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Magnetic fields in neutron stars: A theoretical perspective

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Abstract. We present our view of the main physical ingredients determining the evolution of neutron star magnetic fields. This includes the basic properties of neutron star matter, possible scenarios for the origin of the magnetic field, constraints and mechanisms for its evolution, and a discussion of our recent work on the Hall drift.

INTRODUCTION

In the present conference, Mariano Méndez reviewed our observational knowledge about neutron star magnetic fields, while Don Melrose discussed current models of the pulsar magnetosphere. Therefore, our review focuses on theoretical ideas regarding the magnetic field inside neutron stars, paying particular attention to processes by which this field might evolve in time. We refer to our previous reviews [28, 30] for more basic and general discussions of our knowledge of neutron star magnetic fields.

Our discussion starts with an overview of the properties of matter in neutron star, which are obviously an essential ingredient in any model for the evolution of the magnetic field. We continue with a brief discussion of the state of knowledge regarding the origin of the magnetic field from the main-sequence stage through the hot, proton-neutron star phase, which sets the initial conditions for the subsequent evolution. Then, we quickly present the observational evidence for the evolution of the magnetic field, followed by the basic equations that are likely to describe this evolution and the main processes involved in it. One of these processes, the Hall effect, is singled out for more detailed discussion, where we present recent, yet unpublished analytic solutions for simple geometries, and discuss their implications.
MATTER IN NEUTRON STARS

Protoneutron stars are born through gravitational core collapse in massive stars, which leaves them at a high temperature, $T \sim 10^{11}$K $\sim 10$MeV, comparable to the Fermi energies of their main constituent particles. However, neutrino emission ensures rapid cooling, so all currently observed neutron stars (with ages $\sim 10^3$yr or higher) are expected to have internal temperatures at least two orders of magnitude lower, at which the matter is highly degenerate and likely close to its quantum ground state, although small perturbations from the latter may cause interesting effects [27, 15, 11], some of which are important for the evolution of the magnetic field [24, 12, 16] and will be discussed in what follows.

In this highly degenerate, near-equilibrium state, the properties of matter are determined by a single parameter, for example density or pressure. The density inside neutron stars covers many orders of magnitude, allowing for different regimes in the properties of matter, as discussed below. For a much more detailed discussion, see Ref. [35].

The crust

The neutron star crust is, by definition, the outer part of the star, where atomic nuclei are present. As in ordinary matter, they coexist with electrons, which, due to the high density, are too energetic to be bound to individual nuclei. At $T \sim 10^{10}$K, the nuclei are expected to freeze into a solid, leaving the electrons as the only moving charge carriers.

Traditionally, it has been assumed that the crust is in its absolute ground state, in which one particular kind of nucleus, which minimizes the enthalpy at the local pressure, is present at any given depth in the star, and these nuclei form a near-perfect crystal lattice. This picture has been challenged by Jones [15], who argued that thermodynamic fluctuations at the time of freezing would ensure the presence of several kinds of nuclei at any given pressure, yielding a very impure solid. This result may be very important for the evolution of the magnetic field, as it would substantially increase the resistivity of the neutron star crust, particularly at low temperatures [16].

At densities above “neutron drip”, $\sim 4 \times 10^{11}$g cm$^{-3}$, free neutrons also appear within the crust. At a temperature similar to that for freezing the lattice, these neutrons are expected to form Cooper pairs and turn superfluid, largely decoupling dynamically from the rest of the star. This has been invoked to explain pulsar “glitches”, quick increases of the rotation rate of the externally visible parts of a neutron star, which might be caused by a catastrophic transfer of angular momentum from the more rapidly rotating superfluid neutrons. Given that these neutrons are uncharged and interact only weakly with the electrons, their presence is not likely to be relevant for the magnetic field evolution.

