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Light-like integrability in loop superspace, Kac–Moody central charges and Chern–Simons terms

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We apply the idea of light-like integrability in loop superspace to give a complete derivation of all the constraints of on-shell ten-dimensional supergravity coupled to Yang–Mills. The Yang–Mills Chern–Simons coupling terms are obtained by adding a central extension to the Kac–Moody algebra of the Yang–Mills generators. The form of the loop-superspace covariant derivatives, including the central charge modifications, can also be derived by means of canonical quantization from a Green–Schwarz superstring coupled to a supergravity and super-Yang–Mills background. There the Yang–Mills sector is represented by bosonic group coordinates, which are chiral and describe a gauged Wess–Zumino–Witten model.

1. Introduction

It is well known that in order to describe supersymmetric Yang–Mills or supergravity theories certain constraints must be imposed on the torsion tensors describing the superspace geometry. One way to try to understand the origin of these superspace constraints is to apply the concept of light-like integrability [1]. The basic idea is that the superspace constraints are equivalent to the statement that on light-like lines the supercovariant derivatives obey a flat algebra. In ref. [1] the light-like integrability concept was applied to supersymmetric Yang–Mills theories in four dimensions, and later on generalized to the case $d=10$ in ref. [2]. In view of string applications it is clearly desirable to also understand the $d=10$ supergravity equations in a similar way. Attempts to do this in the context of ordinary superspace were made in refs. [3,4]. However, in contrast to the ten-dimensional supersymmetric Yang–Mills theory, it was found that the requirement of light-like integrability in standard $d=10$ superspace does not lead to all of

the constraints of ten-dimensional supergravity.

Recently, it has been shown that the full set of constraints that describe on-shell ten-dimensional supergravity can be obtained by the requirement of light-like integrability in loop superspace [5]. Loop superspace is a generalization of standard superspace where the coordinates depend on an additional parameter σ parametrizing a circle. In this loop superspace one introduces a generalized set of supercovariant derivatives [6] forming a Kac–Moody algebra. The generalization of that algebra to curved loop superspace was given in ref. [7]. An important feature of this approach is that the two-form gauge superfield of $d=10$ supergravity and its three-form field-strength find their natural place in the formalism of differential geometry.

In this letter we generalize the results of ref. [5], where only $N=1$, $d=10$ supergravity was considered, to include the coupling to a supersymmetric Yang–Mills system. This would be straightforward for a conventional coupling, but in ten dimensions the supergravity–Yang–Mills system [8,9] is known to involve Chern–Simons terms [9]. We shall show how these Chern–Simons terms can be obtained in the framework of differential geometry of loop super-

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space. A distinguishing feature is that the Yang–Mills generators satisfy a Kac–Moody algebra with a central extension. It is this central extension that leads to the relevant Chern–Simons coupling terms in the algebra of the loop-superspace covariant derivatives. Applying the idea of light-like integrability, we then derive the superspace constraints of the coupled supergravity–Yang–Mills system [10].

In the second part of the paper we derive the loop-superspace covariant derivatives, including the central charge modification, by means of canonical quantization from a Green–Schwarz superstring [11] coupled to a supergravity and super-Yang–Mills background. For pure supergravity such a derivation was suggested in ref. [12]. Here we extend this method to include the Yang–Mills background as well. There exist two schemes of coupling Yang–Mills to the superstring. In the first one the Yang–Mills degrees of freedom are represented by fermions [13], in the second one the Yang–Mills sector is a bosonic sigma-model with a WZW term [14,15]. In the fermionic scheme the Chern–Simons supergravity–super-Yang–Mills coupling is justified by *quantum arguments* (anomaly cancellations) [16]. In the bosonic scheme the kappa symmetry [17] of the Green–Schwarz superstring fixes uniquely all the terms in the action involving the group coordinates [18]. The latter are chiral two-dimensional fields and describe a gauged WZW model. After canonical quantization the velocities of the superspace coordinates give rise to the loop-superspace covariant derivatives. The Yang–Mills generators therein contain the momenta conjugate to the group coordinates and automatically satisfy a Kac–Moody algebra with a central extension (compare with ref. [19]). Thus, one can say that the Chern–Simons coupling is a direct consequence of the *classical* kappa-symmetry of the Green–Schwarz superstring.

