2 Representations of Semi-simple Lie Algebras and Differential Modules

The theme of this and the next chapters is to “solve” an absolutely irreducible differential module explicitly in terms of modules of lower dimension and finite extensions of the differential field $K$. In Chapter 1, criteria are given under which this is possible. In the present chapter we present some necessary background on the relation between representations of semi-simple Lie algebras and differential modules. This will be used in Chapter 2 to actually “solve” various types of modules. The results extend the classical work of G. Fano [Fa]. Chapters 2 and 3 are accepted for publication as a joint paper with M. van der Put in the Pure and Applied Mathematics Quarterly in John T. Tate special issue, see [N-vdP].

2.1 Introduction

L. Fuchs posed the problem whether the $n$ independent solutions of a scalar linear differential equation of order $n$ over $K = \mathbb{C}(z)$, under the assumption that these solutions satisfy a non trivial homogeneous equation over $\mathbb{C}$, could be expressed in terms of solutions of scalar linear differential equations of lower order. G. Fano wrote an extensive paper [Fa] on this theme. His tools were an early form of the differential Galois group and an extensive knowledge of low dimensional projective varieties.

In the work of M. F. Singer [Si-3] and in Chapter 1 of this thesis, a powerful combination of Tannakian methods and representations of semi-simple Lie algebras yields a complete solution to Fuchs’ problem. We note that a scalar differential equation is essential for Fuchs’ problem. It is not clear whether this problem makes sense for a linear differential equation in the standard matrix form $Y' = AY$ (or in module form).

The theme of this and the next chapters is to explicitly “solve” a differential equation of order $n$ in terms of differential equations of lower order, whenever possible. This makes sense in terms of differential modules over $K$. It means that one tries to obtain a given differential module $M$ of dimension $n$, by constructions of linear algebra and, possibly, algebraic extensions of $K$, from differential modules of smaller dimension. In the sequel, $K$ will be a finite extension of $\mathbb{C}(z)$ (unless otherwise stated). For actual computer computations, one has to replace $\mathbb{C}$ be a “computable field”, like the algebraic closure of $\mathbb{Q}$. We write $V = V(M)$ for the
solution space of $M$ and $Gal(M) \subset GL(V)$ for the differential Galois group of $M$. Further $(\text{gal}(M), V)$ will denote the Lie algebra of $Gal(M)$ acting on the solution space $V$.

If $M$ admits, for instance, a non trivial submodule $N$, then $M$ is “solved” by $N$, $M/N$ and an element in $Ext^1(M/N, N)$ (corresponding to some inhomogeneous equations $y' = f$ over the Picard-Vessiot field of $N \oplus M/N$). This is the reason that we will only consider irreducible modules $M$.

Fix an algebraic closure $K$ of $K$. A module $M$ over $K$ is called absolutely irreducible if $M \otimes K$ is irreducible. Since $Gal(M) = Gal(M)^o$, this condition is equivalent to the statement that $V$ is an irreducible $Gal(M)^o$-module.

If an irreducible module $M$ over $K$ becomes reducible after tensorization with $K$, then this is a case where $M$ can be expressed, after a finite extension of $K$, into modules of smaller dimension (see [C-W]). We will investigate this phenomenon in a future work and concentrate here (but not exclusively) on absolutely irreducible modules.

Some notation: $\mu_n$ is the subgroup of order $n$ of $C^*$, sometimes identified with scalar multiples of the identity matrix or map; $1$ is the trivial module of dimension one and for a module $M$ of dimension $m$, we write $\text{det} M = \Lambda^m M$; further $M^*$ or $M^{-1}$ denotes the dual of $M$. The condition $\text{det} M = 1$ is equivalent to: the matrix of $\partial$ with respect to a suitable basis of $M$ has trace 0.

For an absolutely irreducible module $M$ with $\text{det} M = 1$ the group $Gal(M)^o$ is semi-simple and so is $\text{gal}(M)$. Moreover $V(M)$ is an irreducible representation of $\text{gal}(M)$. According to [Si-3] and [Ng] (see Theorem 1.2.21), $M$ cannot be solved in terms of modules of lower dimensions and finite field extensions of $K$ if and only if $\text{gal}(M)$ is simple and its representation $V(M)$ has smallest dimension among its non trivial representations. Let a scalar equation $L$ of order $n$ induce a representation of a simple Lie algebra of smallest dimension, then it is still possible that the $n$ independent solutions of $L$ satisfy a non trivial homogeneous relation $F$ over $C$ (in contrast to L. Fuchs’ opinion). Table 2.1.1 below is a combination of Table 1.3.1 and Remark 1.3.5 which gives a complete answer.

We note that: $\mathfrak{so}_3 \cong \mathfrak{sl}_2$, $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$, $\mathfrak{so}_5 \cong \mathfrak{sp}_4$ and $\mathfrak{so}_6 \cong \mathfrak{sl}_4$.

The two cases where $N$ can be solved by modules of smaller dimension and finite field extensions of $K$ are:

(a) $\text{gal}(N) = \mathfrak{g}_1 \times \mathfrak{g}_2$ and $V = V_1 \otimes V_2$, where $V_i$ is an irreducible representation of $\mathfrak{g}_i$ for $i = 1, 2$.

(b) $\mathfrak{g} := \text{gal}(N)$ is simple and the representation $V(N)$ does not have smallest dimension.
In case (a) we produce an algorithm that expresses $N$ (after possibly a finite extension of $K$) as a tensor product $N_1 \otimes N_2$ with $\dim N_i > 1$. In case (b) we produce a differential module $M$ corresponding to a representation of $\mathfrak{g}$ of smallest dimension and a construction of linear algebra by which $N$ is obtained from $M$. In the case that $\text{Gal}(N)$ is not connected a finite (computable) extension of the field $K$ might be needed.

