb{N} \) constrained to take value \( m' \) at time \( t \), with an equal minimally possible mass. Thence, we may talk of a cost of any function \( \varphi(s) \) defined on \([0, t]\) and valued in \([-1, 1]\) such that \( \varphi(t) = m' \).

**Definition 4.1.7.** Let a function \( \varphi(s) \) be such that \( \varphi(t) = m' \). The functional \( E_{m'}(m_0, \beta, \beta') \) defined in Corollary 4.1.6 is called a cost of evolving from an unknown configuration \( \sigma \), such that its magnetization is \( \varphi(0) \), to the fixed configuration \( \eta \).
Therefore, the existence of a phase transition of the joint-model is well captured by the fact whether the solution of the cost-minimizing problem is unique or not. We denote by \( m(s; t, \bar{m}') \) any path started at unknown state \( m_0 \) at the time \( s = 0 \) and ended up at \( \bar{m}' \) at the time \( s = t \). The magnetization \( m(s; t, \bar{m}') \) viewed as a function of time with a constraint on the final point is called a history. If the cost functional reaches its minimum at \( m(s; t, \bar{m}') \), then such a function is denoted by \( m^*(s; t, \bar{m}') \) and called most probable history (or a most probable curve). The value of magnetization \( m^*(0; t, \bar{m}') \) given by the value a most probable curve at zero time is called most probable starting point.

### 4.1.7 Gibbsianness for transformed measures and limiting conditional distributions

In this section we prove the form of the single-site conditional distributions for the transformed measure. The nature of the transformed measure — Gibbsian or non-Gibbsian— strongly depends on the behaviour of these distributions. To see the problem in full detail we first transfer the relevant definitions of Gibbs measure on the transformed finite-volume measure \( \mu'_{\beta, \beta', t, N} \) defined in (4.7).

The definition of a good point reads the same as Definition 4.1.2 replacing \( \gamma_{\beta, t} \) with \( \gamma_{\beta, \beta', t, 1} \).

**Definition 4.1.8.** Let \( \beta, \beta', t \) be given. A point \( \hat{m} \in [-1, 1] \) is said to be good for a mean-field model if and only if:

1. There exists a neighborhood of \( m_{2N}^{N} \) such that, for all \( \alpha \) in this neighborhood the following holds. For all sequences \( \alpha_N \in \{ -1, -1 + \frac{2}{N-1}, \ldots, 1 - \frac{2}{N-1}, 1 \} \) with the property \( \lim_{N \to \infty} \alpha_N = \alpha \) the limit

   \[
   \gamma_{\beta, \beta', 1, \alpha}(\eta_1|\alpha) = \lim_{N \to \infty} \gamma_{\beta, \beta', 1, N}(\eta_1|\alpha_N)
   \]

   exists and is independent of the choice of the sequence \( \alpha_N \).

2. The function \( \alpha \mapsto \gamma_{\beta, \beta', 1, \alpha}(\eta_1|\alpha) \) is continuous at \( \alpha = m_{2N}^{N} \).

The functions \( \gamma_{\beta, \beta', 1, \alpha}(\eta_1|\cdot) \) are called limiting conditional distributions of a spin value \( \eta_1 \). It is conditional because the spins outside a single-site volume are frozen to have a particular magnetization and limiting since the volume we condition on is infinite.

**Definition 4.1.9.** The time-evolved mean-field model \( \{ \mu'_{\beta, \beta', t, N} \} \) with parameters \( \beta, \beta', t \) is called Gibbs if and only if it has no bad points.
We would like to establish the connection between the notion of a solution for the minimization problem (4.26) and the form of limiting conditional distribution. This requires more knowledge for the single-site evolution. We rewrite the form of the single-site linear generator (2.13) according to our needs for fixed $\beta, \beta', t,$ and $m'$, that is

$$(L_i(s; t, m')f)(\sigma_i) = c(\sigma_i, m^*(s; t, m'))(f(-\sigma_i) - f(\sigma_i)), \quad (4.28)$$

with rates which are obtained by substitution of the optimal path for the constrained problem for the empirical magnetization (4.26) into the single-site flip rates. For shortcut we write

rate of flipping from $\begin{pmatrix} + \\ c_s(+) \end{pmatrix} := c(+, m(s; t, m'))$,
rate of flipping from $\begin{pmatrix} - \\ c_s(-) \end{pmatrix} := c(-, m(s; t, m'))$, \quad (4.29)

We first provide the following theorem.

**Theorem 4.1.10. Single-site transition probability.** Fix $\beta, \beta', t, m'$. Consider a Markov jump process on $\{-1, 1\}$ which is defined by the time-dependent generator (4.28) for these values. Then a probability $k_s(\sigma_i, \eta_i; t, m')$ for a single spin to evolve from an initial value $\sigma_i \in \{-1, 1\}$ at time $s = 0$ to $\eta_i \in \{-1, 1\}$ at time $s \leq t$ is given by the expression

$$k_s(+, +; t, m') = 1 - k_s(+, -; t, m') = k_s(-, -; t, m') = 1 - k_s(-, +; t, m')$$

$$= e^{-\int_0^t (c_u(-) + c_u(+)) du} \times \left[ \int_0^s c_u(-) e^{-\int_0^u (c_v(-) + c_v(+)) dv} du + 1 \right] \quad (4.30)$$

**Proof.** We omit the proof of this fact here and refer to Appendix B.4.

In the second place we define a configuration compatible with a minimizing solution $m^*(s; t, m')$

**Definition 4.1.11.** A configuration $\eta_{[1, N]}$ is called consistent with minimizer $m^*(s; t, m')$ at time $s = s_0$, if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \eta_i = m^*(s_0; t, m')$$

We are now ready to give our formula for the limiting conditional distributions of the model started at the inverse temperature $\beta$ and evolved with $\beta'$ during the time $t$.
Theorem 4.1.12. Fix $\beta, \beta', t, m'$. Suppose the constrained variational problem (4.26) for paths $\varphi$, taken over the paths with fixed right endpoint $\varphi(t) = m'$, has a unique minimizing path $s \mapsto m^*(s; t, m')$.

Then the limiting probability kernels of the time-evolved measure $\mu_{\beta, \beta', t, N}$ have a well-defined infinite-volume limit $\gamma_{\beta, \beta', t}(|m'|)$ in the sense of Definition 4.1.8 of the following form

$$
\gamma_{\beta, \beta', t}(\eta_1|m') = \frac{\sum_{\sigma_1 = \pm 1} e^{\sigma_1 \beta m^*(t, \eta_1|m')} k_t(\sigma_1, \eta_1; t, m')} {\sum_{\sigma_1, \eta_1 = \pm 1} e^{\sigma_1 \beta m^*(t, \eta_1|m')} k_t(\sigma_1, \eta_1; t, m')},
$$

(4.31)

Here $k_t(\sigma_1, \eta_1; t, m')$ as in Theorem 4.1.10.

Proof. We start with initial finite-volume configuration $\sigma_{[1,N]}$. With a little abuse of notation we denote by $\sigma_i(s)$ the value of a spin at site $i$ at time $s$. Spins $\eta_i$ of the transformed configuration could be viewed as values of functions $\eta_i(s)$ started at the value $\sigma_i(0) =: \sigma_i$ for corresponding sites.

Take a sequence $\alpha_N \in \{-1, -1 + \frac{2}{N-1}, \ldots, 1 - \frac{2}{N-1}, 1\}$ with the property $\lim_{N \uparrow \infty} \alpha_N = \alpha$. We denote by $m_2^N(s) = \frac{1}{N-1} \sum_{i=2}^N \sigma_i(s)$ the empirical magnetization of the spins of site 2 to $N$ at time $s$ of the evolution. To prove that the promised form for the limiting conditional probabilities is correct we must show that

$$
\lim_{N \uparrow \infty} \mu'_{\beta, \beta', N}(\sigma_1(t) = +|m_2^N(t) = \alpha_N) = \frac{\gamma_{\beta, \beta', t, 1}(\eta_1 = +|\alpha)} {\gamma_{\beta, \beta', t, 1}(\eta_1 = -|\alpha)},
$$

(4.32)

where the terms of the right-hand side are given by (4.31).