2. Differential geometry and central-extended Kac–Moody algebras

We begin by recalling the main points of loop-superspace differential geometry [5]. Loop superspace is the space of maps from the circle S^1 to the superspace M . It can be coordinatized by $z^M(\sigma) = (x^m(\sigma), \theta^\alpha(\sigma))$, where σ ($0 \leq \sigma \leq 2\pi$) is the periodic circle co-

ordinate. Further, one considers a space of functionals $\Phi[z(\sigma)]$ ^{#1}. Then one can define covariant functional derivatives:

$$\mathcal{D}_A(\sigma) = E_A^M \frac{\delta}{\delta z^M(\sigma)} + \omega_A + A_A, \quad (1)$$

where $A = (a, \alpha)$, a and α are vector and spinor tangent-space Lorentz indices. These derivatives involve vielbeins $E_A^M(z(\sigma))$, Lorentz $\omega_A(z(\sigma))$ and Yang–Mills $A_A(z(\sigma))$ connections, which are ordinary superfields (i.e. they depend on the loop coordinate σ only through their arguments $z^M(\sigma)$). The Lorentz and Yang–Mills groups act on a wave functional with external indices $\Phi \dots [z(\sigma)]$ as follows:

$$\delta \Phi \dots = \int d\sigma \lambda^i(z(\sigma)) T^i(\sigma) \Phi \dots \quad (2)$$

Here $\lambda^i(z(\sigma))$ are the local (in superspace) real parameters, and $T^i(\sigma)$ are the anti-hermitean generators of the Kac–Moody Lorentz or Yang–Mills algebras:

$$[T^i(\sigma), T^j(\sigma')] = f^{ijk} T^k(\sigma) \delta(\sigma - \sigma'). \quad (3)$$

Besides the usual gauge transformations (2), the wave functional can undergo specific loop superspace ones:

$$\delta \Phi = \int d\sigma i z'^M(\sigma) \mathcal{A}_M(z(\sigma)) \Phi, \quad z'^M = \frac{dz^M}{d\sigma}. \quad (4)$$

Here the loop-superspace tangent vector iz'^M plays the role of the generators, and $\mathcal{A}_M(z(\sigma))$ are the local parameters of these abelian transformations. The gauge superfield for them is the two-form $B_{MN}(z(\sigma)) = (-)^{MN} B_{NM}(z(\sigma))$, typical for the geometry of ten-dimensional supergravity [20] or for the Green–Schwarz superstring theory [11,21,2]. It transforms under the gauge group (4) as follows:

$$\delta B_{MN} = \partial_M \mathcal{A}_N - (-)^{MN} \partial_N \mathcal{A}_M. \quad (5)$$

It is not hard to see that the modified derivative (1)

$$\mathcal{D}_A(\sigma) = E_A^M \frac{\delta}{\delta z^M(\sigma)} + \omega_A + A_A + \Pi^B(\sigma) B_{BA}(\sigma), \quad (6)$$

^{#1} These functionals correspond to superstring wave functions (see section 4). Note that $\Phi[z(\sigma)]$ does not depend directly on σ .

$$\Pi^A = i z'^M E_M^A, \quad B_{BA} = (-)^{AB+MB} E_A^M E_B^N B_{NM} \quad (6 \text{ cont'd})$$

is covariant,

$$\delta[\mathcal{L}_A(\sigma)\Phi] = \int d\sigma' [\lambda'(z(\sigma')) T^j(\sigma') + i z'^M(\sigma') A_M(z(\sigma'))] [\mathcal{L}_A(\sigma)\Phi], \quad (7)$$

since the two-form B compensates for the local transformations (4). The graded commutator

$$[\mathcal{L}_A(\sigma), \mathcal{L}_B(\sigma')] = (T_{AB}^C \mathcal{L}_C + R_{AB} + F_{AB} + \Pi^C H_{CAB}) \delta(\sigma - \sigma') \quad (8)$$

defines the torsion tensor T_{AB}^C , the Lorentz and Yang–Mills curvatures R_{AB} and F_{AB} , and, notably, the three-form field-strength

$$H_{MNK} = \partial_M B_{NK} + (-)^{M(N+K)} \partial_N B_{KM} + (-)^{K(M+N)} \partial_K B_{MN}, \quad (9)$$

invariant under the gauge transformations (5). All the tensors above satisfy the corresponding Bianchi identities.