Besides using the well known Tannakian methods (often called constructions of linear algebra) a new construction (Theorem 2.2.5) is introduced which can be explained as follows. An irreducible differential module $N$ is called \textit{standard} if $\text{Gal}(N)$ is connected and the $\text{gal}(N)$-module $V(N)$ is faithful of minimal dimension. Further an irreducible differential module $M$ is called \textit{adjoint} if $\text{Gal}(M)$ is connected and the $\text{Gal}(M)$-module $V(M)$ is the adjoint representation. An adjoint differential module $M$ (for a given semi-simple Lie algebra) is obtained as an irreducible submodule of $\text{Hom}(N, N)$ where $N$ is a standard differential module (for the same semi-simple Lie algebra). The new construction uses Lie algebra tools to go in the other direction, i.e., to obtain $N$ from $M$. Together with the Tannakian approach the new construction provides a complete solution for both cases (a) and (b). For special cases there are shortcuts not using adjoint differential modules.

For differential modules of small dimension we rediscover and extend Fano’s work. The finite group $\text{Gal}(M)/\text{Gal}(M)^o$ introduces a technical complication in the method and is responsible for the finite extension of $K$ that are sometimes needed. Our extensive use of representations of semi-simple groups and semi-simple Lie algebras is a link between this paper and several chapters of [Kat].
2.2 Representations of semi-simple Lie algebras

2.2.1 General remarks on computations

(1)  The fields $\mathbb{C}(z)$, $\mathbb{Q}(z)$ and their finite extensions $K$ are $C_1$-fields. In particular a quadratic homogeneous form over $K$ in at least three variables has a non trivial zero and there are algorithms (if $K$ is a “computable field”) producing such a zero, needed in some of the proposed computations.

For finite extensions of the second field there are efficient algorithms, due to M. van Hoeij, et al., implemented in Maple, for finding submodules of a given “input module” $P$ (or, equivalently, for factoring differential operators over $K$). In particular, one can decide whether $P$ is irreducible. This algorithm is less efficient for the verification that $P$ is absolutely irreducible. In the sequel we will suppose that the “input module” $P$ is absolutely irreducible. If $P$ happens to be irreducible but not absolutely irreducible, then the algorithms that we propose will either still work or demonstrate that $P$ is not absolutely irreducible.

(2)  Suppose that the “input module” $P$ is (absolutely) irreducible. Then any module $N$ obtained by a construction of linear algebra from $P$ is semi-simple (i.e., is a direct sum of irreducible submodules). A computation of $\ker(\partial, \text{End}(N))$ (i.e., the rational solutions of $\text{End}(N)$) seems to be an efficient way to produce the direct summands of $N$. The characteristic polynomial of any $K$-linear $f : N \rightarrow N$ with $\partial f = 0$ has coefficients in $\mathbb{C}$. Indeed, it coincides with the characteristic polynomial of the induced map $V(f) : V(N) \rightarrow V(N)$. If $f$ is not a multiple of the identity on $N$, then any zero $\lambda \in \mathbb{C}$ of its characteristic polynomial yields a proper submodule $\ker(f - \lambda, N)$ of $N$.

If $N$ happens to be irreducible but is known to have direct summands after a finite extension $K^+$ of the base field $K$, then one can explicitly compute $K^+$. Consider, as an example (see also the example in Remarks 2.2.3 and the end of Section 3.2), the case where it is a priori known that $N = A_1 \oplus A_2$, where $A_1, A_2$ are non isomorphic irreducible differential modules over $K$. Then $\ker(\partial, \text{End}(\mathcal{N})) = \mathbb{C}p_1 + \mathbb{C}p_2$, where $p_1, p_2$ are the projections onto the two factors $A_1, A_2$ and $p_1 + p_2 = 1$. The group $\text{Gal}(K/K)$ acts as a finite group $H$ (faithfully) on this 2-dimensional vector space and the line $\mathbb{C}(p_1 + p_2)$ is invariant. Therefore, there is another invariant line and $H$ is a cyclic group of order $d > 1$. Thus $\text{End}(N)$ contains precisely two 1-dimensional submodules, a trivial one and a non trivial one $L$ with $L \otimes d = 1$. Then $L = Ke$ with $\partial e = \frac{d}{dg}$ for some $g \in K^*$ and the field $K^+$ equals $K(\sqrt[d]{g})$. 
(3)-(a) Suppose that two irreducible modules $M_1, M_2$ of the same dimension are given. An efficient way to investigate whether $M_1$ and $M_2$ are isomorphic is to compute $\ker(\partial, \text{Hom}(M_1, M_2))$. This space is non zero if and only if $M_1 \cong M_2$.

(b) Suppose that $M_2 \cong M_1 \otimes L$ holds for some unknown 1-dimensional module $L$. In order to find $L$ one considers the module $E := \text{Hom}(M_1, M_2) = \text{Hom}(M_1, M_1) \otimes L$ and observes that $L$ is a 1-dimensional direct summand of $E$. Using the method of (2) above one can produce $L$.