The core

At $\sim 2 \times 10^{14}$g cm$^{-3}$, slightly below standard nuclear density, the nuclei are expected to lose their individual identity, so matter at higher densities consists of a liquid-like
mix of neutrons \((n)\), protons \((p)\), and electrons \((e^-)\), which at progressively higher
densities are joined by additional particles, such as muons, hyperons, and mesons. Chemical
equilibrium among all these particles is established by weak interactions such as
the neutron beta decay \((n \rightarrow p + e^- + \bar{\nu})\) and inverse beta decay \((p + e^- \rightarrow n + \nu)\),
in most cases involving the emission of neutrinos \((\nu)\) or antineutrinos \((\bar{\nu})\), which easily
leave the star. In this phase, the strong Pauli blocking of final states reduces the cross
sections for, e.g., electron-proton collisions, yielding a very high electrical conductivity
\[[3]\]. Here, all particles are movable (within constraints to be discussed below), making
the evolution of the magnetic field much more complicated.

Models predict that the strongly interacting particles in this phase \((n, p, \text{possibly hyperons})\) again form Cooper pairs and enter a superfluid state, in which the neutron
torticity is concentrated in quantized vortex lines of microscopic thickness, much smaller
than their average spacing, and the magnetic flux may be similarly concentrated in proton
vortices. The transition temperatures for these superfluid states are highly uncertain.
Their effect on the thermal evolution of the neutron star, although potentially quite
important, is not obviously shown by the observations of cooling, young neutron stars (e.g.,
Fig. 1 of Ref. \[[42]\]), although better fits to the (quite uncertain) data have been obtained
by allowing for neutron stars with a range of superfluid properties (with mass as the
controlling, free parameter) \[[42, 23]\]. On the other hand, the core superfluids are not
expected to be as easily decoupled as the crustal neutron superfluid, and therefore play
no role in glitches. Therefore, we consider it safe to say that there is no strong evidence
for the existence of superfluids in the neutron star core. This conclusion, in addition to
the fact that superfluids introduce a new element of uncertainty in the already highly
complicated models of magnetic field evolution, motivate us to ignore it in most of what
follows, although it has been discussed in detail by other authors \[[20, 34]\].

ORIGIN

The range of observed surface magnetic field strengths in degenerate stars is quite large,
from below \(10^4\)G to \(B_{\text{max}} \sim 10^9\)G on white dwarfs, and from \(10^{11}\)G up to \(B_{\text{max}} \sim 10^{15}\)G
for the dipole fields on young neutron stars, including classical radio pulsars, anomalous
X-ray pulsars, and soft gamma-ray repeaters (the field on millisecond pulsars may have
been decreased by processes related to accretion, and not be related to the original field).
The measurable dynamic range on massive main sequence stars is much smaller, but it
is suggestive that the largest inferred magnetic fluxes in all three kinds of objects are
similar, \(\Phi = \pi R^2 B_{\text{max}} \sim 3 \times 10^{27}\)Gcm\(^2\), where \(R\) is the stellar radius \[[28]\]. This implies
that, in principle, flux freezing from the main sequence is enough, and no additional
magnetic field generation mechanisms need to be invoked to explain even the strongest
fields observed on the compact stars.

On the other hand, stellar evolution, specially in the massive stars that eventually
become neutron stars, is quite eventful, so the field geometry (and perhaps its strength)
is likely to become modified along this evolution, possibly coupled to the evolution of
the angular momentum. In particular, the vigorous convection occurring in a newborn,
perhaps rapidly rotating, protoneutron star is an ideal setting for a dynamo to operate,
which might increase even a fairly weak initial field up to $\sim 10^{15}$ G. 

On the other hand, thermomagnetic instabilities driven by the heat flow through the neutron star crust [39, 4, 41] can at most account for pulsar-like field strengths $\sim 10^{12}$ G, making them much less likely to be relevant.

Thus, although no detailed scenario has been convincingly worked out, it appears likely that a strong magnetic field, permeating much of the core of the star, is present at least from the hot, protoneutron star phase onwards.