The differential-geometry formalism described above naturally accommodates the two- and three-forms of ten-dimensional supergravity. However, it does not yet describe the peculiar Chern–Simons coupling of ten-dimensional supergravity to super-Yang–Mills discussed in the Introduction. To this end we have to modify the Yang–Mills Kac–Moody algebra (3) by introducing a central charge:

$$[T^i(\sigma), T^j(\sigma')] = f^{ijk} T^k(\sigma) \delta(\sigma - \sigma') + 2in\delta^{ij} \partial_\sigma \delta(\sigma - \sigma'). \quad (10)$$

The central charge n must be integer, in order for the representations of (10) to be well-defined [19]. Gauging the new algebra (10) requires a further modification of the covariant derivative (6). The Yang–Mills connection in (6) has the form

$$A_A(\sigma) = A_A^i(z(\sigma)) T^i(\sigma). \quad (11)$$

It is easy to see that because of the new term in (10) the connection term (11) is not sufficient to compensate for the local transformations (2) any more. Indeed, commuting the generator $T^j(\sigma')$ with $T^i(\sigma)$ from $\mathcal{L}_A(\sigma)$ in order to obtain the right-hand side of (7), one obtains central-charge terms containing de-

derivatives of the parameters $\lambda'(z(\sigma'))$. The solution to this problem involves two steps. First, one has to modify the Yang–Mills generator in (11):

$$A_A(\sigma) = A_A^i(z(\sigma)) [T^i(\sigma) + n\Pi^B(\sigma) A_B^i(z(\sigma))]. \quad (12)$$

Second, one has to ascribe to the two-form in (6) the following compensating Yang–Mills transformation law [8,9]:

$$\delta B_{MN}(\sigma) = n[A_M^i(\sigma) \partial_N(\sigma) \lambda_i(\sigma) - (-)^{MN} A_N^i(\sigma) \partial_M(\sigma) \lambda_i(\sigma)]. \quad (13)$$

After that it is not hard to check that (7) does take place.

The graded commutator of the modified covariant derivatives (6), (12) gives

$$[\mathcal{L}_A(\sigma), \mathcal{L}_B(\sigma')] = \{T_{AB}^C(\sigma) \mathcal{L}_C(\sigma) + R_{AB}(\sigma) + F_{AB}^i(\sigma) [T^i(\sigma) + 2n\Pi^C(\sigma) A_C^i(\sigma)] + \Pi^C(\sigma) \mathcal{H}_{CAB}(\sigma)\} \delta(\sigma - \sigma'). \quad (14)$$

Here

$$\mathcal{H}_{CAB} = H_{CAB} + n\omega_{CAB}, \quad (15)$$

and ω_{CAB} is the Yang–Mills Chern–Simons form

$$\omega_{CAB} = A_C^i F_{AB}^i + (-)^{C(A+B)} A_A^i F_{BC}^i + (-)^{B(A+C)} A_B^i F_{CA}^i - f^{ijk} A_C^i A_A^j A_B^k. \quad (16)$$

Note that in the Yang–Mills curvature term in (14) the generator is again modified by a central-charge term, but with a different coefficient compared to the connection term (12). As in the case of the covariant derivative, one could derive this coefficient by looking for a modification of the field-strength term $F_{AB}^i(\sigma) T^i(\sigma)$, which is covariant with respect to the central-charge algebra (10).