(c) Suppose that $M_1, M_2$ are absolutely irreducible and that $\overline{M}_1 \cong \overline{M}_2$. Then $\ker(\partial, \overline{K} \otimes \text{Hom}(M_1, M_2))$ is a 1-dimensional vector space $C\xi$. Then $\text{Gal}(\overline{K}/K)$ acts as a cyclic group of order $d$ on $C\xi$. Thus $\overline{K}\xi \subset \overline{K} \otimes \text{Hom}(M_1, M_2))$ is invariant under $\text{Gal}(\overline{K}/K)$ and induces a 1-dimensional submodule $L$ of $\text{Hom}(M_1, M_2)$ with the properties $L^{\otimes d} = 1$ and $M_2 \cong M_1 \otimes L$. Thus we can apply method (b) to find $L$. As in (2), $L$ defines a cyclic extension $K^+ \supset K$ of degree $d$ and $K^+ \otimes_k M_1 \cong K^+ \otimes_k M_2$.

(4) Properties of a differential module $M$ over $K$ are often translated into properties of the (faithful) representation $(\text{Gal}(M), V(M))$ (and visa versa). By inverse Galois theory, any faithful representation of any linear algebraic group over $C$ occurs for a differential field $K$ which is a finite extension of $C(z)$.

2.2.2 A table of irreducible representations

We present here a list of irreducible representations $V$, $\dim V = d$, of semi-simple Lie algebras, including the decomposition of $\Lambda^2 V$ and $\text{sym}^2 V$. We adopt here and in the sequel of the paper the efficient notation of the online program [LiE] for irreducible representations. This is the following. After a choice of simple roots $\alpha_1, \ldots, \alpha_d$, the Dynkin diagram (with standard numbering of the vertices by the roots) and the fundamental weights $\omega_1, \ldots, \omega_d$ are well defined. The irreducible representation with weight $n_1 \omega_1 + \cdots + n_d \omega_d$ is denoted by $[n_1, \ldots, n_d]$. In particular, $[0, \ldots, 0]$ is the trivial representation of dimension 1.

For the $\mathfrak{sl}_d$ with $n > 2$ we have omitted duals of representations. Further we have left out symmetric cases. The decompositions of the second symmetric power and the second exterior power are useful to distinguish the various cases. We are here especially interested in those representations which can be expressed in terms of representations of lower dimension. In dimensions $7 - 11$, one finds for the new items of this sort (here we omit the case $\mathfrak{sl}_2$ which is fully treated in Section 3.1 and again we omit duals and symmetric situations) the list:

- $\mathfrak{sl}_3$ with $[1,1]$ (dim 8), with $[3,0]$ (dim 10);
Table 2.2.1  Irreducible representations of dimension $d \leq 6$.

- $\mathfrak{sl}_2$ with $[2,0,0]$ (dim 10);
- $\mathfrak{sl}_3$ with $[0,1,0,0]$ (dim 10);
- $\mathfrak{so}_7$ with $[0,0,1]$ (dim 8);
- $\mathfrak{sp}_4$ with $[2,0]$ (dim 10);
- $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $[1] \otimes [3]$ (dim 8), with $[2] \otimes [2]$ (dim 9), with $[1] \otimes [4]$ (dim 10);
- $\mathfrak{sl}_2 \times \mathfrak{sl}_3$ with $[2] \otimes [1,0]$ (dim 9);
- $\mathfrak{sl}_2 \times \mathfrak{sl}_4$ with $[1] \otimes [1,0,0]$ (dim 8);
- $\mathfrak{sl}_2 \times \mathfrak{sp}_4$ with $[1] \otimes [1,0]$ (dim 8), with $[1] \otimes [0,1]$ (dim 10);
- $\mathfrak{sl}_3 \times \mathfrak{sl}_3$ with $[1,0] \otimes [1,0]$ (dim 9);
- $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $[1] \otimes [1] \otimes [1]$ (dim 8).

<table>
<thead>
<tr>
<th>$d$</th>
<th>Lie algebras</th>
<th>Representations</th>
<th>$\Lambda^2$</th>
<th>$\text{sym}^2$</th>
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<tbody>
<tr>
<td>2</td>
<td>$\mathfrak{sl}_2$</td>
<td>[1]</td>
<td>[0]</td>
<td>[2]</td>
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<tr>
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<td>[2]</td>
<td>[2]</td>
<td>[4, [0]]</td>
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<td>[0,1]</td>
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</tr>
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<td>[3]</td>
<td>[4, [0]]</td>
<td>6, [2]</td>
</tr>
<tr>
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<td>$\mathfrak{sl}_4$</td>
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<td>[0,1,0]</td>
<td>2,0</td>
</tr>
<tr>
<td>4</td>
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<td>[0,1,0,0]</td>
<td>2,0,0</td>
</tr>
<tr>
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</tr>
<tr>
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<td>$\mathfrak{sl}_2$</td>
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<td>[2,1]</td>
<td>4,0, [2]</td>
</tr>
<tr>
<td>6</td>
<td>$\mathfrak{sl}_4$</td>
<td>[0,1,0]</td>
<td>[1,0,1]</td>
<td>0,2,0, [0,0,0]</td>
</tr>
<tr>
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<td>$\mathfrak{sl}_6$</td>
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<td>[0,1,0,0,0]</td>
<td>2,0,0,0,0</td>
</tr>
<tr>
<td>6</td>
<td>$\mathfrak{sp}_6$</td>
<td>[1,0,0]</td>
<td>[0,1,0,0,0]</td>
<td>2,0,0</td>
</tr>
<tr>
<td>6</td>
<td>$\mathfrak{sl}_2 \times \mathfrak{sl}_2$</td>
<td>[1] $\otimes$ [2]</td>
<td>[0] $\otimes$ [0], [0] $\otimes$ [4], [2] $\otimes$ [2]</td>
<td>0 $\otimes$ [2], [2] $\otimes$ [0], [2] $\otimes$ [4]</td>
</tr>
<tr>
<td>6</td>
<td>$\mathfrak{sl}_2 \times \mathfrak{sl}_3$</td>
<td>[1] $\otimes$ [1,0]</td>
<td>[0] $\otimes$ [2,0], [2] $\otimes$ [0,1]</td>
<td>0 $\otimes$ [0,1], [2] $\otimes$ [2,0]</td>
</tr>
</tbody>
</table>