We drop the subscripts of temperatures and time for the conditional probabilities and the transformed measure during the proof of the theorem. The subscript explicitly identifying finiteness of both objects is kept.

$$
\begin{align*}
\gamma_{1,N}(\eta_1 = +|\alpha_N) &= \mu'_{N}(\eta_1 = +|m_2^N(t) = \alpha_N) = \mu'_{N}(\eta_1 = +|m_2^N(t) = \alpha_N) = \\
\gamma_{1,N}(\eta_1 = -|\alpha_N) &= \mu'_{N}(\eta_1 = -|m_2^N(t) = \alpha_N) = \\
\int \mathbb{P}_N(dz_{\sigma_{[1,N]}}|m_2^N(t) = \alpha_N) \mu'_{N}(\eta_1 = +z_{\sigma_{[1,N]}}) = \\
\int \mathbb{P}_N(dz_{\sigma_{[1,N]}}|m_2^N(t) = \alpha_N) \mu'_{N}(\eta_1 = -z_{\sigma_{[1,N]}}) = \\
\int \mathbb{P}_N(dz_{\sigma_{[1,N]}}|m_2^N(t) = \alpha_N) \sum_{\delta = \pm 1} \tilde{\mu}_N(\eta_1 = +|\sigma_1 = \tilde{\sigma}_1, z_{\sigma_{[1,N]}}) \mu_N(\sigma_1 = \tilde{\sigma}_1|z_{\sigma_{[1,N]}}) = \\
\int \mathbb{P}_N(dz_{\sigma_{[1,N]}}|m_2^N(t) = \alpha_N) \sum_{\delta = \pm 1} \tilde{\mu}_N(\eta_1 = +|\sigma_1 = \tilde{\sigma}_1, z_{\sigma_{[1,N]}}) \mu_N(\sigma_1 = \tilde{\sigma}_1|z_{\sigma_{[1,N]}}) =
\end{align*}
$$

(4.33)
Let’s compute \( \mu_N(\sigma_1 = \tilde{\sigma}_1 | z_{\sigma[1,N]}) \).

\[
\mu_N(\sigma_1 = + | z_{\sigma[1,N]}) = \sum_{\sigma[2,N]} \frac{e^{\frac{\beta}{2N} (\sum_{i=2}^n \sigma_i + 1)^2}}{e^{\frac{\beta}{2N} (\sum_{i=2}^n \sigma_i - 1)^2} + e^{\frac{\beta}{2N} (\sum_{i=2}^n \sigma_i - 1)^2}} |_{z_{\sigma[1,N]}}
\]

\[
\sum_{\sigma[2,N]} \frac{1}{1 + \exp \left\{ -2\frac{\beta}{N} \sum_{i=2}^n \sigma_i \right\}} |_{z_{\sigma[1,N]}} = \sum_{\sigma[2,N]} \exp \left\{ -2 \frac{\beta}{N} \sum_{i=2}^n \sigma_i \right\} |_{z_{\sigma[1,N]}}
\]

\[
(4.34)
\]

Generalizing the last statement to any value of \( \sigma_1 \), we write:

\[
\mu_N(\sigma_1 = \tilde{\sigma}_1 | z_{\sigma[1,N]}) = \sum_{\sigma[2,N]} \exp \left\{ \tilde{\sigma}_1 \beta \left( z_{\sigma[1,N]}(0) - \frac{\tilde{\sigma}_1}{N} \right) \right\} \frac{Z}{Z}
\]

\[
(4.35)
\]

While taking an infinite-volume limit, we make use of the assumption on the uniqueness of the solution of the constrained path large deviation principle made in the statement of the theorem. Under our assumption the distribution \( \mathbb{P}_{\beta,\beta',N}(z_{\sigma_N}(s) | m^N_s(t) = \alpha_N) \) concentrates exponentially fast on the unique deterministic trajectory \( m^* : s \mapsto m^*(s; t, m') \) as \( N \) tends to infinity. This collapses the outer expected value and simplifies the formula a lot. Next we have that whenever \( z_{\sigma[1,N]} \to m^*(s; t, m') \), we are guaranteed that at each moment of time \( s \in [0, t] \) there exists a unique (up to permutations) configuration \( \eta_{[1,N]} \), that is consistent with \( m^*(s; t, m') \), meaning that summation over \( \sigma[2,N] \) in (4.35) counts just the number of permutations of the configurations consistent with \( m^*(0; t, m') \) at time \( s = 0 \). Finally, we have that the single-site Markov chain describing the time-evolution of the spin at site 1, conditional on the path of the empirical mean of the other \( N - 1 \) spins and its initial value at time \( s = 0 \), converges to the Markov chain with deterministic but time-dependent generator (4.28). The corresponding transition probabilities converge to the limiting expression from the theorem and we have

\[
\lim_{N \to \infty} \mu_N(\sigma_1 = \tilde{\sigma}_1, z_{\sigma[1,N]}) = k_t(\tilde{\sigma}_1, \eta_1; t, m')
\]

\[
(4.36)
\]

Therefore combining these three ingredients ensued from the assumption we
made we get
\[
\int \mathbb{P}_N(dz_{\sigma[1,N]}) \left[ m_N(t) = \alpha_N \right] \sum_{\tilde{\sigma} = \pm 1} \tilde{\mu}_N(\eta_1 = +|\sigma_1 = \tilde{\sigma}_1, z_{\sigma[1,N]}) \mu_N(\sigma_1 = \tilde{\sigma}_1|z_{\sigma[1,N]}) \\
\int \mathbb{P}_N(dz_{\sigma[1,N]}) \left[ m_N(t) = \alpha_N \right] \sum_{\tilde{\sigma} = \pm 1} \tilde{\mu}_N(\eta_1 = -|\sigma_1 = \tilde{\sigma}_1, z_{\sigma[1,N]}) \mu_N(\sigma_1 = \tilde{\sigma}_1|z_{\sigma[1,N]})
\]
(4.37)

\[
\begin{align*}
\frac{\# \{ \sigma[2,N]: \lim_{N \to \infty} \frac{1}{N} \sum_{i=2}^N \sigma_i = m^*(0;m',t) \}}{\# \{ \sigma[2,N]: \lim_{N \to \infty} \frac{1}{N} \sum_{i=2}^N \sigma_i = m^*(0;m',t) \}} & \sum_{\sigma = \pm 1} k_l(\tilde{\sigma}_1, +; t, m') e^{\tilde{\sigma}_1 \beta m^*(0;t,m')} \\
& \sum_{\tilde{\sigma} = \pm 1} k_l(\tilde{\sigma}_1, -; t, m') e^{\tilde{\sigma}_1 \beta m^*(0;t,m')}
\end{align*}
(4.38)

Canceling counting constants and adding normalizing constants to numerator and denominator completes the proof.

\[\square\]

4.2 Main result

Hitherto, we gave the notion of Gibbsianness for the transformed model and established the link between the single-site conditional probabilities and a solution of the variational problem (4.26). The latter singles out the set of relevant parameters determining the regime of the system. These parameters are an initial temperature \( \beta^{-1} \), a temperature of the dynamics \( \beta'^{-1} \), a value \( \hat{m} \) tested to be a continuity point, and a time of evolution \( t \). The importance of their combinations is reflected in our main result of this chapter.

**Theorem 4.2.1.** Consider the time-evolved Curie-Weiss model with initial and dynamical temperatures \( \beta^{-1}, \beta'^{-1} \). Then the following holds.

1. **Initial high temperature, any temperature of the dynamics.**
   If \( \beta^{-1} \geq 1 \) then the time-evolved model is Gibbs for all \( t \geq 0 \).