Thus we see that the peculiar Chern–Simons modifications to the two-form transformation law and to the three-form field-strength, encountered in the supergravity-super-Yang–Mills coupling in ten dimensions [8,9], are naturally explained by a central-charge extension of the Yang–Mills Kac–Moody algebra in loop superspace. However, in the context of differential geometry alone these modifications are purely optional. The necessity to make such steps becomes clear in the context of the Green–Schwarz superstring, as we shall see in section 4. Note also that

everything above can apply not only to the Yang–Mills group, but to the Lorentz one equally well.

3. Light-like integrability and the supergravity-super-Yang–Mills constraints

In ref. [5] it was shown that the requirement of light-like integrability in loop superspace is equivalent to the constraints of ten-dimensional supergravity. Here we shall extend that result to the coupled supergravity-super-Yang–Mills system as well.

Light-like integrability in loop superspace means the following. Consider a set of Lorentz vectors $(u_a^+(\sigma), u_a^-(\sigma), u_a^i(\sigma))$, $i=1, \dots, 8$, which form an $SO(1, 9)$ matrix [22]:

$$\begin{aligned} u^+ u_a^- &= 1, \quad u^{ia} u_a^i = -\delta^{ij}, \\ u^+ u_a^+ &= u^- u_a^- = u^\pm u_a^\pm = 0. \end{aligned} \quad (17)$$

Note that the vectors $u_a^\pm$ are light-like. With their help one can covariantly project ten-dimensional Lorentz vector and spinor objects onto light-cone covariant ones. For instance, the covariant derivatives $\mathcal{D}_A(\sigma)$ become

$$\begin{aligned} \mathcal{D}_a(\sigma) &\rightarrow \mathcal{D}^\pm(\sigma) = u^{\pm a}(\sigma) \mathcal{D}_a(\sigma), \\ \mathcal{D}^i(\sigma) &= u^{ia}(\sigma) \mathcal{D}_a(\sigma), \\ \mathcal{D}_\alpha(\sigma) &\rightarrow \mathcal{D}^{(\pm\alpha)}(\sigma) = (\Gamma^\pm)^{\alpha\beta}(\sigma) \mathcal{D}_\beta(\sigma). \end{aligned} \quad (18)$$

Here

$$\Gamma^\pm(\sigma) = u^{\pm a}(\sigma) \Gamma_a \quad (19)$$

are nilpotent matrices, which split the ten-dimensional spinor indices into two independent halves. Then the requirement of light-like integrability means that the $+$ projections of the commutation relations (14) should look like the flat ones [5]:

$$\begin{aligned} \{ \mathcal{D}^{(+\alpha)}(\sigma), \mathcal{D}^{(+\beta)}(\sigma') \} \\ = 2i(\Gamma^+)^{\alpha\beta}(\sigma) [\mathcal{D}^+(\sigma) - e^K \Pi^+(\sigma)] \delta(\sigma - \sigma'), \end{aligned} \quad (20)$$

$$[\mathcal{D}^+(\sigma), \mathcal{D}^{(+\alpha)}(\sigma')] = 0. \quad (21)$$

In fact, the only deviation from the flat $+$ projected algebra is the scale factor e^K in (20). Here $K(z(\sigma))$ is the dilaton superfield of ten-dimensional supergravity. Putting $K=0$ would make the constraints fol-

lowing from (20), (21) too strong^{#2}. Working out the consequences of the integrability conditions (20), (21) goes on in exactly the same way as in the case of pure supergravity [5]. The only difference is that the three-form H_{ABC} is replaced everywhere by the Chern–Simons modified three-form (15):

$$\begin{aligned} T_{\alpha\beta}{}^\gamma &= 0, \quad T_{\alpha\beta}{}^c = -i(\Gamma^c)_{\alpha\beta}, \\ T_{\alpha b}{}^c &= -\frac{1}{2}(\Gamma_b \Gamma^c D)_\alpha K, \\ \mathcal{H}_{\gamma\alpha\beta} &= 0, \quad \mathcal{H}_{\alpha\beta}{}^c = -ie^K(\Gamma_c)_{\alpha\beta}, \quad \mathcal{H}_{abc} = 0, \\ F_{\alpha\beta} &= 0 \Rightarrow F_{\alpha b} = (\Gamma_b)_\alpha W^\beta. \end{aligned} \quad (22)$$

$$(23)$$

Thus, the form of the torsion and three-form constraints following from light-like integrability is unchanged by the supergravity-Yang–Mills coupling, nor is that of the Yang–Mills curvature constraints. This conclusion coincides with the one reached in ref. [10] by different reasoning. We stress that light-like integrability alone is not sufficient to justify the necessity of the Chern–Simons (or central charge) modifications described above.

