Geometric approximation of curves and singularities of secant maps
Ghosh, Sunayana

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2010

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):
Chapter 2

Differential geometry of curves

2.1 Introduction

Geometric modeling which includes computer graphics and computer-aided geometric design draws on ideas from differential geometry. This chapter presents the mathematical preliminaries of differential geometry of curves in plane and space which are used in our results that we present in Chapters 3 and 4. For a detailed discussion and introduction to this area we refer to [30, 56, 109]. Differential geometry of curves is an area of geometry that involves the study of smooth curves in the plane and in Euclidean space using the methods of differential and integral calculus. One of the most important tools used to analyze a curve is the Frenet-Serret frame, a moving frame that provides a coordinate system at each point of the curve that describes the curve at a point completely. The theory of regular curves in a Euclidean space has no intrinsic geometry. A curve is said to be regular if its derivative is well defined and non zero on the interval on which the curve is defined. Moreover, different space curves are distinguishable from each other by the way they bend and twist. Formally, these are the differential-geometric invariants called the curvature and the torsion of a curve. The fundamental theorem of curves [30] asserts that the knowledge of these invariants completely determines the curve. Another area of differential geometry that we are concerned with is called affine differential geometry. The differential invariants in this case are invariants under area preserving affine transformations. The name affine differential geometry follows from Klein’s Erlangen program. The basic difference between affine and Euclidean differential geometry is that in the affine case we are concerned with area preserving transformations in the plane whereas in the Euclidean case we are concerned with rigid motions. The affine differential geometry of curves is the study of curves in an affine...
space, and specifically the properties of such curves which are invariant under the area preserving affine transformations in the plane. As stated above, in Euclidean geometry of curves, the fundamental tool is the Frenet-Serret frame, in affine geometry, the Frenet-Serret frame is no longer well defined, but it is possible to define another canonical moving frame, which we call the affine Frenet-Serret frame, which plays a similar decisive role. The theory was developed in the early 20th century, largely due to the efforts of Blaschke [13].

Overview. In Section 2.2 we present the preliminaries of differential geometry of curves in the plane. In particular, we introduce affine arc length and affine curvature, which are invariant under equiaffine transformations. Moreover, we introduce the notion of affine Frenet-Serret frame followed by derivation of affine curvature of curves with arbitrary parametrization. As stated above conic arcs are the only curves in the plane having constant affine curvature, which explains the relevance of these notions from affine differential geometry. Section 2.3 reviews the differential geometry of generalized and circular helices in three-space. There is a derivation of the Darboux vector. Another normal frame for a space curve is derived other than the Frenet-Serret frame, this is a more natural frame for the parametric expression of a helix. Furthermore, the condition for a space curve to be a generalized helix is reviewed.

2.2 Planar Curves and Conics

Circular arcs and straight line segments are the only regular smooth curves in the plane with constant Euclidean curvature. Conic arcs are the only smooth curves in the plane with constant affine curvature. The latter property is crucial for our approach, so we briefly review some concepts and properties from affine differential geometry of planar curves. See also Blaschke [13].

2.2.1 Affine curvature

Recall that a regular curve $\alpha : J \to \mathbb{R}^2$ defined on a closed real interval $J$, i.e., a curve with non-vanishing tangent vector $T(u) := \alpha'(u)$, is parameterized according to Euclidean arc length if its tangent vector $T$ has unit length. In
2.2. Planar Curves and Conics

In this case, the derivative of the tangent vector is in the direction of the unit normal vector \( N(u) \), and the Euclidean curvature \( \kappa(u) \) measures the rate of change of \( T \), i.e., \( T'(u) = \kappa(u) N(u) \). Euclidean curvature is a differential invariant of regular curves under the group of rigid motions of the plane, i.e., a regular curve is uniquely determined by its Euclidean curvature, up to a rigid motion.