2. **Heating from an initial low-temperature, with a either high-temperature or a low-temperature dynamics.**
   For any \( \beta' \) there exists a value \( \beta_{SB}^{-1}(\beta') < \beta'^{-1} \) (which is explicitly computable, see below) such that the following is true. Assume that \( 0 < \beta^{-1} < \min\{\beta'^{-1}, 1\} \).
   a) If \( \beta_{SB}^{-1}(\beta') \leq \beta^{-1} \) then
      - for all \( 0 \leq t \leq t_{nGS}(\beta,\beta') := \frac{\ln \frac{\beta'-\beta}{1-\beta'}}{4(1-\beta')} \) the time-evolved model is Gibbs.
4.2 Main result

- for all $t > t_{nGS}(\beta, \beta')$ the model is not Gibbs and the time-evolved conditional probabilities are discontinuous at $\hat{m} = 0$ and continuous at any $\hat{m} \neq 0$.

b) If $0 < \beta^{-1} < \beta_{SB}^{-1}(\beta')$ there exist sharp values $0 < t_0(\beta, \beta') < t_1(\beta, \beta') < \infty$ such that

- for all $0 \leq t \leq t_0(\beta, \beta')$ the time-evolved model is Gibbs,
- for all $t_0(\beta, \beta') < t < t_1(\beta, \beta')$ there exists $\hat{m} = \hat{m}_c(\beta, \beta'; t) \in (0, 1)$ such that the limiting conditional probabilities are discontinuous at the points $\pm \hat{m}_c$, and continuous otherwise,
- for all $t > t_1(\beta, \beta')$ the limiting conditional probabilities are discontinuous at $\hat{m} = 0$ and continuous at any $\hat{m} \neq 0$.

3. Cooling from initial low temperature.
For $\beta'^{-1} < \beta^{-1} < 1$ there exists a time-threshold $t_{per}(\beta, \beta')$ such that,

- for all $0 \leq t \leq t_{per}(\beta, \beta')$ the time-evolved model is Gibbs.
- for all $t > t_{per}(\beta, \beta')$ the model is not Gibbs and the time-evolved conditional probabilities are discontinuous at non-zero configurations $\hat{m}_c$ (and continuous at $\hat{m} = 0$).

![Figure 4.1: Division between Gibbs and non-Gibbs area for low-temperature dynamics, the thick curve is obtained by computation, the dots are given by numerics](image)

Note that for high-temperature dynamics $\beta'^{-1} > 1$ the Region 3 of initial temperatures in Figure 4.1 is empty. Part 2 of the theorem generalizes the structure which we already know from the independent spin-flip dynamics $\beta' = 0$ (see [36]) which is contained as a special case. This means that a
symmetric (w.r.t. starting measure) bad point \( m_0 = 0 \) will appear after a sharp transition time if the initial temperature is not too low (see Subregion 2a). For lower temperatures (in Subregion 2b) symmetry-breaking in the set of bad configurations for the time-evolved measure appears in an intermediate time-interval: at the beginning of this interval a symmetric pair of bad configuration appears which merges at the end of the time interval.

It is remarkable that the picture we observe in Region 2 is similar to the independent spin-flip case. This is even true for low temperatures \( \beta' < 1 \) of the dynamics. As we will see, we can moreover compute the symmetry-breaking inverse temperature \( \beta_{SB} \) in terms of \( \beta' \) as the largest solution of the following cubic equation

\[
4\beta_{SB}^3 + 12\beta_{SB}\beta' - 6\beta_{SB}^2(1 + \beta') - \beta'(3 + 3\beta' - \beta'^2) = 0
\]  

(4.39)

In the independent spin-flip case \( \beta' = 0 \) we get exactly \( \beta^{-1} = \frac{2}{3} \), which was already found in [36]. We will also give an explicit expression of the critical time in Subregion 2a, for all \( \beta' \).

In Region 3 of cooling from an already low initial temperature we observe an entirely new mechanism for the production of non-Gibbsian points. These are related to periodic orbits of the flow of the \( \beta' \)-dependent vector field which is created by the Euler-Lagrange equations for the variational problem (4.26).

4.3 Deterministic analysis

The key feature that allows to treat our problem of initially stochastic nature from the deterministic point of view is the concentration property of possible evolution ways on an optimal path for large \( N \). The optimality of a path means that it delivers a minimum to the functional (4.26). The connection established by the Theorem 4.1.12 shows that the knowledge of the optimal path is sufficient to judge whether the transformed model retains its Gibbs nature or not. This distinction depends on whether the prescribed end-state there was only one possibility for the system to start from or several. Many options in this case we consider as rather a defect, because they imply the lack of quasilocality for the transformed system. While looking for the most probable starting state of the system which is conditioned to be in a certain state, we have to look for the whole evolution path. This approach does not only answer the question of the most probable starting state, but develops a strong insight into hidden mechanisms of the evolution of the system.
4.3 Deterministic analysis

4.3.1 Variational problem

Fix $\beta, \beta', t, m'$. We look at the constrained variational problem (4.26) taken over the paths $\varphi$ with $\varphi(t) = m'$ with the aim to find (the) minimizing path(s) $s \mapsto m^*(s; m', t)$. With a little abuse of notation we shall further write this problem in the form:

$$E_{m'}(m_0, \beta, \beta') = \Phi(m_0) + I(m_0) + \inf_{\varphi, \varphi(0) = m_0} \int_0^t \mathcal{L}_{\beta'}(\varphi, \dot{\varphi}) ds - \text{Const}(m'),$$

where the function $\varphi(s)$ may viewed as the infinite-volume magnetization functionally dependent on time, when the system is observed. As a reminder, we say that $\Phi$ is the original mean-field interaction and $I$ is a large deviations function of the initial measure.

To bring the functional into a more usual form for calculus of variations, we drop the irrelevant constant, embed the constants depending on the starting value into infimum and further under the integral sign. Let us abbreviate $U(\varphi(s)) = \Phi(\varphi(s)) + I(\varphi(s))$, whenever it is clear from the context we shall write $U(\varphi)$ and for $U(\varphi)$ computed at time $s = r$ we shall write $U(\varphi(r))$, then the updated form of the functional reads:

$$E_{m'}(m_0, \beta, \beta') = \inf_{m, m(t) = m'} \int_0^t \left\{ \mathcal{L}_{\beta'}(\varphi, \dot{\varphi}) - \frac{dU(\varphi)}{d\varphi} \frac{\dot{\varphi}}{t} + \frac{U(\varphi(r))}{\varphi(t)} \right\} ds \quad (4.40)$$

We may formulate the deterministic problem in the following way: find an extremal $\varphi(s)$ fixed at time $s = t$ to take value $m'$ and free at time $s = 0$ delivering minimum to the cost functional (4.40).

We shall equivalently refer to a time $s = 0$ as the left end and to a time $s = t$ as the right end. The end is fixed if an extremal $\varphi(s)$ has to attain a prescribed value on it and open in the opposite case. Thus, the problem of finding extrema of (4.40) is the problem of calculus of variations with an open left end.

It is known in the calculus of variations [25, see Chapter 3, Section 14] that a necessary condition for an extremum is given by the corresponding Euler-Lagrange equation and an additional transversality condition for the free left end of the form

$$\frac{d}{ds} \mathcal{L}_{\beta'}(\varphi(s), \dot{\varphi}(s)) - \mathcal{L}_{\beta'}(\varphi(s), \dot{\varphi}(s)) = 0 \quad \text{for all } s \in [0, t]$$

$$\mathcal{L}_{\beta'}(\varphi(s), \dot{\varphi}(s)) - \Phi_{\beta'}(\varphi(s)) - I_{\beta'}(\varphi(s))|_{s=0} = 0$$

$$\varphi(t) = m' \quad (4.41)$$

Here we have dropped the subscript $\beta'$ for the function $L(\varphi, \dot{\varphi})$ and written subscripts to denote partial derivatives.
To stress that these equations describe the behaviour of histories \( m(s) \) which were defined previously, we shall explicitly write \( m(s) \) in the place of \( \varphi(s) \).

For a general probability distribution \( p_{\beta'}(m) \) the first equation of (4.41) reads:

\[
\dot{m} = 8(2p_{\beta'}(m) - 1) \frac{dp_{\beta'}(m)}{dm}
\]

The above equation does not contain explicitly the time variable \( s \), thus it possesses a first integral. This integral is the preserved total energy of the system whose existence is a consequence of non-dissipativity of the system.

\[
\dot{m}^2 + 16p_{\beta'}(m)(1 - p_{\beta'}(m)) = \mathcal{E}
\]

This equation is integrable in quadratures.

\[
t = m_0 \pm \int_0^m \frac{d\zeta}{\sqrt{\mathcal{E} - 16p_{\beta'}(\zeta)(1 - p_{\beta'}(\zeta))}}
\]

where \( m_0 \) and \( \dot{m}_0 \) are the unknown initial conditions.