4. Loop superspace from canonical quantization of the Green–Schwarz superstring

The concepts of loop superspace, together with the central-charge modifications, were introduced in section 2 in an abstract, purely geometrical way. There exists an alternative way to arrive at that geometric picture. It consists of deriving the form of the covariant derivatives (6), (12) by means of canonical quantization from a Green–Schwarz superstring action coupled to a supergravity and super-Yang–Mills background. As explained in the Introduction, the Yang–Mills sector we shall use will be represented by a gauged bosonic sigma-model. Various elements of the coupling of the (super)string to a curved and Yang–Mills background via a non-abelian gauged bosonic sigma-model have appeared in a number of papers [14,15,18]. We shall use the results of ref. [18], where the decisive role of the specific fermionic

^{#2} Note that recently an alternative derivation of the $d=10$ supergravity constraints has been proposed [23]. There one uses integrability along pure spinor lines, instead of light-like lines. In that approach the dilaton superfield K does not appear in the integrability conditions.

kappa-symmetry of the Green–Schwarz superstring [17,11] is clearly explained. Here we shall just give the form of the action and of the kappa-symmetry transformations, adjusting the notation to our purposes. The action is

$$S = \frac{1}{2} \int d^2 \xi \left[\sqrt{-g} g^{\mu\nu} e^K E_\mu^a E_\nu^a - \epsilon^{\mu\nu} \partial_\mu z^M \partial_\nu z^N B_{NM} \right. \\ \left. + n(\sqrt{-g} g^{\mu\nu} + \rho^{\mu\nu}) J_\mu^I J_\nu^J g_{IJ} \right. \\ \left. - n \epsilon^{\mu\nu} (2 \partial_\mu y^I \partial_\nu z^M A_{M I J}^J + \partial_\mu y^I \partial_\nu y^J b_{IJ}) \right]. \quad (24)$$

The first two terms in (24) represent the usual Green–Schwarz superstring action in a supergravity background. In it $\xi^\mu = (\tau, \sigma)$ are the world-sheet coordinates, $g^{\mu\nu}$ is the world-sheet metric, $E_\mu^a \equiv \partial_\mu z^M(\xi) E_M^a(z)$. The vielbeins $E_M^a(z)$ and the two-form $B_{NM}(z)$ were introduced in section 2. The presence of the dilaton superfield $K(z)$ in (24) is optional, but we need it for the purpose of light-like integrability (section 3).

The third term in (24) is a non-abelian gauged sigma model, described by bosonic group coordinates y^I ($I=1, \dots, N$) for the Yang–Mills group G , coupled to a background gauge superfield. The gauge invariant current is

$$J_\mu^I = \partial_\mu y^I(\xi) - \partial_\mu z^M(\xi) A_{M I}^I, \quad (25)$$

where $A_{M I}^I \equiv A_{M I}^I(z) L_I^I(y)$; $A_{M I}^I(z)$ are the Yang–Mills superfields of section 2. The left-invariant Killing vectors $L_I^I(y)$ and the metric g_{IJ} for the group G satisfy the relations

$$\partial_I L_J^k - \partial_J L_I^k = -L_I^l L_J^j f_{ij}^k, \quad L_I^l L_J^j = g_{IJ}. \quad (26)$$

The field $\rho^{\mu\nu}(\xi)$ is a self-dual Lagrange multiplier:

$$\rho^{\mu\nu} = \rho^{\nu\mu} = P_+^{\mu\lambda} P_+^{\nu\kappa} \rho_{\lambda\kappa}, \\ P_+^{\mu\nu} \equiv \frac{1}{2} \left(g^{\mu\nu} + \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu} \right), \quad \epsilon^{01} = 1.$$

Its role is to make the group-coordinate bosons chiral [24], and it also helps to maintain kappa-symmetry (see (27)).