The larger group of \textit{equi-affine transformations} of the plane, i.e., affine transformations with determinant one (in other words, area preserving linear transformations), also gives rise to a differential invariant, called the \textit{affine curvature} of the curve. To introduce this invariant, let \( I \subset \mathbb{R} \) be an interval, and let \( \gamma : I \to \mathbb{R}^2 \) be a smooth, regular plane curve. We shall denote differentiation with respect to the parameter \( u \) by a dot: \( \dot{\alpha} = \frac{d\alpha}{du}, \ddot{\alpha} = \frac{d^2\alpha}{du^2} \), and so on. Then regularity means that \( \ddot{\alpha}(u) \neq 0 \), for \( u \in I \). Let the reparameterization \( u(r) \) be such that \( \gamma(r) = \alpha(u(r)) \) satisfies

\[
[\gamma'(r), \gamma''(r)] = 1. \tag{2.1}
\]

Here \([v, w]\) denotes the determinant of the pair of vectors \( \{v, w\} \), and derivatives with respect to \( r \) are denoted by dashes. The parameter \( r \) is called the \textit{affine arc length} parameter. If \([\dot{\alpha}, \ddot{\alpha}] \neq 0\), in other words, if the curve \( \alpha \) has non-zero curvature, then \( \alpha \) can be parameterized by affine arc length, and (2.1) implies that

\[
[\dot{\alpha}(u(r)), \ddot{\alpha}(u(r))] u'(r)^3 = 1. \tag{2.2}
\]

Putting

\[
\varphi(u) = [\dot{\alpha}(u), \ddot{\alpha}(u)]^{1/3}, \tag{2.3}
\]

we rephrase (2.2) as

\[
u'(r) = \frac{1}{\varphi(u(r))}. \tag{2.4}
\]

From (2.1) it also follows that \([\gamma'(r), \gamma''(r)] = 0\), so there is a smooth function \( k \) such that

\[
\gamma'''(r) + k(r) \gamma'(r) = 0. \tag{2.5}
\]
The quantity $k(r)$ is called the **affine curvature** of the curve $\gamma$ at $\gamma(r)$. It is only defined at points of non-zero Euclidean curvature. A regular curve is uniquely determined by its affine curvature, up to an equi-affine transformation of the plane.

From (2.1) and (2.5) we conclude $k = [\gamma'', \gamma''']$. The affine curvature of $\alpha$ at $u \in I$ is equal to the affine curvature of $\gamma$ at $r$, where $u = u(r)$.

### 2.2.2 Affine Frenet-Serret frame

The well known Frenet-Serret identity for the Euclidean frame, namely

$$
\dot{\alpha} = T, \quad \dot{T} = \kappa N, \quad \dot{N} = -\kappa T,
$$

(2.6)

where the dot indicates differentiation with respect to Euclidean arc length, have a counterpart in the affine context. More precisely, let $\alpha$ be a strictly convex curve parameterized by affine arc length. The **affine Frenet-Serret frame** $\{t(r), n(r)\}$ of $\alpha$ is a moving frame at $\alpha(r)$, defined by $t(r) = \alpha'(r)$, and $n(r) = \dot{t}(r)$, respectively. Here the dash indicates differentiation with respect to affine arc length. The vector $t$ is called the affine tangent, and the vector $n$ is called the affine normal of the curve. The affine frame satisfies

$$
\alpha' = t, \quad t' = n, \quad n' = -kt.
$$

(2.7)

Furthermore, we have the following identity relating the affine moving frame $\{t, n\}$ and the Frenet-Serret moving frame $\{T, N\}$.

**Lemma 2.2.1.** 1. The affine arc length parameter $r$ is a function of the Euclidean arc length parameter $s$ satisfying

$$
\frac{dr}{ds} = \kappa(s)^{1/3}.
$$

(2.8)

2. The affine frame $\{t, n\}$ and the Frenet-Serret frame $\{T, N\}$ are related by

$$
t = \kappa^{-1/3} T, \quad n = -\frac{1}{3} \kappa^{-5/3} \dot{\kappa} T + \kappa^{1/3} N.
$$

(2.9)

*Here $\dot{\kappa}$ is the derivative of the Euclidean curvature with respect to Euclidean arc length.*
2.2. Planar Curves and Conics