2.2.3 Comparison of the representations of \( G \) and \( \mathfrak{g} \)

For a connected semi-simple group \( G \) with Lie algebra \( \mathfrak{g} \) one considers the categories \( \text{Repr}_G \) of the representations of \( G \) on finite dimensional vector spaces over \( \mathbb{C} \) and \( \text{Repr}_\mathfrak{g} \), the category of the representations of \( \mathfrak{g} \) on finite dimensional vector spaces over \( \mathbb{C} \). Any representation of \( G \) on a vector space induces a representation of \( \mathfrak{g} \) on the same vector space. This defines a functor \( T : \text{Repr}_G \to \text{Repr}_\mathfrak{g} \), which is fully faithful, i.e., \( \text{Hom}_G(V_1, V_2) \to \text{Hom}_\mathfrak{g}(V_1, V_2) \) is a bijection. Further, \( G \) is simply connected if and only if \( T \) is an equivalence.

For a representation \( W \in \text{Repr}_G \), we write \( \{\{W\}\} \) for the Tannakian subcategory generated by \( W \) (i.e., the objects of this subcategory are obtained from \( W \) by constructions of linear algebra). The action of \( G \) is faithful if and only if \( \{\{W\}\} = \text{Repr}_G \). Similarly, for an object \( W \in \text{Repr}_\mathfrak{g} \) one writes \( \{\{W\}\} \) for the smallest Tannakian subcategory generated by \( W \).

Suppose that \( \mathfrak{g} \) acts faithfully on \( W \), then in general \( \{\{W\}\} \neq \text{Repr}_\mathfrak{g} \). Indeed, let \( G^+ \) be the simply connected group with Lie algebra \( \mathfrak{g} \). Then \( W \) has a unique structure as \( G^+ \)-module compatible with its structure as \( \mathfrak{g} \)-module. The kernel \( Z' \) of the action of \( G^+ \) on \( W \) is a finite group. Put \( H := G^+/Z' \). Then \( W \) is a faithful \( H \)-module and generates \( \text{Repr}_H \). Thus the subcategory \( \{\{W\}\} \) of \( \text{Repr}_\mathfrak{g} \) is the image under \( T \) of \( \text{Repr}_H \).

**Example 2.2.1.** \( G = \text{SL}_3 \) is simply connected and \( \mathfrak{g} = \mathfrak{sl}_3 \). There is only one other connected group with Lie algebra \( \mathfrak{sl}_3 \), namely \( \text{PSL}_3 = \text{SL}_3/\mu_3 \). Let \( V \) be the standard representation of \( \text{SL}_3 \) with \( T \)-image \((\mathfrak{sl}_3, [1,0])\). Then \( \text{sym}^3 V \) is a faithful representation for \( \text{PSL}_3 \) and its image under \( T \) is \( W := (\mathfrak{sl}_3, [3,0]) \). Then \( \{\{V\}\} = \text{Repr}_{\mathfrak{sl}_3} \) and \( \{\{W\}\} \) is the full subcategory of \( \text{Repr}_{\mathfrak{sl}_3} \) for which the irreducible objects are the \([a,b]\) with \( a \equiv b \mod 3 \).

**Consequences for differential modules:** Let the “input module” \( P \) be an absolutely irreducible differential module with \( \det P = 1 \) which induces \( W := (\mathfrak{sl}_3, [a,b]) \) with, say, \( [a,b] \neq [1,0], [0,1] \). Now, we do not assume that \( \text{Gal}(P) \) is connected. There exists, as we know, a differential module \( M \) of dimension 3 inducing \((\mathfrak{sl}_3, [1,0])\) such that \( P \) is obtained from \( M \) by constructions of linear algebra and possibly a finite field extension of \( K \).

If \( a \equiv b \mod 3 \), then \([1,0]\) is obtained from \( W \) by a construction \( \text{cst}_1 \) of linear algebra and \([a,b]\) is (of course) obtained by a construction \( \text{cst}_2 \) from \([1,0]\). Let \( M \) be obtained from \( P \) by construction \( \text{cst}_1 \). Then \( \text{cst}_2 \) applied to \( M \) yields a module \( \tilde{P} \) which is isomorphic to \( P \) over the algebraic closure of \( K \). Case (3)(c) of Subsection 2.2.1 solves this.

If \( a \equiv b \mod 3 \), then one obtains by a construction of linear algebra a module \( Q \) which induces \((\mathfrak{sl}_3, [1,1])\). The step from \( Q \) to a module which induces \((\mathfrak{sl}_3, [1,0])\) cannot be done by constructions of linear algebra. This problem is an example for the main theme of the remainder of this section.
In general, the problem has its origin in the possibilities for the connected groups with a given (semi-) simple Lie algebra \( \mathfrak{g} \). There is a simply connected group \( G \) with Lie algebra \( \mathfrak{g} \). Its center \( Z \) is a finite group. Any connected group with Lie algebra \( \mathfrak{g} \) has the form \( G/Z' \) where \( Z' \) is a subgroup of \( Z \). The list of the groups \( Z \) that occur is, see [Hu-2, page 231]:

- \( \mathbb{Z}/(n+1)\mathbb{Z} \) for \( A_n \);
- \( \mathbb{Z}/2\mathbb{Z} \) for \( B_\ell, C_\ell, E_7 \);
- \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) for \( D_\ell \) with \( \ell \) even;
- \( \mathbb{Z}/4\mathbb{Z} \) for \( D_\ell \) with \( \ell \) odd;
- \( \mathbb{Z}/3\mathbb{Z} \) for \( E_6 \);
- 0 for \( E_8, F_4, G_2 \).