The last term in (24) is the familiar WZW sigma-model term written down as a two-dimensional integral [19]. There $b_{IJ}(y) = -b_{JI}(y)$ is a two-form, the curl of which gives the structure constant of G :

$$\partial_I b_{JK} + \partial_K b_{IJ} + \partial_J b_{KI} = f_{IJK} = L_I^l L_J^j L_K^k f_{ljk}.$$

The last two terms in (24), together with the B_{NM} term from the Green–Schwarz action, form a gauge invariant combination, provided the two-form B_{NM} undergoes the compensating gauge transformations (13) [14,15]. The coefficient n of the sigma-model terms is just the central charge of the Kac–Moody Yang–Mills algebra, as we shall show below (see also ref. [19]).

A crucial feature of the action (24) is its kappa-symmetry [17,11]. The transformation laws of the various fields are as follows:

$$\delta z^M E_M^a = 0, \quad \delta z^M E_M^\alpha = 2(\Gamma_a)^{\alpha\beta} E_\mu^a \kappa_\beta^\mu, \\ \delta y^I - \delta z^M A_{M I}^I = 0, \\ \delta(\sqrt{-g} g^{\mu\nu}) = 4\kappa_\alpha^\mu [-2i\sqrt{-g} P_+^{\nu\lambda} E_\lambda^\alpha \\ + n e^{-K} (\sqrt{-g} g^{\nu\lambda} - \epsilon^{\nu\lambda} + \rho^{\nu\lambda}) J_\lambda^I W^{J\alpha} g_{IJ}], \\ \delta \rho^{\mu\nu} = -\delta(\sqrt{-g} g^{\mu\nu}). \quad (27)$$

Here the parameter $\kappa_\alpha^\mu(\xi)$ is self-dual,

$$\kappa_\alpha^\mu = P_+^{\mu\nu} \kappa_{\alpha\nu}. \quad (28)$$

The necessary and sufficient conditions for the invariance of (24) under (27) are the supergravity-super-Yang–Mills constraints (22), (23) of section 3.

The kappa-symmetry is decisive for the superstring to have the right physical spectrum [11]. Besides, as has been shown in ref. [18], kappa-symmetry fixes uniquely all the terms in the action (24) and implies the same constraints on the background as the ones following from light-like integrability. Then it is not surprising that straightforward canonical quantization of the action (24) leads to the loop-superspace covariant derivative of section 2, as we shall now show.

We start by fixing the gauges

$$g^{\mu\nu} = \eta^{\mu\nu}, \quad \rho^{\mu\nu} = 0. \quad (29)$$

The first of them is a conformal gauge, and the second one has to do with the specific Lagrange multiplier gauge invariance for chiral sigma-models [24]. Then, from the action (24) we find the conjugate momenta to the superspace and group coordinates:

$$P_M = \frac{\delta L}{\delta \dot{z}^M} \\ = e^{K/2} E_N^a E_M^a - n A_{M I}^I (J_\tau^I - y'^J) g_{IJ} - z'^N B_{NM}, \quad (30)$$

$$Q_I = \frac{\delta L}{\delta \dot{y}^I} = n(J^J_{\tau} g_{IJ} - z'^M A^J_M g_{IJ} - y'^J b_{IJ}). \quad (31)$$

Here $\dot{}$ and $'$ mean $d/d\tau$ and $d/d\sigma$. From (31) we can solve for $nJ^J_{\tau} g_{IJ}$ and insert it in (30):

$$\begin{aligned} e^K \dot{z}^N E^a_N E^a_M = P_M \\ + A^I_M [Q_I - n y'^J (g_{IJ} - b_{IJ}) + n z'^N A^J_N g_{IJ}] \\ + z'^N B_{NM}. \end{aligned} \quad (32)$$

After quantization the momenta in (3) become derivatives, acting on the string wave functional $\Phi[z(\sigma), y(\sigma)]$:

$$P_M = -i \frac{\delta}{\delta z^M(\sigma)}, \quad Q_I = -i \frac{\delta}{\delta y^I(\sigma)}. \quad (33)$$