In the situation of the present chapter, we are led to choose the distribution \( p_{\beta'}(m) \) in the following to match the rates in the generator \( \hat{L}_{\beta',N} \):

\[
p_{\beta'}(m) = \frac{c_-(m)}{c_-(m) + c_+(m)} = \frac{e^{2\beta'm}(1 - m)}{e^{2\beta'm}(1 - m) + (1 + m)},
\]

where \( m = m(s) \).

Substituting the form of \( p_{\beta'}(m) \) and \( \mathcal{L}(m, \dot{m}) \) in the equations (4.41) after computations we get:

\[
\begin{align*}
\dot{m} & = 16e^{2\beta'm}\frac{(1+m)e^{-2\beta'm}(1-m)}{(1+m)+e^{2\beta'm}(1-m)} \\
\dot{m}
|_{s=0} & = g(m)|_{s=0} \\
m(t) & = m' 
\end{align*}
\]

with the function

\[
g(m) = 2\frac{(1 + m)e^{-2\beta m} - (1 - m)e^{2m(\beta - \beta')}}{(1 + m)e^{-2\beta'm} + (1 - m)}
\]

We call the function \( g(m) \) which gives the relationship between initial point and initial slope of the solution curve the curve of the “allowed” initial configurations (ACC). We note that it is independent from the final value \( m' \).
4.3.2 Typical paths for non-interacting time-evolution

Let us start with a discussion of the independent time-evolution — when $\beta' = 0$.

(i) For $\beta' = 0, \beta = 0$, the system becomes

$$\begin{align*}
\ddot{m}(s) &= 4m(s) \\
\dot{m}(s)\bigg|_{s=0} &= 2m(s)\bigg|_{s=0} \\
m(t) &= m'
\end{align*}$$

and the solution becomes $m(s) = m' e^{2(s-t)}$. This describes how a curve which is conditioned to end in $m'$ away from zero is built up from the initial condition $m' e^{-2t}$ close to zero.

(ii) For independent dynamics $\beta' = 0$ and initial inverse temperature $\beta \neq 0$ the simplified system is

$$\begin{align*}
\ddot{m}(s) &= 4m(s) \\
\dot{m}(s)\bigg|_{s=0} &= e^{-2\beta m(s)}(1 + m(s)) - e^{2\beta m(s)}(1 - m(s))\bigg|_{s=0} \\
m(t) &= m'
\end{align*}$$

In this case the general solution is a linear combination of the $e^{\pm 2s}$. Looking at the right-end condition one gets

$$m(s) = (m' - C_2 e^{2t}) e^{2(t-s)} + C_2 e^{2s},$$

where $C_2$ is a constant and must be determined by the left-end condition. This can be done numerically.

It is possible to match the current approach with the one of [36] by plugging the solution curves with an initial condition $m(0) = m_0$ which are given by

$$m(s) = \frac{m_0 e^{2t} - m'}{e^{2t} - e^{-2t}} e^{-2s} + \frac{m' - m_0 e^{-2t}}{e^{2t} - e^{-2t}} e^{2s}, s \in [0, t]$$

into the rate function and carrying out the time integral explicitly. This gives

$$E_m(m_0, \beta, 0) = H(m_0) + I(m_0)$$

$$+ \frac{1}{4} \left( 4t + \ln\left[ \frac{1 - m'^2}{1 - m_0^2} \right] + 2m' \ln\left[ \frac{R - C_1 e^{-2t} + C_2 e^{2t}}{1 - m'} \right] \right)$$

$$- 2m_0 \ln\left[ \frac{R - C_1 + C_2}{1 - m_0} \right] + \ln\left[ \frac{1 - R - 2C_1 m' e^{-2t}}{1 + R - 2C_1 m' e^{-2t}} \cdot \frac{1 + R - 2C_1 m_0}{1 - R - 2C_1 m_0} \right],$$

where $R = \sqrt{1 - 4C_1 C_2}, \ C_1 = \frac{m_0 e^{2t} - m'}{e^{2t} - e^{-2t}}, \ C_2 = \frac{m' - m_0 e^{-2t}}{e^{2t} - e^{-2t}}$.
In the approach of [36] a related function called $\Psi_{\beta,t,m'}(m_0)$ was obtained by Hubbard-Stratonovich transformation, whose minimizers with a given conditioning $(t, m')$ correspond to the most probable initial conditions. This provides an opportunity to check if the results of the present analysis done via path large deviations coincide with the approach employing the function $\Psi_{\beta,t,m'}(m_0)$.

It is known that the functions $\Psi_{\beta,t,m'}(m_0)$ (4.3.2) and $E_m'(m_0, \beta, 0)$ have the same set of extrema (see [48] in a more general context). In Figure 4.2 is the plot of these functions (after normalization to have zero as a minimum) for the same set of parameters $(\beta, \beta' = 0, m', t)$ which shows that the minima appear in fact at the same value.

The form (4.51) of the curves delivering minimum to the cost functional induces fast relaxation and fast concentration properties for the magnetization of the system being transformed\(^1\). In other words, the evolution time could be split into three stages: the magnetization 1) relaxes in a short time to a value close to zero, 2) stays close to zero for a long time, and 3) at time $s = t$ quickly — just before time $s = t$ — approaches the prescribed value $m'$. A simple proof of this fact is given in Appendix B.3.

Further we turn to the case of interacting dynamics $\beta' \neq 0$. In this case trajectories can only be obtained numerically. Before we go on, let us discuss

\(^1\text{Proposed by R.Fernández at Nature-Nurture workshop, 12-13.01.2009, University of Groningen}\)
in more detail the geometrical properties of the vector field and the allowed-configurations curve.

4.3.3 Geometric interpretation of Euler-Lagrange vector-field and curve of allowed initial configurations.

Since the Euler-Lagrange density $L(m(s), \dot{m}(s)) (4.20)$ does not contain an explicit dependence on the time $s$, the generalized energy given by the Legendre transformation of (4.20) is the system first integral of motion

$$L(m(s), \dot{m}(s)) - \dot{m}(s)L_{\dot{m}}(m(s), \dot{m}(s)) = \mathcal{E} \quad (4.53)$$

This can be rewritten as

$$\frac{e^{4\beta'm}(1 - m)^2\dot{m}^2 + (1 + m)^2\dot{m}^2 + 2e^{2\beta'm}(1 - m^2)(8 + \dot{m}^2)}{(1 + e^{2\beta'm}(1 - m) + m)^2} = \mathcal{E} \quad (4.54)$$

and explicitly solved for the velocity

$$\dot{m} = \pm \sqrt{\mathcal{E} + \frac{16e^{2\beta'm}(m^2 - 1)}{(1 - e^{2\beta'm}(m - 1) + m)^2}} \quad (4.55)$$

Looking at the integral curves in phase space we get some geometric intuition.

First let us understand what it means to have several equiprobable initial states (for the system) which could be led by the evolution to the same final state in terms of our differential equations. Phase diagrams for different values of $\beta'$ help us.

![Phase Portrait](image)

Figure 4.3: Phase portrait for several values of $\beta'$

Two initial states $m_{0,1}$, $m_{0,2}$ (necessarily lying on the ACC) are equiprobable for the final prescribed state $m'$ if the corresponding points $(m_{0,1}, g(m_{0,1}))$, $(m_{0,2}, g(m_{0,2}))$ in the phase space are transferred by the phase flows at time $s = t$ to points having equal projections on the $m$-axis (see Figure 4.3). This corresponds to the fact that the solution-functions started at different magnetizations collapse in $m'$ after time $t$ with different speeds (slopes).
We would like to identify areas on the phase portrait where a possibility to start with two different points and after some time to end up with the transferred points having equal projections on \( m \)-axis is excluded. The only requirement we have for the initial points is that they have to lay on the graph of a function. Nonetheless, this requirement suffices to find “safe” areas (see Figure 4.4). Plot a graph of a function \( f \) crossing only “safe” regions, take any two magnetization values (w.l.g) \( m_{0,1} < m_{0,2} \), then the corresponding points lying on the graph of \( f \) will be \((m_{0,1}, f(m_{0,1}))\), \((m_{0,2}, f(m_{0,2}))\). Safety of the filled regions is a combination of three facts: 1) a driving force is always bigger for a point with a greater \( m \)-coordinate in absolute value, 2) the driving force is a smooth function of \( m \), and 3) for (w.l.g) \( m > 0 \) the phase flow keeps the same direction of a drift for both of the starting points. Thus, the phase flow leaves no possibility for the point with the projection \( m_{0,1} \) to \( m \)-axis to speed up and catch up with the another one and for the point with the projection \( m_{0,2} \) to slow down and let the first one to reach it. On the other hand, the empty areas in Figure 4.4 suggest the very possibility excluded in the “safe” areas. Areas with a periodic motion are not “safe” because of the nature of the motion itself.