The following proposition, closely related to [Kat, Corollary 2.2.2.1], is a nonconstructive solution to our problem.

**Proposition 2.2.2.** Let \( G^+ \rightarrow G \) be a surjective morphism of connected linear algebraic groups over \( \mathbb{C} \) having a finite kernel \( Z \). Let \( M \) be a differential module over \( K \) with \( \text{Gal}(M) = G \). Suppose that \( K \) is a \( C_1 \)-field. Then there exists a differential module \( N \) over \( K \) with \( \text{Gal}(N) = G^+ \), such that the faithful representation \( (\text{Gal}(N), V(N)) \) has minimal dimension and such that \( M \in \{ N \} \).

**Proof.** The Picard-Vessiot ring \( \text{PVR} \) of \( M \) is a \( G \)-torsor over \( K \). Since \( K \) is a \( C_1 \)-field and \( G \) is connected, this torsor is trivial and thus \( \text{PVR} = K \otimes_{\mathbb{C}} \mathbb{C}[G] \), where \( \mathbb{C}[G] \) is the coordinate ring of \( G \). The \( G \)-action on \( \text{PVR} \) is induced by the \( G \)-action on \( \mathbb{C}[G] \). The differentiation, denoted by \( \partial \), commutes with the \( G \)-action, but is not explicit. Now \( \mathbb{C}[G] = \mathbb{C}[G^+]^2 \) and this yields an embedding \( \text{PVR} \subset R := K \otimes_{\mathbb{C}} \mathbb{C}[G^+] \). The differentiation \( \partial \) on \( \text{PVR} \) extends in a unique way to \( R \) since \( \text{PVR} \subset R \) is a finite étale extension. The extended differentiation commutes with the action of \( G^+ \). Further, \( R \) has only trivial differential ideals since \( R \) is finite over \( \text{PVR} \) and \( \text{PVR} \) has only trivial differential ideals. Let \( W \) be a faithful representation of \( G^+ \) of minimal dimension \( d \). Then one writes \( \mathbb{C}[G^+] = \mathbb{C}\{X_{i,j}\}_{i,j=1,...,d; \frac{1}{D}}/J \), where \( D = \det(X_{i,j}) \) and \( J \) is the ideal defining \( G^+ \) as subgroup of \( \text{GL}(W) \). Write \( x_{i,j} \) for the image of \( X_{i,j} \) in \( \mathbb{C}[G^+] \). Define the matrix \( A \), with entries in \( R \), by \( (\partial x_{i,j}) = A(x_{i,j}) \). Then \( A \) is invariant under the action of \( G^+ \) and therefore its entries are in \( K \). It now follows that \( R \) is the Picard-Vessiot ring for the differential equation \( Y' = AY \). This equation defines the required differential module \( N \) over \( K \). \( \square \)
2.2 Representations of semi-simple Lie algebras

**Remarks 2.2.3 (Non connected linear algebraic groups case).** Let \( G \) be a linear algebraic group such that \( G' \) is semi-simple and \( G \neq G' \). Let \( \mathfrak{g} \) denote the Lie algebra of \( G' \). As explained above, the functor \( T : \text{Repr}_{G'} \to \text{Repr}_{\mathfrak{g}} \) induces an equivalence of the first category with a well described full subcategory of the second one. The forgetful functor \( F : \text{Repr}_{G} \to \text{Repr}_{G'} \) is not fully faithful. Indeed, one can easily construct an irreducible \( G \)-module \( W \) (i.e., a finite dimensional representation of \( G \)) which is, as \( G' \)-module, the direct sum of several copies of an irreducible \( G' \)-module.

A more delicate situation occurs when \( \text{Out}(G') := \text{Aut}(G')/\text{Inner}(G') \) is not trivial. The group \( \text{Out}(G') \) permutes the irreducible representations of \( G' \). The action by conjugation of \( G \) on \( G' \) induces a homomorphism \( G/G' \to \text{Out}(G') \). If this homomorphism is not trivial, then one can construct an irreducible \( G \)-module \( W \) such that \( W \), seen as a \( G' \)-module, is a direct sum of distinct irreducible \( G' \)-modules forming a single orbit under the action of \( G/G' \).

We recall that for a connected simple \( H \) the group \( \text{Out} \) is equal to the automorphism group of the Dynkin diagram of its Lie algebra \( \mathfrak{h} \). According to [Ja, Theorem 4, page 281], one has:

- \( \text{Out} = S_3 \) for \( \mathfrak{so}_8 \);
- \( \text{Out} = \mathbb{Z}/2\mathbb{Z} \) for \( \mathfrak{sl}_n \) \((n > 2)\), for \( \mathfrak{so}_{2n} \) \((n \geq 3, n \neq 4)\) and for \( \varepsilon_6 \).

For the other simple Lie algebras \( \text{Out} \) is trivial. From this list one deduces \( \text{Out} \) for any semi-simple Lie algebra, e.g., \( \text{Out}(\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2) = S_3 \).

**Example 2.2.4.** \( \text{Out}(\mathfrak{sl}_4) \) is generated by the element \( A \mapsto (A')^{-1} \). This non trivial element changes a representation \( [a,b,c] \) of \( \mathfrak{sl}_4 \) (or of \( \mathfrak{sl}_4 \)) into its dual \( [c,b,a] \). Choose a group \( G \) with \( G' = \mathfrak{sl}_4 \), \( [G : G''] = 2 \) and \( G/G' \to \text{Out}(G') \) is bijective. Then there is an irreducible \( G \)-module \( W \) which induces the \( \mathfrak{sl}_4 \)-module \( [1,0,0] \oplus [0,0,1] \). Thus \( W \) is reducible as \( G' \)-module.