Then the right-hand side of (32), multiplied by iE^M_A , defines the loop-superspace covariant derivative^{#3}

$$\mathcal{D}_A = E^M_A \frac{\delta}{\delta z^M} + A^i_A (T^i + n \Pi^B A^i_B) + \Pi^B B_{BA}, \quad (34)$$

where $A^i_A = E^M_A A^i_M L^i$. The Yang–Mills generators T^i in (34) are realized as differential operators on the group manifold of G:

$$T_i = L^i_i \left(\frac{\delta}{\delta y^i} - i n y'^J (g_{IJ} - b_{IJ}) \right). \quad (35)$$

It is easy to check that the commutator of the generators (35) produces exactly the central-extended Kac–Moody algebra (10). The origin of the central charge term in the algebra can be traced back to the term $n \epsilon^{\mu\nu} \partial_\mu y^I \partial_\nu z^M A^J_M q_{IJ}$ in the action (24). Its presence there can be explained in two ways. On the one hand, it is needed to restore the gauge invariance of the WZW term, which in turn is responsible for the correct quantum behaviour of the sigma-model [19]. The alternative explanation is purely classical, it is based on the kappa-symmetry of the superstring. Indeed, the proof of kappa-symmetry relies on the self-duality of κ (28). One forms a chiral projector P_+ from the kappa-variations of the gauge fields in the $g^{\mu\nu}$ and $\epsilon^{\mu\nu}$ terms in (24). Thus, the $\epsilon^{\mu\nu}$ terms, including the WZW term, are actually needed for kappa-

symmetry. We see that kappa-symmetry supplies an additional piece of information, compared to light-like integrability in the abstract loop superspace of section 2. In the latter context we had to add the central-charge terms by hand.

Another comment we would like to make concerns the covariant derivative (34). Compared to the abstract one (6), (12), it obviously lacks the Lorentz connection term. The reason for that is clear. The wave functional $\Phi[z(\sigma), y(\sigma)]$ of this section, unlike the one in (2), does not carry external Lorentz or Yang–Mills indices. The presence of Yang–Mills generators in (34) is due to the non-trivial dependence of $\Phi[z(\sigma), y(\sigma)]$ on the coordinates y^I of G. Since we have not introduced coordinates for the Lorentz group, Φ is completely inert under that symmetry. Then the covariant derivative (34) is good enough for such scalar wave functionals. However, there is a problem when trying to commute two derivatives (34), in order to obtain the invariant tensors (14). The point is that the index A of $\mathcal{D}$ (34) is by definition a local Lorentz index, so one does need a Lorentz connection, as in (6)^{#4}.

One might guess that the above problem could be easily solved by introducing coordinates for the Lorentz group, just as we did with the Yang–Mills group. Then canonical quantization would produce Lorentz generators similar to (35), which would satisfy a Lorentz Kac–Moody algebra with a central charge. Correspondingly, the three-form H would be modified by a Lorentz Chern–Simons form, as in (15). This is in fact expected to happen in an anomaly-free superstring theory [25]. However, there is an obvious obstacle for this. The Lorentz group coordinates would give rise to unwanted new propagating degrees of freedom. Some of them would even have negative energy, because of the non-compactness of the Lorentz group. A possible solution to the problem could be to introduce the Lorentz coordinates as auxiliary harmonic variables, as we did in section 3. Progress in this direction will be reported elsewhere.

^{#3} Note that eq. (32) implies $\mathcal{D}_\alpha = 0$, which is a typical superstring hamiltonian constraint (see, e.g. ref. [12]).

^{#4} Note, however, that the essential supergravity constraints, which are necessary for kappa-symmetry, involve only torsion components, for which the Lorentz connections vanish [12].

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Note added

After the completion of this work we received a preprint by Howe [26] in which the constraints of $d=10$ supergravity coupled to super Yang–Mills, including Chern–Simons terms, are studied from the viewpoint of the differential geometry of loop superspace. The author also discusses the constraints of $=11$ supergravity.

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