Chapter 5

Biarc approximation of plane curves

5.1 Introduction

In this chapter we are concerned with approximation of parametric curves in the plane by biarcs. A biarc consists of two circular arcs which are tangent where they meet. By considering two points on a given curve and the tangents at these two points one get a one parameter family of biarcs. A great many of the papers on curve fitting with biarcs seem to be concerned with how to choose the remaining parameter so as to be able to minimize the error. In this chapter we address this problem and prove for the first time that the complexity - the number of elements - of an optimal biarc spline approximating a space curve to within Hausdorff distance $\varepsilon$ is of the form $O(\varepsilon^{-1/3})$. Furthermore, we give the first global proof of computing the unique optimal biarc approximating a spiral curve in the plane under the prescribed conditions. A spiral curve in the plane is a curve with monotonically increasing/decreasing curvature.

In Computer Numerically Controlled (CNC) manufacturing it is very important to control the speed of the tool along its path precisely. For this reason the curves with simple arc length function are very suitable for CNC manufacturing. The traditional approach is to use curves composed of linear and circular segments, for which the arc length function can be easily expressed. These kind of applications need a tight hold on the error bound for the Hausdorff metric when the interpolant is curved, thus it is of interest to study approximations with biarcs.
5.1.1 Related Work

Biarcs appeared in the engineering design literature as early as the 1970s [103, 6, 15], and a great deal of research in curve fitting with biarcs has been done [103, 84, 83, 85]. In [83] Meek and Walton consider the problem of approximating planar spirals with biarcs, they show that for a particular case the Hausdorff distance between a spiral curve $\alpha$ and a biarc $\beta$ approximating it is $\delta_H(\alpha, \beta) = \frac{1}{324} \kappa_0' \sigma^3 + O(\sigma^4)$, where $\sigma$ is the length of the spiral arc $\alpha$. However, they do not prove that this choice is indeed the optimal one. We prove this for the first time in this chapter, and show that it is a direct consequence of the Approximation Theorem proved in Chapter 4. In [85] Meek and Walton show there is a one parameter family of biarcs that can match a pair of points and tangents at the these point in the plane. This paper considers the range of $G^1$ Hermite data that can be matched, identifies the region in which the members of the family of biarcs lie, and shows that there is exactly one member biarc passing through each point in that region. In the current chapter we prove a similar result, for approximation of a spiral arc by biarcs, which is more intuitive as we exploit the differential geometric characteristics of this approximation. We show in this chapter that for a spiral arc there exists a one parameter family of biarcs and they lie side by side. Furthermore, we prove that the distance function for this one parameter family of biarcs also lie side by side thus giving a unique biarc which minimizes the Hausdorff distance. Also we show that the distance function of the optimal unique biarc is equioscillatory.

5.1.2 Results of this chapter

Complexity of biarc approximation. We show that the complexity of an optimal biarc spline approximating a plane curve to within Hausdorff distance $\varepsilon$ is of the form $c \varepsilon^{-1/3} + O(1)$, where we express the value of the constant $c$ in terms of derivative of curvature. Note that this expression matches the expression for optimal bihelix approximation of space curves. The error bound is derived as a consequence of the Approximation Theorem, proved in Chapter 4.
5.2. Local approximation of plane curves with biarcs

Algorithmic issues. For curves with monotone curvature, called spirals, we consider biarcs tangent to the curve at its endpoints, and show that the locus of junction points of these biarcs is a circle which we define as the junction circle. The junction circle intersects the spiral arc at the endpoints and we prove that it intersects the spiral arc exactly at one more point in between the endpoints. Furthermore, we show that in this family of biarcs there is a unique one minimizing the Hausdorff distance. If $\alpha : I \to \mathbb{R}^2$ is an affine spiral, its displacement function $d : I \to \mathbb{R}$ measures the signed distance between the spiral and the optimal biarc along the normal lines of the spiral. The displacement function $d$ has an equioscillation property: there are two parameter values $u_+, u_- \in I$ such that $d(u_+) = -d(u_-) = \delta_H(\alpha, \beta_{opt})$, where $\beta_{opt}$ is the optimal biarc. Furthermore, the Hausdorff distance between a section of the spiral and its optimal approximating biarc is a monotone function of the arc length of the spiral section. This useful property gives rise to a straightforward bisection algorithm for the computation of an optimal interpolating tangent continuous biarc spline. Although we did not implement this algorithm it follows in the same fashion as the algorithm for an optimal conic spline approximation as discussed in Chapter 3.