Proof. 1. Let $\gamma(r)$ be the parametrization by affine arc length, and let $\alpha(s) = \gamma(r(s))$ be the parametrization by Euclidean arc length. Then $\dot{\alpha} = T$ and $\ddot{\alpha} = \kappa N$. Again we denote derivatives with respect to Euclidean arc length by a dot. Since $\gamma' = t$ and $t' = \gamma'' = n$, we have

$$T = \dot{\alpha} = \dot{r}t, \quad \text{and} \quad N = \kappa^{-1} \ddot{\alpha} = \kappa^{-1}(\dot{\gamma}t + (\dot{r})^2n) \quad (2.10)$$

Since $[T, N] = 1$, and $[t, n] = 1$, we obtain $1 = \kappa^{-1}\dot{r}$. This proves the first claim.

2. The first part of the lemma implies $\dddot{r} = \frac{1}{3} \kappa^{-2/3} \kappa'$. Plugging this into the identity (2.10) yields the expression for the affine Frenet-Serret frame in terms of the Euclidean Frenet-Serret frame.

The affine Frenet-Serret identities (2.7) yield the following values for the derivatives—with respect to affine arc length—of $\alpha$ up to order five, which will be useful in the sequel:

$$\alpha' = t, \quad \alpha'' = n, \quad \alpha''' = -kt, \quad \alpha^{(4)} = -k't - kn, \quad \alpha^{(5)} = (k^2 - k'')t - 2k'n. \quad (2.11)$$

Combining these identities with the Taylor expansion of $\alpha$ at a given point yields the following affine local canonical form of the curve.

**Lemma 2.2.2.** Let $\alpha : I \to \mathbb{R}^2$ be a regular curve with non-vanishing curvature, and with affine Frenet-Serret frame $\{t, n\}$. Then

$$\alpha(r_0 + r) = \alpha(r_0) + (r - \frac{1}{3!} k_0 r^3 - \frac{1}{4!} k_0' r^4 + O(r^6)) t_0 + \left(\frac{1}{2} r^2 - \frac{1}{3!} k_0 r^4 - \frac{2}{5!} k_0' r^5 + O(r^6)\right) n_0,$$

where $t_0, n_0, k_0$, and $k_0'$ are the values of $t, n, k$ and $k'$ at $r_0$.

Furthermore, in its affine Frenet-Serret frame the curve $\alpha$ can be written locally as $x t_0 + y(x) n_0$, with

$$y(x) = \frac{1}{2} x^2 + \frac{1}{8} k_0 x^4 + \frac{1}{40} k_0' x^5 + O(x^6).$$

The first identity follows directly from (2.11). As for the second, it follows from the first by a series expansion. Indeed, write

$$x = r - \frac{1}{3!} k_0 r^3 - \frac{1}{4!} k_0' r^4 + O(r^6).$$
2. Differential geometry of curves

Computing the expansion of the inverse function gives

\[ r = x + \frac{1}{3!} k_0 x^3 + \frac{1}{4!} k'_0 x^4 + O(x^6). \]

Plugging in \( y = \frac{1}{2} r^2 - \frac{1}{4} k_0 r^4 - \frac{3}{8} k'_0 r^5 + O(r^6) \) gives the result.

2.2.3 Affine curvature of curves with arbitrary parametrization

The following proposition gives an expression for the affine curvature of a regular curve in terms of an arbitrary parameterization. See also [13, Chapter 1.6].

**Proposition 2.2.3.** Let \( \alpha : I \to \mathbb{R}^2 \) be a regular \( C^4 \)-curve with non-zero Euclidean curvature. Then the affine curvature \( k \) of \( \alpha \) is given by

\[
k = \frac{1}{\varphi^3} \left[ \dot{\alpha}, \ddot{\alpha} \right] + \frac{\ddot{\varphi} \varphi - 3 \varphi^2}{\varphi^4}, \tag{2.12}
\]

where \( \varphi = [\dot{\alpha}, \ddot{\alpha}]^{1/3} \).