**Consequence for differential modules:** Let \( M \) be an absolutely irreducible differential module with \( \det M = 1 \) such that the \( \text{gat}(M) \)-module \( V(M) \) has a non trivial direct sum decomposition. Then, in general, a finite field extension of \( K \) is needed to obtain a corresponding direct sum decomposition of \( M \).

### 2.2.4 Standard and adjoint differential modules

Let \( G \) be simply connected semi-simple linear algebraic group over \( \mathbb{C} \) with Lie algebra \( \mathfrak{g} \). One writes \( \mathfrak{g} = G_1 \times \cdots \times G_s \) and \( \mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_s \) for the decompositions into simple objects. The **standard representation** of \( G \) is the direct sum \( V = \bigoplus_{i=1}^s V_i \), where each \( V_i \) is the faithful representation of \( G_i \) of smallest dimension. For the groups \( \mathfrak{sl}_n, n > 2 \), both the standard representation and its dual have smallest dimension. This ambiguity in the above definition is of no importance for the sequel.
The $G$-module $\text{End}(V) = V^* \otimes V$ has $\mathfrak{g}$ as irreducible submodule. The action of $G$ on $\mathfrak{g}$ is the adjoint representation. The kernel of this action is the finite center $Z$ of $G$ and $G/Z$ is the adjoint group.

A differential module $M$ over $K$ is called standard (adjoint resp.) for $G$ if $\text{Gal}(M)$ is connected, $\det M = 1$ and $(\text{Gal}(M), V(M))$ is isomorphic to the standard (adjoint resp.) representation of $G$. For any standard differential module $M$, the module $\text{End}(M)$ contains a unique direct summand which is an adjoint differential module.

Let $G$ be a simply connected semi-simple group with Lie algebra $\mathfrak{g}$ and let $V$ be the standard representation of $G$. An explicit standard module for $G$ is the following. Write $M = K \otimes_C V$. Then $\mathfrak{g}$ is identified, as before, with a subspace of $\text{End}(V)$ and $g(K) = K \otimes \mathfrak{g}$ is identified with the $K$-vector space $K \otimes \mathfrak{g} \subset \text{End}_K(M)$. Define the derivation $\partial_0$ on $M$ by $\partial_0 h$ is zero on $V$. For any element $S$ of $g(K)$ one defines the derivation $\partial_S := \partial_0 + S$ on $M$. According to [vdP-S Prop. 1.31], the differential Galois group of $(M, \partial_S)$ is contained in $G$. We call $M = (M, \partial_S)$ an explicit standard module for $G$ if the differential Galois group is equal to $G$. According to [vdP-S Sect. 1.7], the differential Galois group is equal to $G$ for sufficiently general $S$. On the other hand, under the assumption that $K$ is a $C_1$-field, every standard differential module for $G$ has an explicit representation $(M, \partial_S)$ by [vdP-S Corollary 1.32].

For any explicit standard module $(M, \partial_S)$, one considers the direct summand $N := K \otimes_C \mathfrak{g}$ of $\text{End}_K(M)$. Define the derivation $\partial_0$ on $N$ by $\partial_0$ is zero on $\mathfrak{g}$. One easily verifies that $(M, \partial_S)$ induces on $N$ the derivation $A \mapsto \partial_0(A) + [A, S]$. In this way $(M, \partial_S)$ induces an adjoint differential module.

**Theorem 2.2.5.** Let $N$ be an adjoint differential module for $G$. The $\mathbb{C}$-Lie algebra structure of $\mathfrak{g} = V(N)$ induces a $K$-Lie algebra structure $[,]$ on $N$ satisfying $\partial [a, b] = [\partial a, b] + [a, \partial b]$ for all $a, b \in N$. This structure is unique up to multiplication by an element in $\mathbb{C}^\ast$.

The assumption that $K$ is a $C_1$-field implies that there exists an isomorphism of $K$-Lie algebras $\phi : N \rightarrow K \otimes_C \mathfrak{g}$. After choosing $\phi$, there exists a unique $S \in g(K)$ such that $N$ is isomorphic to the adjoint module induced by the explicit standard module $(M, \partial_S)$.

**Proof.** By definition, $(\text{Gal}(N), V(N))$ is the adjoint action of $G$ (or $G/Z$) on $\mathfrak{g}$. The morphism of $G$-modules $\Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g}$, given by $\Lambda^2 \mathfrak{g} \rightarrow [A, B]$, is $G$-equivariant and therefore is induced by a morphism of differential modules $F : \Lambda^2 N \rightarrow N$. The map $F$ is a non zero element of $\ker(\partial, \text{Hom}(\Lambda^2 N, N))$, unique up to multiplication by an element in $\mathbb{C}^\ast$. Define for $a, b \in N$ the expression $[a, b] = F(a \wedge b)$. The statement $\partial F = 0$ is equivalent to $\partial [a, b] = [\partial a, b] + [a, \partial b]$ for all $a, b \in N$.

Let $PVR(N)$ denote the Picard-Vessiot ring for $N$. The canonical isomorphism
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\( PVR(N) \otimes_{\mathbb{C}} V(N) \rightarrow PVR(N) \otimes_K N \) is, by construction, compatible with the Lie algebra structures on \( N \) and \( V(N) = g \). Since \( Gal(N) \) is connected and \( K \) is a \( C_1 \)-field, the \( Gal(N) \)-torsor \( Spec(PVR(N)) \) over \( K \) is trivial (see [vdPS]). This means that there is a \( K \)-algebra homomorphism \( e : PVR(N) \rightarrow K \). One applies \( e \) to both sides of the above canonical isomorphism and finds an isomorphism of \( K \)-Lie algebras \( \phi : K \otimes_{\mathbb{C}} g \rightarrow N \).