Regarding its geometrical configuration, recent magneto-hydrodynamic (MHD) simulations by Braithwaite & Spruit [7] show that, in a stably stratified star, a complicated, random, initial field generally evolves on an Alfvén-like time scale to a relatively simple, roughly axisymmetric, large-scale configuration containing a toroidal and a poloidal component of comparable strength, both of which are required in order to stabilize each other. Only the poloidal component crosses the surface of the star, yielding an external field that is roughly dipolar, though perhaps somewhat offset from the center, as is observed in magnetic A stars and white dwarfs.

We should note, however, that both A stars and white dwarfs are of essentially uniform composition (except for the discontinuity between the hydrogen-burning, convective core and the chemically pristine radiative envelope in A stars), so they are stably stratified only through the entropy gradient in their radiative regions. This means that, if stable stratification is an essential ingredient in stabilizing the configurations found in Ref. [7], their lifetime would be limited by the exchange of entropy between regions of different field strengths [17]. If, as seems to be the case, this time is shorter than the lifetime of the stars, the field could be confined only by compositional gradients (discussed below for the case of neutron stars), as in the core-crust transition of upper main-sequence stars.

In any case, it appears plausible that fields of configurations similar to those of Ref. [7], with poloidal field strengths $\sim 10^{11-15}$ G, and perhaps with their toroidal component enhanced by the initial differential rotation [38], could be present in early neutron stars.

**EVIDENCE FOR MAGNETIC FIELD EVOLUTION**

Since the earliest days of the study of pulsars, it has been claimed [13, 22] that the distribution of these objects on the period ($P$) - period derivative ($\dot{P}$) plane (sometimes combined with other variables, such as their position with respect to the Galactic plane, their proper motion, and their luminosity) would indicate a magnetic field decay on a time scale comparable to a pulsar lifetime. A safe proof or disproof of such statements is very difficult to make, since various selection effects are important, but recent work has generally not confirmed them [5, 26]. Nevertheless, in the past it provided the motivation for a fair amount of theoretical work trying to explain the claimed decay.

Another weak argument for magnetic field evolution is the fact that the braking indices $n \equiv \dot{\Omega}/\Omega^2$ measured in very young pulsars have always turned out to be smaller than the value ($n = 3$) expected from a magnetic dipole torque [35] with a constant dipole moment. This may indicate that the magnetic field increases in very young pulsars, or else that the dipole formula is not completely adequate (and therefore does not give a
very reliable estimate of the dipole field strength [19]).

A stronger, “classical” argument for field decay is the observation that old neutron stars, such as found in low-mass X-ray binaries and millisecond pulsars, tend to have much weaker fields ($\sim 10^8-9$ G) than the younger classical pulsars and high-mass X-ray binaries ($\sim 10^{12}$ G). It is still not clear if this is spontaneous decay happening over the long lifetime of these objects, or whether it is due to the accretion process diamagnetically screening the field [33, 9] or increasing the resistivity to cause a faster decay. In all these cases, it seems surprising that the field in all millisecond pulsars appears to reach a bottom value $\sim 10^8$ G and does not decay beyond that. A possible explanation might be that the original field of the star is completely screened or lost from the star, whereas new magnetic flux is carried onto the neutron star by the accretion flow, limited by the condition that the fluid stresses be strong enough to compress the field onto the star, which yields a maximum field strength of about the right order of magnitude.

Finally, but most interestingly, soft gamma-ray repeaters and anomalous X-ray pulsars appear to radiate substantially more power than available from their rotational energy loss. At the same time, their dipole field inferred from the observed torque is stronger than in all other neutron stars, $\sim 10^{14-15}$ G (although a couple of pulsars with $B \sim 10^{14}$ G are also known [33, 21, 18]). This makes it natural to accept the argument of Thompson & Duncan [37] that these sources are in fact “magnetars”, powered by the dissipation of their magnetic energy (which requires an rms internal field even stronger, though not by very much, than the inferred dipole field). In addition, torque changes of both signs, associated with outbursts from these sources, also argue for a changing magnetic field structure. These are, at the moment, the only objects in which there is strong evidence for spontaneous evolution of the magnetic field.