5.1.3 Overview

In Section 5.2, we derive the asymptotic error expression for approximation of a planar spiral arc with the optimal biarc with respect the Hausdorff distance. Section 5.3 presents the global results on biarc approximation of planar spiral arcs. We show that there exists a unique biarc with minimal distance to the spiral arc. The equioscillatory nature of the distance function for the unique optimal biarc is also prove in this section.

5.2 Local approximation of plane curves with biarcs

Let $\alpha : [0, \sigma] \to \mathbb{R}^2$ be a $C^3$-curve, parametrized by arc-length, and let $\beta : [0, \sigma] \to \mathbb{R}^2$ be a curve of the form

$$\beta(s) = \alpha(s) + f(s) N(s),$$
where $N(s)$ is the unit normal of the curve $\alpha$ at the point $\alpha(s)$, and where $f(s)$ satisfies the condition of Lemma 4.5.3. In particular, $f$ is of the form $f(s) = s^2 P(s)$. Then a short computation shows that $\kappa'_\beta(0) = \kappa'_0 + 6 P'(0)$, and $f'''(0) = 6 P'(0)$, so $f'''(0) = -\kappa'_0$. Therefore, Lemma 4.5.3 implies that the Hausdorff distance between the curves $\alpha$ and $\beta$ is

$$\delta_H(\alpha, \beta) = \max_{0 \leq s \leq \sigma} || \beta(s) - \alpha(s) || = || f ||_\infty \geq \frac{1}{324} |\kappa'_0| \sigma^3 + O(\sigma^4).$$

The well-known construction of optimal biarcs shows that this inequality is sharp, in the sense that there is a biarc $\beta$ tangent to $\alpha$ at its endpoints, such that the Hausdorff distance between $\alpha$ and $\beta$ satisfies

$$\delta_H(\alpha, \beta) = \frac{1}{324} |\kappa'_0| \sigma^3 + O(\sigma^4).$$

In case of equality we have $f'(\frac{1}{2} \sigma + O(\sigma^2)) = -\frac{1}{24} \kappa'_0 \sigma^2 + O(\sigma^3)$, from Approximation theorem it follows that the junction tangent is given by

$$\beta'(\frac{1}{2} \sigma) = T(\frac{1}{2} \sigma) - \frac{1}{24} \kappa'_0 \sigma^2 N(\frac{1}{2} \sigma) + O(\sigma^4).$$

Theorem 4.6.1 also follows in the two-dimensional case and implies that the junction point is

$$\beta(\frac{1}{2} \sigma + O(\sigma^2)) = \alpha(\frac{1}{2} \sigma + O(\sigma^2)) + O(\sigma^4).$$

### 5.3 Geometry of Biarcs

In this section we discuss some of the global results in approximation of plane curves with biarcs. We consider $C^3$, regular curves in the plane with monotone Euclidean curvature, such curves are called spiral curves. Spiral curves in the plane have some interesting properties, here we state those that are of interest to us.

The result stated below is due to Kneser and originally proved in [64], we also refer to [56] for this result.

**Theorem 5.3.1** (Kneser’s Theorem). *Any osculating circle of a spiral arc contains every smaller osculating circle of the arc in its interior and in its turn is contained in the interior of every circle of curvature of greater radius.*
5.3. Geometry of Biarcs

Figure 5.1: Nested osculating circles of a spiral arc

For the proof of the following proposition on spiral arcs we refer to [66]

**Proposition 5.3.2.** A curve with increasing curvature intersects its every circle of curvature from right to left (and from left to right if curvature decreases).