After choosing \( \phi \) and identifying \( N \) with \( K \otimes_{\mathbb{C}} g \), one can define the derivation \( \partial_0 \) on \( N \) by \( \partial_0 \) is zero on \( g \). Clearly, \( \partial_0[a,b] = [\partial_0a,b] + [a,\partial_0b] \) for all \( a, b \in N \). Therefore \( \partial - \partial_0 \) is a \( K \)-linear derivation of the semi-simple Lie algebra \( N \) and there is a unique \( S \in N = g(K) \) such that \( \partial - \partial_0 = [ , S] \) (see [Ja, Theorem 9, page 80]). Thus we find that \( \partial \) on \( N \) is induced by \( (M, \partial_S) \).

\[ \text{Comments 2.2.6.} \] The computation of the Lie algebra structure \( F \) on \( N \) amounts to computing a rational solution (i.e., with coordinates in \( K \)) of the differential module \( \text{Hom}(\Lambda^2 N, N) \). The computation of \( S \in g(K) \) is an easy exercise on Lie algebras. The computation of an isomorphism \( \phi \) seems more complicated. It is essentially equivalent to the computation of a Cartan subalgebra of \( N \) which is “defined” over \( \mathbb{C} \). As an example we consider here the case \( g = sl_2 \).

Since \( N \cong K \otimes_{\mathbb{C}} sl_2 \), we know that there exists an element \( h \in N \) such that the eigenvalues of \( ad(h) \), acting upon \( N \), are 0, \( \pm 2 \). Such an element \( h \) can be found by solving some quadratic equation over \( K \). Any other candidate \( h' \) is conjugated to \( h \) by an automorphism of the \( K \)-Lie algebra \( N \). We choose an element \( e_1 \) with \( [h, e_1] = 2e_1 \) and an element \( e_2 \) with \( [h, e_2] = -2e_2 \). The last element is multiplied by an element in \( K^* \) such that moreover \( [e_1, e_2] = h \) holds. The \( \mathbb{C} \)-subspace of \( N \) generated by \( h, e_1, e_2 \) is isomorphic to \( sl_2 \) and we have found an isomorphism \( \phi \).

The element \( h \), which generates a Cartan subalgebra for \( N \), defined over \( \mathbb{C} \), is essentially unique. The vectors \( e_1, e_2 \) can however be replaced by \( fe_1, f^{-1}e_2 \) for any \( f \in K^* \). This reflects the observation that the differential module \( M \), with \( \det M = 1 \) and differential Galois group \( SL_2 \), that induces the adjoint module \( N \) is not unique. In fact, one can replace \( M \) by \( Ke \otimes_K M \) where the 1-dimensional module \( Ke \) is given by \( df = \frac{f'}{f} \) for \( f \in K^* \).

\[ \text{2.2.5 A general solution to the problem} \]

We recall the following. \( \text{Diff}_K \) will denote the (neutral) Tannakian category of all differential modules over \( K \). The Tannakian group of this category is an affine group scheme \( U \) (this is the universal differential Galois group), i.e., we have an equivalence \( \text{Diff}_K \rightarrow \text{Repr}_U \) of Tannakian categories. For any differential module \( P \) over \( K \), we denote by \( \{ P \} \) the full Tannakian subcategory generated by \( P \). The module \( P \) corresponds a representation \( \rho : U \rightarrow GL(V(P)) \). Its image is \( Gal(P) \).
By differential Galois theory, there is a natural equivalence of Tannakian categories \( \{\{P\}\} \to \text{Repr}_{\text{Gal}(P)}. \)

We note that the constructions of linear algebra in the category \( \text{Repr}_g \) for a semi-simple \( g \) are known and can be found explicitly by, for instance, the online program [LiE].

**The problem.** \( P \) is an input differential module. Find a differential module of smallest dimension \( M \) such that \( P \in \{\{M\}\} \) or, more precisely, some differential modules \( M_1, \ldots, M_r \) with \( \max\{\dim M_i\} \) as small as possible such that \( P \) lies in the Tannakian subcategory \( \{\{M_1, \ldots, M_r\}\} \) generated by \( \{M_1, \ldots, M_r\} \).

**A solution to the problem.** Suppose that the “input module” \( P \) has the properties absolutely irreducible, \( \det P = 1 \) and \( \text{Gal}(P) \) is connected. Then \( g := \text{gal}(P) \) is semi-simple. Let \( G \) be, as before, the simply connected group with Lie algebra \( g \). Then \( \text{Gal}(P) = G/Z' \) for some subgroup \( Z' \) of the center \( Z \) of \( G \). The adjoint representation \( (G/Z,g) \) lies in \( \text{Repr}_{G/Z} \). The construction of linear algebra \( \text{csrt}(1) \) from \( (G/Z',V(P)) \to (G/Z,g) \) can be read off in the equivalent subcategory \( \{\{g,V(P)\}\} \) of \( \text{Repr}_g \). Using the equivalence of \( \{\{P\}\} \) and \( \text{Repr}_{G/Z} \) one can apply \( \text{csrt}(1) \) to \( P \). This yields \( N \in \{\{P\}\} \) which maps to the adjoint representation \( (G/Z,g) \). Thus \( N \) is an adjoint differential module for \( G \). Theorem 2.2.5 provides a standard module \( M \) for \( G \) which induces \( N \).