**Proof.** Identity (2.4) implies \( \gamma'(r) = \Gamma(u(r)) \), where \( \Gamma(u) = \frac{1}{\varphi(u)} \dot{\alpha}(u) \). We denote differentiation with respect to \( u \) by a dot, like in \( \dot{\alpha} \), and differentiation with respect to \( r \) by a dash, like in \( \gamma' \). Then \( \gamma''(r) = u''(r) \tilde{\Gamma}(u(r)) \), and \( \gamma'''(r) = u'''(r) \tilde{\Gamma}(u(r)) + u''(r)^2 \tilde{\Gamma}(u(r)) \). From the definition of \( \Gamma \) we obtain

\[
\tilde{\Gamma} = -\frac{\ddot{\varphi}}{\varphi^2} \dot{\alpha} + \frac{1}{\varphi} \dddot{\alpha}, \quad \text{and} \quad \tilde{\Gamma} = \left( 2 \frac{\dot{\varphi}^2}{\varphi^5} - \frac{\dddot{\varphi}}{\varphi^2} \right) \dot{\alpha} - 2 \frac{\dot{\varphi}}{\varphi^2} \dddot{\alpha} + \frac{1}{\varphi} \dddot{\alpha}.
\]

Furthermore, since \( u''(r) = \frac{1}{\varphi(u(r))} \), it follows that \( u'''(r) = -\frac{\dddot{\varphi}(u(r))}{\varphi^2(u(r))} \). Therefore,

\[
\gamma''(r) = -\frac{\ddot{\varphi}}{\varphi^2} \dot{\alpha} + \frac{1}{\varphi^2} \dddot{\alpha}, \quad \text{and} \quad \gamma'''(r) = \left( 3 \frac{\dot{\varphi}^2}{\varphi^6} - \frac{\dddot{\varphi}}{\varphi^3} \right) \dot{\alpha} - 3 \frac{\dot{\varphi}}{\varphi^3} \dddot{\alpha} + \frac{1}{\varphi^3} \dddot{\alpha},
\]

where we adopt the convention that \( \varphi, \alpha, \) and their derivatives are evaluated at \( u = u(r) \). Hence, the affine curvature of \( \alpha \) at \( u \in I \) is given by

\[
k(u) = \left[ \gamma'', \gamma''' \right] = \frac{1}{\varphi^3} \left[ \dot{\alpha}, \dddot{\alpha} \right] - \left( 3 \frac{\dot{\varphi}^2}{\varphi^5} - \frac{\dddot{\varphi}}{\varphi^2} \right) \left[ \dot{\alpha}, \dddot{\alpha} \right] + 3 \frac{\dot{\varphi}^2}{\varphi^7} \left[ \dot{\alpha}, \dddot{\alpha} \right] - \frac{\dot{\varphi}}{\varphi^5} \left[ \dot{\alpha}, \dddot{\alpha} \right]
\]

\[
= \frac{1}{\varphi^3} \left[ \dot{\alpha}, \dddot{\alpha} \right] + \frac{\ddot{\varphi}}{\varphi^3} \left[ \dot{\alpha}, \dddot{\alpha} \right] - \frac{\dddot{\varphi}}{\varphi^3} \left[ \dot{\alpha}, \dddot{\alpha} \right].
\]
From (2.3) it follows that $[\dot{\alpha}, \ddot{\alpha}] = \varphi^3$ and $[\dot{\alpha}, \dddot{\alpha}] = 3 \varphi^2 \dot{\varphi}$. Using the latter identity we obtain expression (2.12) for the affine curvature of $\alpha$.

**Remark 2.2.4.** Proposition 2.2.3 gives the following expression for the affine curvature $k$ in terms of the Euclidean curvature $\kappa$:

$$k = \frac{9 \kappa^4 - 5 (\dot{\kappa})^2 + 3 \kappa \ddot{\kappa}}{9 \kappa^{8/3}},$$

where $\dot{\kappa}$ and $\ddot{\kappa}$ are the derivatives of the Euclidean curvature with respect to arc length. This identity is obtained by observing that, for a curve parameterized by Euclidean arc length, the function $\varphi$ is given by $\varphi = \kappa^{1/3}$. This follows from the Frenet-Serret identities (2.6) and the definition (2.3) of $\varphi$. Substituting this expression into (2.12) yields the identity for $k$ in terms of $\kappa$.