Let us go back to the notion of the ACC (4.47) on which all possible “allowed” starting conditions lie. The curve of allowed initial configurations for different combinations of initial and dynamical temperatures crosses both “safe” and not “safe” regions. In figure Figure 4.5 there are several ACC’s drawn which correspond to different values of \( \beta \), but the same value of the dynamical inverse temperature \( \beta' = \frac{3}{2} \), which is relatively low. The production of discontinuities of the limiting conditional probabilities will be related to the time-evolution of the curve of allowed initial configurations under the Euler-Lagrange vector field, as we will describe now.

Let us first give a definition of a bad quadruple of initial temperature, dynamical temperature, time, and final magnetization in terms of dynamical-systems quantities. We start by defining candidate quadruples making use of the Euler-Lagrange flow in the following way.

**Definition 4.3.1.** The quadruple \((\beta, \beta', t, m_{pb})\) is called pre-bad iff there ex-
exists a pair \( m_{0,1} \neq m_{0,2} \) of initial magnetizations s.t. the solution of the initial value problem of the Euler-Lagrange equations started in the corresponding points \((m_{0,1}, g(m_{0,1}))\) and \((m_{0,2}, g(m_{0,2}))\) on the allowed-configurations curve for \( \beta, \beta' \) has the same magnetization value \( m_{pb} \) at time \( t \), that is

\[
m(t; m_{0,1}, g(m_{0,1})) = m(t; m_{0,2}, g(m_{0,2})) = m_{pb}
\]

While this first definition refers only to the existence of overhangs of the time-evolved allowed-configurations curve, the next definition involves also the value of the cost (4.40), which makes it much more restrictive.

**Definition 4.3.2.** The pre-bad quadruple \( (\beta, \beta', t, m_{bad}) \) is called bad if and only if the two different paths started at the corresponding \( m_{0,1} \neq m_{0,2} \) are both minimizers for the cost, i.e.

\[
E_{m_{bad}}(m_{0,1}, \beta, \beta') = E_{m_{bad}}(m_{0,2}, \beta, \beta') = \inf_m E_{m_{bad}}(m, \beta, \beta')
\]  

(4.56)

We will exploit both definitions both to gain geometric insight as well as numerical results. The important connection to non-Gibbsian behaviour of the time-evolved measure lies in the fact that \( m_{bad} \) of a bad quadruple will (generically) be a bad configuration for \( \gamma_{\beta,\beta',t}(\cdot|m) \). Indeed, to see this, let us go back to the explicit expression of the limiting conditional probabilities, given by

\[
\gamma_{\beta,\beta',t}(\eta_1|m') = \frac{\sum_{\sigma_1 = \pm 1} e^{\sigma_1 \beta m^*_{\sigma_1}(0;m',t)} k_t(\sigma_1, \eta_1; m', t)}{\sum_{\sigma_1, \tilde{\eta}_1 = \pm 1} e^{\sigma_1 \beta m^*_{\sigma_1}(0;m',t)} k_t(\sigma_1, \tilde{\eta}_1; m', t)}
\]  

(4.57)

Note that the function \( m^*(0; m', t) \) is not well defined for \( m' = m_{bad} \) itself since at time \( t \) there are two minimizing paths available, one from \( m_{0,1} \) to
$m_{\text{bad}}$ and one from $m_{0,2}$ to $m_{\text{bad}}$. Varying however around $m_{\text{bad}}$ the paths will become unique and we might select the minimizing paths (and hence their initial points) by approaching the bad configuration from the right or left, obtaining (say) $\lim_{m' \rightarrow m_{\text{bad}}} m^*(0; m', t) = m_{0,1}$ and $\lim_{m' \uparrow m_{\text{bad}}} m^*(0; m', t) = m_{0,2}$. Note that we also expect that (generically) $\lim_{m' \downarrow m_{\text{bad}}} k_t(\sigma_1, \tilde{\eta}_1; m', t) \neq \lim_{m' \uparrow m_{\text{bad}}} k_t(\sigma_1, \tilde{\eta}_1; m', t)$. This follows since the $k_t$ are probabilities for two different single-particle Markov chains, one depending on the path starting from $(m_{0,1}, g(m_{0,1}))$, the other one on the path starting from $(m_{0,2}, g(m_{0,2}))$. We note that, knowing the paths entering the $k_t$'s, an explicit formula for $k_t$ in terms of time-integrals can be written, and so, given (numerical) knowledge of the minimizing path, the $\gamma_{\beta,\beta',t}(\eta_1|m')$ can be obtained by simple integrations. Unless these two discontinuities compensate each other (which is generically not happening and which can be quickly checked by numerics) we will have that $\lim_{m' \downarrow m_{\text{bad}}} \gamma_{\beta,\beta',t}(\eta_1|m') \neq \lim_{m' \uparrow m_{\text{bad}}} \gamma_{\beta,\beta',t}(\eta_1|m')$. Consequently the model will be non-Gibbs at the time $t$.

Conversely, if $(\beta, \beta', t, m_{\text{pb}})$ is not bad, then $m' \mapsto \gamma_{\beta,\beta',t}(\eta_1|m')$ is a continuity point. This follows since in that case all $m'$-dependent terms in (4.31) deform in a continuous way. So the absence of bad points (and a fortiori the absence of pre-bad points) implies Gibbsianness at $(\beta, \beta', t)$.

### 4.3.4 Time-evolved allowed initial configurations

We just saw that non-Gibbsianness is produced by multiple histories which means in other words the production of overhangs in the time-evolved curve of allowed initial configurations. To get an intuition for this let us discuss the Regions 2) and 3) of the Theorem 4.2.1 in more detail. Let us begin with the phase-space picture for the non-interacting dynamics $\beta' = 0$. We are starting with the Region 2a) of non-symmetry-breaking non-Gibbsianness i.e. $\frac{2}{3} = \beta_{\text{SB}}^{-1}(\beta' = 0) \leq \beta^{-1} < \min\{\beta^{-1}, 1\} = 1$.

![Figure 4.6: Non-symmetry-breaking mechanism, $\beta' = 0, \beta^{-1} = 0.8$](image)
4.3 Deterministic analysis

The time-evolved allowed-configurations curve for \( t = t_{\text{NGS}}(\beta, \beta' = 0) \) is shown at the left plot of Figure 4.6 where it acquires a vertical slope at zero. The right plot shows the time-evolved allowed-configurations curve for \( t > t_{\text{NGS}}(\beta, \beta' = 0) \) where it has two symmetric overhangs. In particular \((\beta, \beta' = 0, t, m' = 0)\) is pre-bad. It is also bad, since the preimages of the upper and lower time-evolved allowed-configurations curve which intersect the vertical axis have paths with the same cost, by the symmetry of the model. Note that \((\beta, \beta' = 0, t, m')\) is pre-bad for a whole interval of values of \(m'\), but (as the study of the cost shows and as it was proved in [36]) there are no other bad points. We note that \(m' = 0\) is easily checked to be indeed a bad configuration (discontinuity point) of \(\gamma_{\beta,\beta'=0,t}(\cdot|m')\) since there are no cancellations of discontinuities in this case, as we will explain now. Indeed, \(k_t(\sigma_1, \eta_1; m', t)\) does not depend on the trajectory of the empirical magnetization and is given by the independent spin-flip at the site 1 between plus and minus with rate 1,

\[
\gamma_{\beta,\beta'=0,t}(\eta_1|m') = \frac{\sum_{\sigma_1 = \pm 1} e^{\lambda_{\sigma_1 m'}(0;m',t) k_t(\sigma_1, \eta_1)}}{\sum_{\sigma_1, \eta_1 = \pm 1} e^{\lambda_{\sigma_1 m'}(0;m',t) k_t(\sigma_1, \eta_1)}} \tag{4.58}
\]

where \(k_t(\cdot, \cdot) = \frac{1}{2}(1 + e^{-2t})\), and \(k_t(\cdot, \cdot) = k_t(-\cdot, -\cdot) = 1 - k_t(\cdot, \cdot) = 1 - k_t(-\cdot, \cdot).\) So, a discontinuity under variation of \(m'\) is entering the formula only through \(m'(0; m', t)\), and hence \(m' \mapsto \gamma_{\beta,\beta'=0,t}(\eta_1|m')\) is discontinuous if and only if \(m \mapsto m'(0; m', t)\) is discontinuous.