Another interesting result about spiral arcs in the plane is due to Vogt, for the proof we refer to Guggenheimer [56, page 49]

**Theorem 5.3.3** (Vogt’s Theorem). Let $A$ and $B$ be the endpoints of a spiral arc, the curvature nondecreasing from $A$ to $B$. The angle $\beta$ of the tangent to the arc at $B$ with the chord $AB$ is not less than the angle $\alpha$ of the tangent at $A$ with $AB$. $\alpha = \beta$ only if the curvature is constant.

**Corollary 5.3.4.** A convex spiral arc and a circle have at most three points of intersection without contact or one point of contact and one noncontact intersection.

*Proof.* Let $\gamma : [c_1, c_2] \to \mathbb{R}^2$ be a spiral arc with strictly increasing curvature from $\gamma(c_1)$ to $\gamma(c_2)$. Let $C_\gamma$ be a circle tangent to $\gamma(c_1)$ and $\gamma(c_2)$, this is a contradiction to Vogt’s theorem, hence $C_\gamma$ is a circle through $\gamma(c_1)$ and tangent to $\gamma(c_2)$ or it is a circle through $\gamma(c_2)$ and tangent to $\gamma(c_1)$. $\square$
In the scheme of biarc approximation, the locus of all junction points of one parameter family of biarcs approximating a planar arc is a circle, as shown in Lemma 5.3.5. We call this circle the Junction Circle.

**Lemma 5.3.5.** Let $v$ and $w$ be unit vectors in the plane, with $v \neq \pm w$. For all unit vectors $t$, with $t \neq v, w$, the angle between $v - t$ and $w - t$ is constant on each of the intervals of $S^1$ defined by $v$ and $w$. The angle of the first interval is the complement of the angle on the second interval (i.e., the sum of their values is $\pi$).

For a proof of Lemma 5.3.5 we refer to ACS technical report [20] by Chazal and Vegter. From the results above we have the following theorem.

**Theorem 5.3.6.** Let $\gamma : [c_1, c_2] \to \mathbb{R}^2$, be a regular, $C^3$ spiral arc with strictly increasing/decreasing Euclidean curvature, then the Junction Circle through $\gamma(c_1)$ and $\gamma(c_2)$ intersects the curve $\gamma$ at exactly one more point $\gamma(c)$, for $c \in (c_1, c_2)$.

**Proof.** Let $C_\gamma$ denote the Junction Circle for $\gamma$. In [112], Šir et. al, show that $C_\gamma$ intersects $\gamma$ at least at a point $\gamma(c)$, for $c \in (c_1, c_2)$. Since $\gamma$ is a spiral arc, it follows from Corollary 5.3.4 that $C_\gamma$, passing through $\gamma(c_1)$ and $\gamma(c_2)$ intersects $\gamma$ at exactly one point in between.

In the following proposition we show that two distinct biarcs lie side by side.
5.3. Geometry of Biarcs

**Proposition 5.3.7.** Let $\gamma : [c_1, c_2] \to \mathbb{R}^2$, be a regular, $C^3$, spiral arc. Let $C_\gamma$ be the Junction Circle of $\gamma$ in the given interval. The set of biarcs is denoted by $B_\gamma$. 1. Suppose $u, u' \in C_\gamma$ and $u \neq u'$, then the corresponding biarcs $B_u, B_{u'} \in B_\gamma$ lie side by side.
Proof. A biarc is composed of a left and a right circular arc. Thus $B_u$ is a biarc composed of $C_{L_u}$ and $C_{R_u}$, and $B_{u'}$ is a biarc composed of two circular arcs $C_{L_{u'}}$ and $C_{R_{u'}}$. To prove our claim that two biarcs with distinct junction points $u$ and $u'$ lie side by side, we prove the following,
(i.) Two left circular arcs $C_{L_u}$ and $C_{L_{u'}}$ lie side by side.
(ii.) Two right circular arcs $C_{R_u}$ and $C_{R_{u'}}$ lie side by side.
(iii.) $C_{L_u}$ and $C_{R_u}$ intersect in a tangent continuous manner at the junction point $u$ and at no other point.
(iv.) The two circular arcs $C_{L_u}$ and $C_{R_{u'}}$ do not intersect at all.

Let us define $p_L := \gamma(c_1)$ and $p_R := \gamma(c_2)$.