### 2.3 Space Curves and Helices

In Chapter 4 we consider the approximation of curves in space with circular helices. In this section we present the principal notions about differential geometry of space curves and review the fact that circular helices are regular curves in space with constant curvature, and non-zero constant torsion.

Generalized and circular helices in three-space have nice properties, that are easily derived by considering a special moving frame based on the Darboux vector. The equations of motion of this moving frame are similar to the well-known Frenet-Serret frame associated with regular space curves.

#### 2.3.1 The Darboux vector

Let $\alpha : I \to \mathbb{R}^3$ be a regular curve, parametrized by arc length, and let $\{T, N, B\}$ be its standard Frenet-Serret frame. Recall that this moving frame satisfies the equations

$$
\begin{align*}
T' &= \kappa N \\
N' &= -\kappa T - \tau B \\
B' &= \tau N
\end{align*}
$$

(2.13)
Here $\kappa$ and $\tau$ are the curvature and torsion of the curve $\alpha$, and they are the differential invariants of $\alpha$. Let $A$ be the skew-symmetric Frenet-Serret matrix

$$
A = \begin{pmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & -\tau \\
0 & \tau & 0
\end{pmatrix}.
$$

The vector $(\tau, 0, -\kappa)^T$, corresponding to $\tau T - \kappa B$, spans the kernel of $A$. Normalization gives the unit vector

$$V = \tau T - \kappa B,$$  

with $\beta = \sqrt{\kappa^2 + \tau^2}$, $\pi = \kappa / \beta$ and $\sigma = \tau / \beta$. Note that $\pi$ and $\sigma$ only depend on the ratio $\tau / \kappa$. Since $V$ is orthogonal to the normal $N$, we introduce the orthonormal frame $\{V, N, W\}$, with

$$W = V \times N = \pi T + \sigma B.$$  

A straightforward calculation shows that

$$T = \sigma V + \pi W \text{ and } B = -\pi V + \sigma W.$$  

The vector $\Omega = \beta V$ is the Darboux vector, so $\Omega = \tau T - \kappa B$, cf [65]. It satisfies the identities $T' = T \times \Omega$, $N' = N \times \Omega$ and $B' = B \times \Omega$. These identities are easily derived from the Frenet-Serret equations.

**Lemma 2.3.1.** The system $\{N, W, V\}$ is a positively oriented orthonormal moving frame, i.e., $N \times W = V$. The motion of this frame is determined by

$$
\begin{align*}
N' &= -\beta W \\
W' &= \beta N - \gamma V \\
V' &= \gamma W
\end{align*}
$$

where

$$\gamma = \frac{(\tau/\kappa)'}{1 + (\tau/\kappa)^2} = \frac{\pi'}{\pi}.$$
2.3. Space Curves and Helices

Proof. Since \( V \) and \( N \) are orthogonal unit vectors, and \( W = V \times N \), it follows that the system \( \{N, W, V\} \) is a positively oriented orthonormal frame.

The third identity of (2.17) is a straightforward consequence of the second Frenet-Serret identity in (2.13). Using \( \tau T' - \kappa B' = 0 \) we see that

\[
V' = \tau' T - \kappa' B
= \tau' (\tau V + \kappa W) - \kappa' (-\tau V + \tau W)
= (\tau' \tau + \kappa' \kappa) V + (\tau' \kappa - \kappa' \tau) W
= \frac{\kappa \tau' - \kappa' \tau}{\kappa^2 + \tau^2} W
= \gamma W.
\]