![Figure 4.7: Symmetry-breaking mechanism, \(\beta' = 0, \beta^{-1} = 0.4\)](image)

Let us now look at region 2b) of symmetry-breaking non-Gibbsianness i.e. \(\beta^{-1} < \beta_{\text{SB}}^{-1}(\beta' = 0)\)

The left plot of Figure 4.7 shows the time-evolved allowed-configurations curve at \( t = t_0(\beta, \beta' = 0) \) where it acquires a vertical slope away from zero. The right plot shows the time-evolved allowed-configurations curve for \(t_0(\beta, \beta') < t < t_1(\beta, \beta')\) where it has two symmetric overhangs away from zero. This means that \((\beta, \beta' = 0, t, m')\) is pre-bad for a whole range of values of final magnetizations \(m'\). Due to the lack of symmetry it is not clear to identify in the picture which of the \((\beta, \beta' = 0, t, m')\)'s will be bad. It turns
out that it is precisely one such value \((\beta, \beta' = 0, t, m_c)\), and this can be found looking numerically at the cost.

Perturbations of these pictures stay true for \(\beta'^{-1} > 1\), where they describe the only mechanism of non-Gibbsianness. Perturbations of these pictures also stay true for \(\beta'^{-1} < 1\), but then there is also the Region 3 of the main theorem which describes the cooling from an initial low temperature. We choose \(\frac{2}{3} = \beta'^{-1} < \beta^{-1} = 0.85 < 1\). Then the vector field has periodic orbits which are intersected by the allowed-configurations curve, and the time-evolution will create overhangs and smear out the allowed-configurations curve over time.

\[
\text{Figure 4.8: Non-Gibbsianness by periodicity, } \beta'^{-1} = \frac{2}{3}, \beta^{-1} = 0.85
\]

The left plot of Figure 4.8 shows the time-evolved allowed-configurations curve at \(t = t_{\text{per}}(\beta, \beta')\) where it acquires a vertical slope away from zero inside the area of periodic motion.

The right plot shows the time-evolved allowed-configurations curve for a time \(t > t_0(\beta, \beta')\) where it has overhangs. Again, from the interval of pre-bad points, the bad point has to be selected by looking at the cost. When time gets larger more overhangs are created and the trajectory is smeared out. The corresponding potential function \(m \rightarrow E_{m'}(m, \beta, \beta')\) will acquire more and more local extrema as \(t\) increases. Then, by fine-tuning of the \(m'\) while keeping the \(\beta, \beta', t\) fixed, equality of the depths of the two lowest minima can be achieved. Since the number of available minima is increasing with \(t\) we conjecture that there will be also an increasing number of bad \(m'\)'s which becomes dense as \(t\) increases. To prove this conjecture however, more investigation is needed.

### 4.3.5 Emergence of bad points as a function of time

The notion of a bad point can be viewed from two different standpoints. A pre-bad point in the time-space diagram is a point where two (or more) histories collide. If the costs computed along these paths are equal, then a
Determine that the pre-bad point is a bad point. In the phase space this means that the phase flow transported two (or more) points originally lying on the curve of allowed initial configurations to the same space-position within equal time but with different speeds. Two (or more) points have the same space-position if their projections to the $m$-axis are equal, as seen in Figures (4.6), (4.7), and (4.8). How can we identify analytically the first time $t$ where time-evolved initial points from the curve of allowed initial configurations will obtain the same projection to the $m$-axis? As intuition suggests one has to look when the transported curve of allowed configurations acquires a vertical slope for the first time. This discussion brings us to the following computation.

Writing $v = \dot{m}$ for the velocity, let us consider the flow $m(t; m_0, v_0)$, $v(t; m_0, v_0)$ of our system under the Euler-Lagrange equations,

$$
\begin{align*}
\dot{m} &= v \\
\dot{v} &= f_{\beta'}(m)
\end{align*}
$$

(4.59)

We take the curve of allowed initial configurations to be transported by the flow $v_0 = g_{\beta,\beta'}(m_0)$ where we write in short $f = f_{\beta'}$ and $g = g_{\beta,\beta'}$. We are then interested in the projections to the $m$-axis of the time-evolved curves in phase space, that is the curves $m_0 \mapsto m(t; m_0, g(m_0))$, as they evolve with $t$. Restricted to suitable neighbourhoods this curve becomes a function, and we view it as a potential function with state variable $m_0$ and parameter $t$ (keeping also $\beta, \beta'$ as fixed parameters.)

Doing so we see that the derivatives of the flow with respect to the initial conditions obey at the threshold time $t$ that

$$
0 = F_{\beta,\beta'}(t, m_0) := \frac{dm(t; m_0, g(m_0))}{dm_0} = \frac{\partial m(t; m_0, v_0)}{\partial m_0} + \frac{\partial m(t; m_0, v_0)}{\partial v_0} g'(m_0)
$$

$$
0 = \frac{d^2 m(t; m_0, g(m_0))}{(dm_0)^2}
$$

(4.60)

The first equation means that in the $(m, v)$ plane the time-evolved curve will obtain a vertical slope which is clear by the interpretation of the variable $m_0$ as a parametrization of the curve of allowed initial configurations.

Moreover we have that the second derivative will also vanish, since a minimum and a maximum of $m_0 \mapsto m(s; m_0, g(m_0))$ collide for $s \downarrow t$, in a fold bifurcation.

4.3.6 The threshold time for non-symmetry-breaking non-Gibbsianness for dependent dynamics

We can use these equations to obtain quantitative information about the threshold time for non-symmetry-breaking non-Gibbsianness also for depend-
Mean-field models

ent dynamics. For this it suffices to look at the dynamics locally around the origin \((m, \dot{m}) = (0, 0)\) in phase space which is a stationary point for the dynamics independently of \(\beta'\).

Linearizing \(f_{\beta'}\) we get

\[
\begin{pmatrix}
\dot{m} \\
\dot{v}
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
4(1 - \beta')^2 & 0
\end{pmatrix} \begin{pmatrix}
m \\
v
\end{pmatrix}
\]

(4.61)

Nonetheless the linearizing procedure provides corrections of third order. The eigenvalues of the matrix are \(\lambda_{1,2} = \pm 2(1 - \beta')\), these eigenvalues are real and have different signs, so \((m, \dot{m}) = (0, f_{\beta'}(0)) = (0, 0)\) is a saddle point. This ensures that the nature of solutions close to \((0; 0)\) stays the same whatever \(\beta'\) is taken.

Let us now discuss the phase flow around the origin \((0, 0)\). At this point non-Gibbsianness without symmetry-breaking occurs, by the following argument. Suppose a symmetric pair of initial conditions \((m_0, v(m_0))\) and \((-m_0, v(-m_0)) = (-m_0, -v(m_0))\) is given which has the same time-evolved magnetization \(0\) at time \(t\). This corresponds to the fact that the transported curve will have overhangs at the points \((0, v_1(m))\) and \((0, -v_1(m))\). If we look at the phase portraits of the dynamics as a function of time we see that for times larger than but very close to the first time where this occurs the speed \(v_1(m)\) will be very close to \(0\). It converges to \(0\) when \(t\) approaches the transition time for Gibbsianness. Indeed, the whole path was evolving in an arbitrarily small neighborhood of the origin and hence it suffices to look at the linearized dynamics. We also note that there is no need to look at the cost functional in this case, due to the symmetry of the paths. As time becomes larger than the transition-time (as in the right picture of Figure 4.6) the intersection points of the time-evolved curve with the vertical axis will move away from zero and so it would not be sufficient to use the linearization of the dynamics to compute the relation between bad magnetization values and time.