(i.) By definition $C_{L_u}$ and $C_{L_{u'}}$ are tangent to $\gamma$ and hence to each other at $p_L$. Suppose $C_{L_u}$ and $C_{L_{u'}}$ intersect each other at a point distinct from $p_L$, then we have that $C_{L_u} = C_{L_{u'}}$. Furthermore, since $p_L, u, u'$ lie on $C_\gamma$ we have that $C_{L_u} = C_{L_{u'}} = C_\gamma$. Therefore $C_\gamma$ is tangent to $\gamma$ at $p_L$, passes through $p_R$ and intersects it at another point $\gamma(c)$, for $c \in (c_1, c_2)$. This is a contradiction to Corollary 5.3.4. Therefore we conclude that $C_{L_u}$ and $C_{L_{u'}}$ do not intersect at any other point other than $p_L$.

(ii.) Proof to this part follows in the same way as proof to part (i.).

(iii.) Let $C_{L_u}$ and $C_{R_u}$ be the left and right circular arcs for the biarc $B_u$. By definition $C_{L_u}$ and $C_{R_u}$ intersect each other at the junction point $u$ in a tangent continuous fashion. Suppose $C_{L_u}$ and $C_{R_u}$ intersect each other at another point distinct from $u$, then $C_{L_u} = C_{R_u}$. Thus $C_{L_u}$ and $C_{R_u}$ contain the points $p_L, p_R$ and $u$, hence $C_{L_u} = C_{R_u} = C_\gamma$, hence the junction circle is tangent to $\gamma$ at $p_L$ and $p_R$, which is a clear contradiction to Vogt’s Theorem 5.3.3. Therefore we conclude that $C_{L_u}$ and $C_{R_u}$ intersect each other at no other point other than $u$.

(iv.) Without loss of generality let us suppose that $||p_L - u|| < ||p_L - u'||$, then $C_{L_u}$ lies inside the region of $C_{L_{u'}}$ completely. Since $C_{R_{u'}}$ lies outside the region of $C_{L_{u'}}$, only having a contact at the point $u'$ we conclude that $C_{R_{u'}}$ does not intersect $C_{L_u}$.

\qed
Since biarcs for a spiral curve lie side by side, we conclude that the corresponding distance functions between biarcs and the spiral curve also lie side by side.

\[ \text{Equioscillatory} \]

Since the distance functions lie side by side there is a unique distance function which is \textit{Equioscillatory} and hence a unique biarc minimizing the Hausdorff distance exists. The proof of this and the monotonicity of the Hausdorff distance follows exactly as in the conic case as discussed in Chapter 3.

5.4 Conclusion

In the previous section we prove the monotonicity property of the Hausdorff distance for a biarc approximation of a space curve. Although, we did not implement any algorithm, a bisection algorithm, similar to the one discussed in Chapter 3 on conics can be implemented to verify the theoretical results obtained in this chapter.

In the Chapter 4 on helical arcs, we considered the problem of approximation of curves in space with bitangent bihelical arcs. Although we found the local expression for asymptotically optimal approximation of curves in space by bihelical splines, the global side of the problem still remains open. Some
the remaining challenges are the following:
- we need to prove the existence of a bitangent bihelical arc from the three-parameter family, which minimizes the Hausdorff distance.
- We conjecture that a space curve with monotone curvature and non-zero torsion can be defined to be a spiral curve in space, furthermore, we believe that for such a curve we can prove Kneser type theorem for osculating helices of a space curve. We also conjecture that the osculating helices of a spiral space curve lie side by side on a surface embedded in $\mathbb{R}^3$.

We believe that the last conjecture will be useful in giving a construction of a bitangent bihelix which minimizes the Hausdorff distance.

In the planar case, it would be interesting to find a geometric characterization of the point of the optimal bitangent biarc which lies on the given planar spiral curve. Also approximation of curves in higher dimension with generalizations of helices is another interesting problem to study. In similar lines it would be of interest to consider approximation of space curves with curves in space with constant affine curvature and affine torsion. Furthermore, similar tools considered in this thesis could be developed to study approximations of surfaces in $\mathbb{R}^3$ by patches of quadrics or spheres.