The fourth equality in the latter derivation follows from \( \kappa^2 + \tau^2 = 1 \), so \( \kappa \kappa' + \tau \tau' = 0 \). The expression for \( W' \) follows by differentiating both sides of the identity \( W = V \times N \):

\[
W' = V' \times N + V \times N'
= \gamma W \times N - \beta V \times W
= -\gamma V + \beta N.
\]

Finally, the equality \( \gamma = \frac{\tau'}{\kappa} \) follows from

\[
\tau' = \frac{d}{ds} \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} = \kappa \frac{\kappa \tau' - \kappa' \tau}{(\kappa^2 + \tau^2)^{3/2}} = \kappa \gamma.
\]

\( \square \)

2.3.2 Generalized helices

As a first illustration of the versatility of the frame \( \{N, W, V\} \) in our further study of helices we consider generalized helices, which are space curves with non-zero curvature for which the ratio of torsion and curvature is constant. We first derive an expression for the derivative of this ratio.

Lemma 2.3.2. Let \( \alpha : I \to \mathbb{R}^3 \) is a space curve parametrized by arc length, then the derivative of its torsion to curvature ratio is given by

\[
\left( \frac{\tau}{\kappa} \right)' = -\frac{\text{Det}(\alpha'', \alpha''', \alpha^{(4)})}{\kappa^5}.
\]
In particular, \( \alpha \) is a generalized helix iff 
\[
\det(\alpha'', \alpha''', \alpha^{(4)}) = 0.
\]

**Proof.** Since \( \alpha' = T \), \( \alpha'' = \kappa N \) and \( \alpha''' = \kappa' N + \kappa N' = \kappa' N - \beta \kappa W \), we see that 
\[
\alpha'' \times \alpha''' = -\beta \kappa V.
\]
Therefore, 
\[
\det(\alpha'', \alpha''', \alpha^{(4)}) = \langle \alpha'' \times \alpha''', \alpha^{(4)} \rangle = -\beta \kappa^2 \langle V, \alpha^{(4)} \rangle.
\]
Furthermore, \( \langle V, \alpha^{(4)} \rangle = 0 \), so 
\[
\langle V, \alpha^{(4)} \rangle = -\langle V', \alpha''' \rangle = -\langle \gamma W, \kappa' N - \beta \kappa W \rangle = \beta \kappa \gamma.
\]
Therefore, 
\[
\det(\alpha'', \alpha''', \alpha^{(4)}) = -\beta^2 \kappa^3 \gamma.
\]
Since \( \gamma = 0 \), the Frenet-Serret equations \((2.17)\) imply that 
\( V \) is constant for generalized helices. This can also be concluded directly from the observation that 
\[
\kappa = \frac{(\tau/\kappa)'}{\sqrt{1 + (\tau/\kappa)^2}} \text{ and } \tau = \frac{1}{\sqrt{1 + (\kappa/\tau)^2}}
\]
are constant for a generalized helix. \( \square \)

**Remark 2.3.3.** Since \( \gamma = 0 \), the Frenet-Serret equations \((2.17)\) imply that 
\( V \) is constant for generalized helices. This can also be concluded directly from the observation that 
\( \kappa = \frac{(\tau/\kappa)'}{\sqrt{1 + (\tau/\kappa)^2}} \text{ and } \tau = \frac{1}{\sqrt{1 + (\kappa/\tau)^2}} \) are constant for a generalized helix.

We show that a generalized helix \( c \) lies on a generalized cylinder. To this end we introduce the function \( \varphi \), which is the primitive function of \( \beta \) with \( \varphi(0) = 0 \), i.e., 
\[
\varphi(s) = \int_0^s \beta(u) \, du = \int_0^s \sqrt{\kappa(u)^2 + \tau(u)^2} \, du,
\]
and the functions \( C \) and \( S \), given by 
\[
C(s) = \int_0^s \cos \varphi(u) \, du \quad \text{and} \quad S(s) = \int_0^s \sin \varphi(u) \, du.
\]