A PHYSICAL MODEL

As explained above, a neutron star core contains a number of mobile particle species. After an initial transient, following the formation of the neutron star, on which all sound and Alfvén waves are damped, the evolution of the magnetic field should be slow enough to make the inertia of the particles negligible, leading to the equation of diffusive motion for the particles of each species $i$,

$$0 = -\nabla \mu_i - m_i^* \nabla \psi + q_i \left( \vec{E} + \frac{\vec{v}_i}{c} \times \vec{B} \right) - \sum_j \gamma_{ij} n_j (\vec{v}_i - \vec{v}_j),$$  

(1)

where $\mu_i, m_i^*, q_i, \vec{v}_i$ are their chemical potential (Fermi energy), effective mass (including relativistic corrections due to random motions and interactions), electric charge, and mean velocity, $\psi$ is the gravitational potential, $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, and the last term represents the momentum transfer due to collisions with all other particle species $j$, each with number density $n_j$. The collisional coupling strengths are parameterized by the symmetric matrix $\gamma_{ij}$, whose coefficients generally depend on position.
The collision terms have two effects in the context of the evolution of the magnetic field:

(a) They damp the relative motion of positive and negative charge carriers, leading to a resistive diffusion of the magnetic field, opposed by the induced electric field. For a magnetic field of spatial scales comparable to the neutron star radius, the induction is large and the resistance is small, as in most astrophysical settings, so this process is slow and unlikely to have any effect over the lifetime of any neutron star [3]. It may, however, be important if the magnetic field is created within a thin surface layer (which does not fit with the likely formation scenarios we described) or if small-scale structure is created by other processes, as discussed below.

(b) Collisions tend to keep the different particle species moving with similar velocities, as in the standard MHD approximation. We now proceed to analyze this type of motion.

Adding the forces given by eq. (1) over all particles (of all species) within a volume containing a unit total mass and no net charge, one obtains the equation of MHD equilibrium,

\[ 0 = -\frac{\nabla P}{\rho} \nabla \psi + \frac{\vec{j} \times \vec{B}}{\rho c}, \]  

(2)

where the mass density \( \rho = \sum_n n_i m_i^* \), the current density \( \vec{j} = \sum_n n_i q_i \vec{v}_i = (c/4\pi) \nabla \times \vec{B} \), and the pressure gradient term was obtained from the zero-temperature Gibbs-Duhem relation, \( dP = \sum_i n_i d\mu_i \). Taking the curl of eq. (2), we obtain

\[ \nabla \times \nabla P \times \nabla \rho = \nabla \times \left( \frac{\vec{j} \times \vec{B}}{\rho c} \right). \]  

(3)

The right-hand side is generally non-zero, therefore equilibrium is only possible if the pressure and density gradients appearing on the left-hand side are not parallel. This is not possible in cold matter in chemical equilibrium, since then the density is a unique function of pressure (because all the chemical abundances are also determined by the latter). Therefore, the magnetic field generally perturbs the chemical equilibrium, causing a tiny misalignment between the pressure and density gradients.

A magnetic force distribution that has a horizontal curl component induces compensated up and downward motions of the fluid in different regions, producing the mentioned misalignment of the density and pressure gradients, which chokes the motion. Only the horizontal fluid motions produced by a magnetic force density with a purely vertical curl can proceed essentially unimpeded. This is a manifestation of the stable stratification of the neutron star matter [24, 31, 29], which prevents even a magnetar-strength field from being transported vertically through the star. The only ways by which this constraint can be circumvented are [24, 12]:

(a) elimination of the induced chemical imbalance by weak interaction processes, which is most effective at high temperatures, where reactions are fast, and

(b) relative motion of different particle species.
We now proceed to discuss each of these processes.