Clearly the general solution of the linearized system is

\[
m(s) = C_1 e^{-2(1-\beta')s} + C_2 e^{2(1-\beta')s}
\]

(4.62)

Putting the initial condition to be \((m_0, v_0)\) the phase flow becomes

\[
m(s; m_0, v_0) = \frac{2(1 - \beta')m_0 - v_0}{4(1 - \beta')} e^{-2(1-\beta')s} + \frac{2(1 - \beta')m_0 + v_0}{4(1 - \beta')} e^{2(1-\beta')s}
\]

\[
v(s; m_0, v_0) = \frac{v_0 - 2(1 - \beta')m_0}{2} e^{-2(1-\beta')s} + \frac{v_0 + 2(1 - \beta')m_0}{2} e^{2(1-\beta')s}
\]

(4.63)
Computing the function \( F_{\beta',\beta}(t, m_0) \) (4.60) for this phase flow and setting it to zero having in mind that \( v_0 = g(m_0) \), we solve it w.r.t. time \( t \) and get

\[
t = \frac{1}{4(1 - \beta') \ln g'(m_0) - 2(1 - \beta')} g'(m_0) + 2(1 - \beta')
\]  
(4.64)

Putting \( m_0 = 0 \) we obtain from this for the transition time

\[
t = \frac{1}{4(1 - \beta')} \ln \frac{\beta' - \beta}{1 - \beta}
\]  
(4.65)

By setting \( \beta' = 0 \) for the independent evolution in the last expression, the result \( t = \frac{1}{4} \ln(1 - \beta^{-1}) \) given in [36] is reproduced. We note that the transition time given by formula (4.65) is positive only in the case when \( \beta > 1 \). This confirms the intuition obtained via “safe” regions. For \( \beta \leq 1 \) the curve of allowed configurations (w.l.g. \( m > 0 \)) lies either higher than any branch of separatrix in a “safe” area or coincides with it providing the invariance of ACC under the phase flow.

![Figure 4.9: The symmetry-breaking inverse temperature \( \beta_{SB} \) as a function of \( \beta' \)](image)

To identify for which temperature-values the phenomenon of non-Gibbsian-ness without symmetry-breaking ends, let us look when the function (4.64) starts having several minima. In order to do this we compute the second
derivative of (4.64) with respect to \( m_0 \) in \( m_0 = 0 \) and put it equal to zero. The computations are made easier due to the second equation in (4.60). This results in the equation

\[
4\beta^3 + 12\beta \beta' - 6\beta^2 (1 + \beta') - \beta'(3 + 3\beta' - \beta'^2) = 0
\]  
\[(4.66)\]

In the independent-dynamics case \( \beta' = 0 \) we get exactly \( \beta = \frac{3}{2} \), which was already found in the paper [36]. The algebraic curve (4.66) is plotted in Figure 4.9.

### 4.3.7 Cooling and non-Gibbsianness by periodic orbits

Let us specialize to the case of a low-temperature dynamics \( \beta' > 1 \). In that case the phase space decomposes into the areas of periodic and non-periodic dynamics. The separatrix is given by (4.55) with \( C = 4 \).

\[
f_\pm(m) = \pm 2 \frac{(1 + m) - e^{2\beta m}(1 - m)}{(1 + m) + e^{2\beta m}(1 - m)}
\]  
\[(4.67)\]

Note that the curve \( f_+(m) \) coincides with the curve of the “allowed” configurations (4.47) when \( \beta' = \beta \). This means that it will be stable under the phase flow in that case. In particular the time-evolved curve will not acquire overhangs which corresponds to the fact that the time-evolved measure will be invariant under the dynamics and the model Gibbs.

Note also that the negative branch of the separatrix coincides with the right-hand side of the ODE describing the unconstrained typical evolution (4.18) and so the intersection point with the \( m \)-axis is given by the biggest solution of the ordinary mean-field equation \( m = \tanh(\beta' m) \). Let us first concentrate of the existence of pre-bad points, that is different initial points of the allowed-configurations curve leading to the same projection to the \( m \)-axis after time \( t \).

Now multiple overhangs are created if the allowed curve of initial configurations intersects the periodic motion area, as seen in Figure 4.8. Indeed, this part of the curve will perform periodic motion and while doing so it will acquire more and more overhangs, filling out the part of the periodic motion area which is bounded by its extremal value of the integral of motion over time. It is now interesting to note for which temperatures this phenomenon can happen and this is the content of the following theorem.

**Theorem 4.3.3** (Non-Gibbsianness by periodicity). Suppose \( \beta' > 1 \) and let \( m_1^* \) and \( m_2^* \) be the biggest solutions of the mean-field equations for \( \beta' \) and \( \beta \). Then the following is true.

1. if \( 1 < \beta < \beta' \) (or equivalently \( 0 < m_2^* < m_1^* \)) holds then
The curve of allowed initial configurations for $\beta, \beta'$ has non-zero intersection with the (open) periodic motion area in phase phase for $\beta'$.

Consequently there exists a threshold time $t_{\text{per}}(\beta, \beta')$ such that for all $t > t_{\text{per}}(\beta, \beta')$ there exists pre-bad $(\beta, \beta', t, m')s$.

2. if $1 < \beta < \beta'$ fails, there is either no periodic motion areas, or the curve of allowed-configurations has no intersection with them.

Proof. Denote $f = f_-$ (here we take the branch which bounds the periodic motion area from above), and the curve of the “allowed” configurations by $g(m)$ so that we have

$$f(m) = -2 \frac{(1 + m) - e^{2m\beta}(1 - m)}{(1 + m) + e^{2m\beta}(1 - x)};$$

$$g(m) = 2e^{2\beta m} \frac{(1 + m) - e^{2m(\beta - \beta')}(1 - m)}{(1 + m) + e^{2m\beta}(1 - m)}$$ (4.68)

Previously it was mentioned that periodic motion arises only in the case $\beta' > 1$, and so we will consider this along the proof, also w.l.g. we say that $m > 0$. Let us show what the condition $1 < \beta < \beta'$ means and its equivalence to $0 < m^*_2 < m^*_1$. First, we put $f(m) = 0$ to determine the right border of the periodic motion area, and we get that it’s given by the equation

$$(1 + m) - e^{2\beta m}(1 - m) = 0,$$

which is equivalent to the mean-field equation for $\beta'$. Let its biggest solution be given by $m^*_1$. Second, consider $f(m) = g(m)$ to determine their intersection point. This is simply

$$(1 + m) - e^{2\beta m}(1 - m) = 0,$$

which is again the same mean-field equation, but for $\beta$, where $m^*_2$ has the same meaning as before.

The allowed-configurations curve comes into the region of periodic motion and stays there when the following condition is satisfied

$$-f'(x) \bigg|_{m=0} < g'(m) \bigg|_{m=0} < f'(m) \bigg|_{m=0},$$

which turns out to be just equivalent to

$$-(2\beta' - 2) < 2 - 4\beta + 2\beta' < 2\beta' - 2$$ (4.69)

or $1 < \beta < \beta'$. One can get an intuitive understanding of this mechanism from Figure 4.10. □
Figure 4.10: Allowed-configurations curve for different $\beta$ keeping $\beta'$ constant

4.4 Numerical results

Since the variational problem with fixed endpoint (4.46) cannot be solved in closed form unless the dynamics is independent, let us now describe some of the key features which are seen in a numerical study.

4.4.1 General approach

We start with the describing the numerical procedure we used to discover pre-bad points. These pre-bad points later are examined as to whether they are bad by computing corresponding costs. We look at the variational problem of finding an extremal constrained to take value $m'$ at time $s = t$ from another stand-point. We apply a modification of the shooting method [53, see Section 7.3], when the variational problem is solved for a couple of initial conditions which later on are examined for the collision with each other at any point. The difference with the shooting method is that originally a final value of a solution of the variational problem has to be prescribed.