Using the equivalences \( \{\{M\}\} \to \text{Repr}_G \to \text{Repr}_g \) one finds an explicit construction of linear algebra \( \text{csrt}(2) \) from \( M \) to a differential module \( Q \) with \( (\text{gal}(Q),V(Q)) = (\text{gal}(P),V(P)) \). Now \( P \) and \( Q \) are almost isomorphic.

What we know is that \( \text{csrt}(1) \) applied to \( P \) and \( Q \) produce \( N \). Let \( \rho : \mathcal{U} \to \text{SL}(V(P)) \) and \( \rho' : \mathcal{U} \to \text{SL}(V(Q)) \) denote the representation corresponding to \( P \) and \( Q \). The fact that \( \rho \) and \( \rho' \) yield the same representation \( \rho'' : \mathcal{U} \to \text{SL}(g) \), corresponding to \( N \), implies that \( \rho \) and \( \rho' \) are projectively equivalent, i.e., \( \rho(u) = c(u)\rho'(u) \) for all \( u \in \mathcal{U} \) and with \( c(u) \in \mathbb{C}^* \) (in fact \( c(u)^n = 1 \) where \( n = \dim_K P \)). Let the 1-dimensional differential module \( L \) correspond to the representation \( \mathcal{U} \to \mathbb{C}^*, u \mapsto c(u) \). Then \( P \cong L \otimes_K Q \). Finally, \( L \) can be made explicit by Subsection 2.2.1 part (3). Thus we have explicitly found \( P \in \{\{M,L\}\} \). In the case that \( G \) is semi-simple but not simple we write, as before, \( G = G_1 \times \cdots \times G_s \). The standard module \( M \) is a direct sum \( M_1 \oplus \cdots \oplus M_s \) where each \( M_i \) is a standard module for the simple \( G_i \). Thus we have \( P \in \{\{M_1, \ldots, M_s, L\}\} \) and this is a solution to our problem.

**Comments 2.2.7.** (1) Variations. In many cases there are shortcuts.

1.1 If \( Z' = \{1\} \), then \( \{\{P\}\} \cong \text{Repr}_G \) and there is an explicit construction \( \text{csrt}(3) \) from \( (g,V(P)) \) to the standard module \( (g,V) \). Then \( \text{csrt}(3) \) applied to \( P \) yields the standard differential module \( M \) with \( P \in \{\{M\}\} \).

1.2 If \( Z' = Z \), then there is a construction of linear algebra from the adjoint module \( N \) (obtained from \( P \)) back to \( P \).
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(1.3) If the adjoint $G$-module $\mathfrak{g}$ is not the faithful $G/Z$-module of smallest dimension, then we may use a faithful $G/Z$-module of smallest dimension at the place of $\mathfrak{g}$. For example, for $G = Sp_4$ the group $Z = \{\pm 1\}$ and $Sp_4/Z = SO_5$.

The natural representation of $SO_5$ has dimension 5 and $sp_4$ has dimension 10.

(2) Non connected groups. More generally, we may consider absolutely irreducible differential modules $P$ with $\det P = 1$. This assumption is equivalent to the statement that $V(P)$ is an irreducible $Gal(P)^o$-module. The finite group $Gal(P)/Gal(P)^o$ introduces (in general) two kinds of obstructions to the above method. Consider namely a $Gal(P)$-module $W$, obtained by some construction of linear algebra from $P$ and $V(P)$. The irreducible summands $\{W_i\}$ of $(Gal(P)^o,W)$ can be permuted by $Gal(P)$ because the image of $Gal(P)/Gal(P)^o$ in $Out(gal(P))$ is not trivial. The other possible obstruction can occur when some $W_i$ has multiplicity greater than one. A computable finite extension $\tilde{K}$ of the base field $K$ is needed to make this subspace invariant under the new (smaller) differential Galois group of $\tilde{K} \otimes_K P$ over $\tilde{K}$.

We study this in more detail for the case that $(Gal(P)^o,V(P))$ is the adjoint representation. After identifying $V(P)$ with $\mathfrak{g}$, the group $Gal(P)$ is contained in the group $G^+ = G^{++} \cap SL(\mathfrak{g})$, where $G^{++}$ is the normalizer of $G^o := Gal(P)^o$ in $GL(\mathfrak{g})$. An element $T \in GL(\mathfrak{g})$ belongs to $G^{++}$ if and only if there exists a constant $c \in \mathbb{C}^*$ such that $[TA, TB] = c[A,B]$ for all $A,B \in \mathfrak{g}$. One obtains an exact sequence $1 \rightarrow (\mu_n, G^o)/G^o \rightarrow G^+/G^o \rightarrow Out(G^o) \rightarrow 1$, where $n$ is the dimension of $P$.

If the image of $Gal(P)$ in $Out(G^o)$ is not trivial, then one has to compute a finite field extension of $K$ which kills this part of $Gal(P)$. If the image is trivial, then one can replace $P$ by the direct summand $\tilde{P}$ of $P \otimes P^*$ which is an adjoint representation and one obtains a standard module $\tilde{M}$ with $\tilde{P} \in \{\tilde{M}\}$. Further $P \cong \tilde{P} \otimes L$, where $L$ is a 1-dimensional differential module such that $L^{\otimes n} = 1$. As mentioned in Subsection 2.2.1 there is an easy algorithm for the computation of $L$.

(3) In the next chapter (Sections 3.2–3.5) we investigate special cases of shortcuts and non connected groups. This includes all cases, listed in Subsection 2.2.2 where a differential module can be “solved” in terms of modules of lower dimension and field extensions.