**Bulk motion facilitated by weak interactions**

As described above, the Lorentz force produces non-barotropic perturbations to the pressure, of magnitude\[ \delta P \sim B^2/8\pi \sim \sum n_i \delta \mu_i, \]
where the relation to the chemical potential perturbations again comes from the Gibbs-Duhem relation. If, for definiteness, we consider the simplest possible neutron star matter, composed of neutrons, protons, and electrons (the latter two related by the condition of charge neutrality, \( n_p = n_e = n_c \)), the total chemical imbalance is

\[ \Delta \mu \equiv |\mu_p + \mu_e - \mu_n| \sim B^2/(8\pi n_c) \sim 3B_{15}^2 \text{keV}. \]

where \( B_{15} \equiv B/(10^{15} \text{G}) \). An asymmetry in the weak interaction rates tends to reduce this imbalance. In what follows, we assume that the dominant process are “modified Urca reactions” without Cooper pairing, which give a reasonable fit to the early cooling of neutron stars (see Fig. 1 in Ref. [42]) and possibly to the late, “rotochemical” reheating of millisecond pulsars [11]. As long as \( T \gg \Delta \mu \) [27], the imbalance decays exponentially, with time constant \( t_{\text{mU}} \sim 0.5/T_6^6 \) yr, essentially the cooling time of the star, which is highly sensitive to temperature \( (T = T_9 \times 10^9 \text{K} \approx T_9 \times 86 \text{keV}) \). In this time, the Lorentz force makes the fluid move a small fraction, \( \sim \Delta \mu/\mu_e \sim 2 \times 10^{-5}B_{15}^2 \), of the stellar radius, so the total decay time of the field is

\[ t_{\text{decay}} \sim \frac{\Delta \mu}{\Delta \mu} t_{\text{mU}} \sim \frac{3 \times 10^4}{B_{15}^2 T_6^6} \text{yr}, \]

in principle quite short for strong fields and high temperatures. However, it is important to note that it is longer than the cooling time by the generally quite considerable ratio \( \mu_e/\Delta \mu \). Thus, unless the magnetic field is in the vicinity of \( \sim 10^{17} \text{G} \) (making \( \Delta \mu \sim \mu_e \)), a heat source is required to prevent passive cooling of the neutron star from “freezing” the magnetic field. The most obvious energy source is the magnetic field itself [37], whose energy in this scenario is released by the weak interactions. We note, however, that, in the absence of a chemical imbalance, the neutrino emission produced by these weak interaction leads to the cooling of the star. It is balanced by the heat release only if the chemical imbalance is fairly large, \( \Delta \mu \approx 5.5T_9 \) [11] or, equivalently, \( B_{15} \approx 13T_9^{1/2} \). A highly magnetized neutron star would be born at a high temperature and quickly cool down until this condition is satisfied, after which the temperature would be stabilized (at \( T_9 \sim 0.2[10^4 \text{yr}/t]^{1/7} \)) by the energy injected through the decaying magnetic field,\(^1\) \( B_{15} \sim 6.5(10^4 \text{yr}/t)_{1/14} \). At this fairly high temperature, the collision rates are high, thus the single-fluid MHD approximation is probably adequate.

\(^1\) The coefficient is slightly higher than found by Thompson & Duncan [37], who first obtained essentially the same result.
Relative motion of charged and neutral particles

The process just described is not relevant for stars with internal fields substantially lower than $5 \times 10^{15}$ G, which would enter the photon-cooling epoch before the magnetic field has substantially decayed or contributed to any reheating. For these, the only possibility to have magnetic field decay relies on relative motion of charged particles. We consider a slight extension of the model of npe matter proposed by Goldreich & Reisenegger, in which different linear combinations of the eqs. (1) for the three particle species, together with the induction equation $\frac{\partial \vec{B}}{\partial t} = -c \nabla \times \vec{E}$, yield

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left[ (\vec{v}_n + \vec{v}_A + \vec{v}_H) \times \vec{B} \right] - \nabla \times \left( \frac{c \vec{j}}{\sigma} \right) - \frac{c}{2e} \nabla \left( \frac{\gamma_{en} - \gamma_{pm}}{\gamma_{en} + \gamma_{pn}} \right) \nabla (\mu_p + \mu_e). \quad (6)$$

In the absence of the last two (resistive and battery) terms, which in realistic conditions are generally small, the magnetic field is advected by a combination of three velocities:

(a) the neutron velocity $\vec{v}_n$, corresponding to the bulk motions considered in the previous subsection;

(b) the ambipolar diffusion velocity,

$$\vec{v}_A \equiv \frac{\gamma_{pn}(\vec{v}_p - \vec{v}_n) + \gamma_{en}(\vec{v}_e - \vec{v}_n)}{\gamma_{pm} + \gamma_{en}} = \frac{\vec{j} \times \vec{B}/(n_c e) - \nabla (\Delta \mu)}{(n_n + n_e)(\gamma_{pm} + \gamma_{en})}, \quad (7)$$

representing a relative motion of the charged particles with respect to the neutrons, driven by the Lorentz force, and potentially choked by the gradient it induces in the chemical potential imbalance; and

(c) the Hall drift velocity,

$$\vec{v}_H \equiv \frac{\gamma_{en} - \gamma_{pm}}{\gamma_{en} + \gamma_{pn}} (\vec{v}_p - \vec{v}_e) = \frac{\gamma_{en} - \gamma_{pm}}{\gamma_{en} + \gamma_{pn}} \frac{\nabla \times \vec{B}}{4 \pi n_c e}, \quad (8)$$

due to the relative motion of protons and electrons and proportional to the current density.

The chemical imbalance induced by ambipolar diffusion can in principle be eliminated by weak interactions, on time scales similar to those of the bulk motions, which are short only at high temperatures, at which ambipolar diffusion is strongly suppressed by collisions. The regime in which it may be important is at lower temperatures, where collisions are less constraining, but weak interactions are strongly suppressed. In this case, only the solenoidal mode of ambipolar diffusion (driven by the finite-curl, zero-divergence part of the Lorentz force) can proceed (and even this only in the simplified case of a uniform charged fluid with $n_p = n_e$, without the stabilizing effect of nonuniform relative particle abundances if more charge carriers are present).

The Hall drift is of a very different character, since the drift velocity is directly related to the magnetic field, and independent of any arising fluid forces. In a solid medium such as the neutron star crust, where the electrons are the only free charge carrier, it
is the only active process besides resistive diffusion. This regime has been studied by several authors and is discussed in the following section. The more complicated regime of the fluid core, in which the Hall effect acts in conjunction with (or in opposition to) a highly constrained ambipolar diffusion, is largely an open question [2].

**HALL DRIFT**

In the neutron star crust or another solid, conducting medium, the only moving charges are the electrons, and the evolution of the magnetic field is described by the “Hall equation”,

\[
\frac{\partial \vec{B}}{\partial t} = \nabla \times \left( -\frac{c}{4\pi n_e} \nabla \times \vec{B} \right) \times \vec{B} + \frac{c^2}{4\pi \sigma} \nabla \times \vec{B},
\]

where the two terms on the right-hand side correspond to the Hall drift and resistive diffusion, respectively, whose relative importance in the neutron star crust is still a matter of controversy [10, 16].

Here, we discuss some physical issues that are likely to determine the evolution of magnetic field under the Hall effect, without attempting to cover the many simulations recently performed to address how specific magnetic field configurations might decay in real neutron stars [32, 14].

Goldreich & Reisenegger [12] focused on the more interesting case in which, on large scales (comparable to the crust thickness) the Hall effect is dominant, and argued, by analogy with the Euler equation of fluid dynamics, that the nonlinear Hall term may give rise to a turbulent cascade to small scales. Due to the presence of stable, linear modes, this turbulence would be “weak”, with an energy transfer time generally longer than the typical oscillation period on a given scale, resulting in a power spectrum \( \propto k^{-2} \), slightly different from the Kolmogorov spectrum \( \propto k^{-5/3} \) of fluid turbulence. At small scales, the magnetic energy would finally be dissipated by resistivity. This nonlinear evolution was simulated by Biskamp et al. [6], who found an energy cascade to small spatial scales, but with an even steeper spectrum than predicted, \( \propto k^{-7/3} \).