Fix the initial and dynamical inverse temperatures $\beta$ and $\beta'$. We want to identify all initial magnetizations leading to the same (unknown) pre-bad value of magnetization at (unknown) time $\hat{t}$. Since the model is symmetric, we may consider only positive initial conditions. As before, denote as $m(s)$ a solution of the Euler-Lagrange (EL, shortly) equation.

1. Select $M$, a fine enough discretization of $(0, 1)$. Also choose a partition $T$ of time interval $[0, t]$; $T_i = (t_i, t_{i+1})$
2. For each element $m_0 \in M$ the corresponding $\dot{m}_0$ could be computed from ACC. This defines two initial conditions for the Euler-Lagrange differential equation.

$$m(0) = m_0$$

$$\frac{d}{ds} \bigg|_{s=0} m(s) = \dot{m}_0$$  \hspace{1cm} (4.70)

3. Solve the EL-equation for each couple $(m_0, \dot{m}_0)$ on the time interval $[0, t]$.

4. Set $i = 0$, and $T$ to be the corresponding time-interval, $T = T_i$.

5. For each couple of any two intersecting solutions $m_1(s)$ and $m_2(s)$ on $T$ at $(\tilde{t}, \tilde{m})$ look around for intersection points in $\varepsilon$-neighbourhood. Find all curves intersecting at least one of $m_1(s), m_2(s)$ within this neighbour- hood. Call the union of these curves $C_\varepsilon$.

6. For each curve $m_\infty(s) \in C_\varepsilon$ (w.l.g. let $m_\infty(s)$ and $m_1(s)$ intersect at $(\tilde{t}_\infty, \tilde{m}_\infty)$) perturb the initial condition corresponding to $m_\infty(s)$ until the distance between $(\tilde{t}_\infty, \tilde{m}_\infty)$ and $(\tilde{t}, \tilde{m})$ is sufficiently small. In the degenerate case $m_\infty(s)$ will converge to one of $m_1(s), m_2(s)$. In other case we have found that if started at any of $m_1(0), m_2(0), m_\infty(0)$ with $\beta', \beta^{-1}$ fixed paths will go through $(\tilde{t}, \tilde{m})$ which is, therefore, a pre-bad point.

7. If $T_{i+1}$ does not exceed $[0, t]$, increase $i$ by a unit and go to (5) with the new $T = T_i$, otherwise algorithm stops here.

### 4.5 Typical paths, bad configurations, multiple histories, forbidden regions

We remind the reader that for given conditioning $(\beta', \beta, t, m')$ a solution of (4.46) with this set of parameters is called a history curve. Let us first discuss such curves for the example of independent dynamics. Figure 4.11 shows on the right such history curves conditioned to end at time $t$ at $m'$, for different values of $m'$. There is a jump in the optimal trajectory when we change $m' = 0^+$ to $m' = 0^-$. The associated cost functional at $m' = 0$, depicted on the left, has two symmetric minima, and their minimizers are the two possible initial magnetization values. This is an example of a multiple history scenario. We call the regions showing on the right plot which cannot be visited by any integral curve forbidden regions.
Figure 4.11: Symmetric forbidden regions

Figure 4.12: Non-symmetric forbidden region

Figure 4.12 shows on the right history curves for the independent dynamics with a low initial temperature smaller than $\frac{7}{3}$ where symmetry-breaking in the set of bad configurations takes place. We see on the right two discontinuity points $m'$ and correspondingly two components of forbidden regions for the trajectory. The cost functional corresponding to the positive one of them is depicted on the right. Deformations of these pictures describe the phenomena for all temperatures of the dynamics, as long as the initial temperature is lower.

Finally, Figure 4.13 displays history curves and cost functional at the critical conditioning for an example of cooling dynamics.

Next, let us fix $\beta, \beta'$ and describe the possible change of the set of bad configurations as a function of the time. Again we look at the independent dynamics first.

The top line of Figure 4.14 has an initial temperature in which non-Gibbsian behaviour without symmetry-breaking takes place. In the Figure 4.14(b) we see the bad configurations $m'$ as a function of the time $s$ which
were found numerically depicted by dots. Since \( m' = 0 \) appears at a threshold time and stays to be the only bad configuration from that on, the graph of bad configurations is just a straight line starting at the threshold time. In the Figure 4.14(a) we see the corresponding initial points of the history curves which are conditioned to end at \( m' \).

The lower line of Figure 4.14 has an initial temperature for which non-Gibbsian behaviour with symmetry-breaking takes place, in an intermediate time-interval. The right plot shows the corresponding non-negative branch of bad configurations \( m' \). (By the symmetry of the model, taking the negative of these one obtains the full set of bad configurations.) The left plot shows the corresponding initial points of the history curves which are conditioned to end at the non-negative bad configurations \( m' \) on the right.

Finally, the Figure 4.15 displays the time-evolution of bad configurations and their initial points for a low-temperature dynamics. The lowest line corresponds to heating from very low initial temperature and shows non-Gibbsianness with symmetry-breaking at an intermediate time-interval. The middle line corresponds to heating from an intermediate lower temperature and shows non-Gibbsianness without symmetry-breaking. These two mechanisms are known from high-temperature dynamics. Figures 4.15(a) and 4.15(b) correspond to cooling and shows data from the region of periodic orbits.

Applying numerical integration of the Euler-Lagrange equations from initial conditions chosen on the allowed-configurations curve, check for intersecting trajectories and numerical computation of the cost function we can get (numerical approximations to) the array of bad quadruples \( (\beta, \beta', t, m_{pb}) \), augmented by the possible initial points. With this procedure we re-derived the Gibbs-non-Gibbs phase diagram for \( \beta' = 0 \) (which was obtained earlier in[36]). Based on it we can draw the Gibbs-non-Gibbs phase-diagram at any dynamical temperature \( \beta' \). An example for this was presented in the section

![Figure 4.13: Forbidden region for \( \beta' = \frac{3}{2} \)](image)
where the main result of the present chapter was stated in the Figure 4.1 for a fixed relatively low dynamical temperature.

4.6 Final remarks

The paper [15] which this chapter is based on is to our knowledge the first one where Gibbs properties of a model subjected to a low-temperature dynamics are investigated. Shortly after, the paper of van Enter, A.C.D. et al. [14] appeared where the lattice case was treated. In that paper a large-deviation approach was proposed to understand dynamical transitions in the Gibbs properties for lattice systems, too. While there is a beautiful formalism available for path large deviations of empirical measures of lattice systems on an abstract level, explicit results are very hard and given only for an infinite-temperature dynamics, which underlines also the use of our work, and the necessity of future research.

There are several possible ways for extension of our work. We first mention theoretical issues. As previously conjectured, the set of bad configurations expands, this requires more numerical experiments to be done. Moreover, a similar effect has been seen in Chapter 3. The investigated model was considered in a vanishing field, while it is not always the case and it will be interesting to see which effects persist (if at all) for the model in a field. We expect a shift in the phase diagram leaving less possibilities for non-Gibbsian behaviour. A more challenging generalization and first to think of is to run the same analysis for mean-field Potts model.

Viewing the problem from the standpoint of applications, the questions and methods used should have interest also in models of population dynamics. In such models a population of $N$ individuals, each individual carrying genes from a finite alphabet of possible types, performs a stochastic dynamics which can be described on the level of empirical distributions. Starting the dynamics from a known initial measure corresponds to an a-priori belief (prior distribution) over the distribution of types. Conditioning to a final configuration $m'$ at time $s = t$ corresponds to measuring the distribution of types. The occurrence of multiple histories leading to the same $m'$ (which is responsible for non-Gibbsianness in the spin-model) has the interesting interpretation of a non-unique best estimator for the path explaining the present mix of genes.

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Figure 4.14: Initial points of trajectories (left) and bad configurations as function of time (right), $\beta' = 0$
Figure 4.15: Initial points of trajectories (left) and bad configurations as function of time (right), low-temperature dynamics $-\beta' = \frac{3}{2}$.