**Lemma 2.3.4.** Let \( \alpha : I \to \mathbb{R}^3 \) be a generalized helix, parametrized by arc length. Then the normalized Darboux-vector \( V_0 \) is constant, and \( \alpha \) lies on a generalized cylinder with ruling lines parallel to the normalized Darboux vector \( V_0 \). More precisely, \( \alpha(s) = s \bar{\tau} V_0 + \gamma(s) \), where \( \gamma \) is the curve in the plane perpendicular to \( V_0 \) through \( \alpha(0) \), given by 
\[
\gamma(s) = \alpha(0) + \bar{\tau}(s) \left( C(s) W_0 + S(s) N_0 \right).
\]
Proof. Let \( \{N, W, V\} \) be the moving frame of \( \alpha \). Since the ratio of torsion and curvature is constant, the function \( \gamma \), introduced in Lemma 2.3.1, is zero. Therefore, \( V \) is a constant vector, say \( V = V_0 \), and the vectors \( W \) and \( N \) satisfy the set of differential equations

\[
\begin{align*}
N' &= -\beta W, \\
W' &= \beta N.
\end{align*}
\]  

(2.19)

Then the system (2.19) has solution

\[
\begin{align*}
W(s) &= \cos \varphi(s) W_0 + \sin \varphi(s) N_0 \\
N(s) &= -\sin \varphi(s) W_0 + \cos \varphi(s) N_0.
\end{align*}
\]  

(2.20)

Since \( \alpha'(s) = T(s) = \tau V_0 + \pi W(s) \), it follows that

\[\alpha(s) = s\tau V_0 + \gamma(s),\]

where \( \gamma \) is the curve in the plane through \( p_0 = \alpha(0) \), perpendicular to \( V_0 \), given by

\[
\gamma(s) = \alpha(0) + \pi \int_0^s W(u) \, du = \alpha(0) + \pi \left( \int_0^s W(u) \, du \right).
\]

Therefore, \( \alpha \) is a curve on the generalized cylinder with base curve \( \gamma \) and axis direction \( V_0 \).

Lemma 2.3.4 yields an expression for the generalized helix in terms of the frame \( \{T_0, V_0, T_0 \times V_0\} \), which will be useful in later applications. Note, however, that this frame is not necessarily orthogonal.

Corollary 2.3.5. Let \( \alpha : I \to \mathbb{R}^3 \) be a generalized helix, parametrized by arc length. Then \( \alpha(s) = \gamma(s) + s\tau V_0 \), where \( \gamma \) is a curve given by

\[
\gamma(s) = \alpha(0) + C(s) (T_0 - \tau V_0) + S(s) T_0 \times V_0.
\]

The curve \( \gamma \) lies in the plane through \( \alpha(0) \) perpendicular to \( V_0 \). Furthermore, the unit tangent vector of the generalized helix is given by

\[
T(s) = T_0 + (\cos \varphi(s) - 1) (T_0 - \tau V_0) + \sin \varphi(s) T_0 \times V_0.
\]
Proof. Since \( \kappa W_0 = T_0 - \tau V_0 \) and \( \kappa N_0 = T_0 \times V_0 \), the expression for \( \alpha(s) \) follows from the identity in Lemma 2.3.4. Since \( \langle T_0, V_0 \rangle = \tau \), we see that \( \langle \gamma(s) - \alpha(0), V_0 \rangle = 0 \), so the curve \( \gamma \) lies in the plane through \( \alpha(0) \), perpendicular to \( V_0 \). The expression for the tangent vector follows from \( C'(s) = \cos \varphi(s) \) and \( S'(s) = \sin \varphi(s) \).

Remark 2.3.6. Note that frame \( \{V_0, T_0 - \tau V_0, T_0 \times V_0\} \) is orthogonal, though not necessarily orthonormal. In fact, \( T_0 - \tau V_0 = \kappa W_0 \) and \( T_0 \times V_0 = \kappa N_0 \).

Circular helices. Consider a generalized helix with constant curvature \( \kappa \) and constant torsion \( \tau \). This special case of generalized helix is called a circular helix, since it lies on the circular cylinder. In Chapter 4 we consider approximation of space curves with circular helices hence we present them in greater detail in the same chapter.