A complementary approach is to study analytic solutions of the Hall equation (with or without the resistive term), which has been done by Vainshtein et al. [40], Cumming et al. [10], and by our group [1, 25]. In the latter, which extends and generalizes the work of Ref. [40], we first considered a purely toroidal field, \( \vec{B} = \mathcal{B}(R, z, t) \nabla \phi \), where \( R, \phi, z \) are the standard cylindrical coordinates. When evolved by the Hall equation, the field remains toroidal. In order to describe its evolution, it becomes convenient to introduce a new coordinate \( \chi \equiv c/[4\pi n_e(R, z)R^2] \). Surfaces of constant \( \chi \) (hereafter, \( \chi \)-surfaces) are toroids contained inside the star. A complementary coordinate \( s \) can be defined on each \( \chi \)-surface by \( \partial / \partial s \equiv -R^2 \nabla \phi \times \nabla \chi \cdot \nabla \), in terms of which the Hall equation (now neglecting the resistive term) reduces to the Burgers equation,

\[
\frac{\partial \mathcal{B}}{\partial t} + \mathcal{B} \frac{\partial \mathcal{B}}{\partial s} = 0,
\]

with the well-known, implicit solution \( \mathcal{B} = f(s - \mathcal{B}t) \). Each value of \( \mathcal{B} \) is carried around the corresponding \( \chi \)-surface with a velocity proportional to \( \mathcal{B} \), developing
discontinuities at the (comoving) points where $\partial B / \partial s$ is large. There, the resistive term increases and tends to smooth the discontinuity. Therefore, magnetic energy is dissipated at the (relatively fast) rate as it is fed into the discontinuity by the Hall drift. Eventually, on the characteristic time scale of the Hall drift, $B$ becomes uniform on each $\chi$-surface, so the resulting magnetic field $\vec{B} = \mathcal{B}(\chi) \nabla \phi$ only evolves on the (assumed) much longer resistive time scale.

However, further analysis [25] shows that this field is unstable to small, poloidal perturbations, which grow when different segments of the poloidal field lines, crossing different $\chi$-surfaces, are carried along at different speeds by the Hall drift velocity associated to the toroidal field. So, the question arises: Are there any non-trivial configurations which are not only static, but also stable, under the Hall effect?

A likely important clue is conservation of magnetic helicity,

$$\frac{\partial}{\partial t} (\vec{A} \cdot \vec{B}) + \nabla \cdot (c \phi \vec{B} + c \vec{E} \times \vec{A}) = -2c \vec{E} \cdot \vec{B}. \quad (11)$$

This equation, derived directly from Maxwell’s equations (where $\phi$ and $\vec{A}$ are the standard electromagnetic scalar and vector potentials), shows that the volume integral of the “magnetic helicity density” $\vec{A} \cdot \vec{B}$ is conserved if $\vec{E} \cdot \vec{B} \equiv 0$, which is the case whenever the resistivity is zero, so the generalized Ohm’s law reduces to $\vec{E} + \vec{v} \times \vec{B} / c = 0$, regardless of the specific form of the velocity field $\vec{v}$. In particular, the stable MHD equilibria found by Braithwaite & Spruit [7] most likely minimize energy at a given magnetic helicity, and the Hall effect also conserves this quantity.

Since, dimensionally, the magnetic helicity density is $\sim B^2 L$ (where $B$ and $L$ are a characteristic magnetic field strength and length scale), whereas the magnetic energy density is $\sim B^2$, the helicity tends to reside on large scales, making it much more difficult to dissipate than energy. This could make strongly helical configurations stable and prevent them from decaying under any of the processes considered (aside from the extremely slow resistive diffusion). We are currently engaged in identifying such configurations.

**CONCLUSIONS**

The evolution of the magnetic field in neutron stars is a very challenging subject, with strong observational clues and involving complex physics. Perhaps surprisingly (given the extreme conditions of density, gravity, and field strength), it is likely to involve many of the same processes also emerging in the evolution of magnetic fields in other contexts, from plasma physics laboratories to galaxies, so continued cross-feeding of insights should be beneficial both to the study of neutron stars and of other systems.

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