We couple $N = 4$, $d = 4$ supersymmetric Yang–Mills theory to supergravity. The scalars of the theory parametrize the coset $(\text{SO}(n,6)/[\text{SO}(n) \times \text{SO}(6)]) \times (\text{SU}(1,1)/\text{U}(1))$. Keeping the composite local $\text{SO}(n) \times \text{SO}(6) \times \text{U}(1)$ invariance intact, we gauge an $(n+6)$ parameter subgroup of $\text{SO}(n,6)$ which is either (i) $\text{SU}(2) \times \text{SU}(2) \times \text{H}$ (dim $\text{H} = n$), (ii) $\text{SO}(4,1) \times \text{H}$ (dim $\text{H} = n - 4$) or (iii) $\text{SO}(6,1) \times \text{H}$ (dim $\text{H} = n - 15$). In all these cases the theory has an indefinite potential.

Matter couplings of $N = 1$ supergravity have been widely studied in recent years. Although they have been applied to phenomenology with some success, a great deal of arbitrariness remains in these theories. This is mainly due to the fact that scalar multiplets in arbitrary representations can be coupled to $N = 1$ supergravity, the only restriction being that the scalars of the matter multiplet parametrize a Kähler manifold [1]. This still allows two arbitrary functions of the scalars in the theory. The matter couplings in $N = 2$ supergravity are somewhat more restrictive. In that case, the scalars of the matter multiplets must parametrize a quaternionic Kähler manifold [1] and only one arbitrary function is allowed. On the other hand, supersymmetry breaking from $N = 2$ to $N = 1$ in Minkowski background has been shown to be impossible [3].

Since matter multiplets exist up to $N = 4$, it is natural to consider the matter couplings of $N = 4$ supergravity [4]. In fact, the only $N = 4$ matter multiplet available is the vector multiplet [5] which contains six scalars. Thus, the $N = 4$ supergravity coupled to matter is clearly the largest possible and the most restrictive matter coupled supergravity. The scalar couplings in this theory are described by the manifold $(\text{SO}(n,6)/[\text{SO}(n) \times \text{SO}(6)] \times (\text{SU}(1,1)/\text{U}(1))$, where $n$ is the number of vector multiplets. The $N = 4$ ungauged matter couplings based on this coset structure have been recently constructed by the Roo [6]. For phenomenological purposes, one must gauge an appropriate subgroup of the global $\text{SO}(n,6) \times \text{SU}(1,1)$. Such gaugings require the addition of a potential to the theory. It is important to determine whether such a potential allows the breaking of supersymmetry from $N = 4$ down to $N = 1$.

With these motivations in mind, we have constructed the gauged matter couplings of $N = 4$, $d = 4$ supergravity. In this paper we gauge an $(n+6)$ parameter subgroup of $\text{SO}(n,6)$, and do not consider the gauging of the $\text{U}(1)$ subgroup of $\text{SU}(1,1)$. We find that the gauge group must be either $\text{SU}(2) \times \text{SU}(2) \times \text{H}$ (dim $\text{H} = n$) or $\text{SO}(4,1) \times \text{H}$ (dim $\text{H} = n - 4$) or $\text{SO}(6,1) \times \text{H}$ (dim $\text{H} = n - 15$). In the latter two cases there is a natural Higgs mechanism which breaks $\text{SO}(4,1)$ to $\text{SO}(4)$ and $\text{SO}(6,1)$ to $\text{SO}(6)$, respectively. The potential which we find is indefinite, and it depends on the structure constants of the gauge group and the representative elements of the $(\text{SO}(n,6)/[\text{SO}(n) \times \text{SO}(6)]) \times (\text{SU}(1,1)/\text{U}(1))$ coset scalars. In obtaining these results, we follow closely the methods of refs. [7–9] which were recently applied to $d = 7$ supergravity [10].

In this letter we use a formulation of the $N = 4$, $d = 4$ supergravity multiplet and the $n$ vector multiplets that has manifest global $\text{SU}(4)$ invariances [4]. The field components of the supergravity multiplet
are given by a vierbein $e^m_\mu$, four gravitini $\dot{\psi}_\mu^I$ of positive chirality\(^3\), six vector fields $A^I_\mu$, four spinors $\chi^I$ of negative chirality (i.e. $\gamma_5 \chi^I = -\chi^I$) and one complex scalar $z$ that parametrizes an SU(1,1)/U(1) coset space [4]. On the other hand, an $N = 4$ vector multiplet consists of one vector field $A^I_\mu$, four spinors $\chi^I$ of positive chirality (i.e. $\gamma_5 \chi^I = +\chi^I$) and six scalars. In the coupled system the scalars of the $n$ vector multiplets parametrize the coset $SO(n,6)/[SO(n) \times SU(4)]$. Recall that $SU(4) \approx SO(6)$.

The $6n$ scalars of the $n$ vector multiplets are conveniently described by an $(n+6) \times (n+6)$ matrix $L^A_I (I, A = 1, \ldots, n+6)$, which is a representative element of the coset space $SO(n,6)/[SO(n) \times SU(4)]$ [8]. The index $A$ can be decomposed into a 6 of SU(4) and an $n$ of SO(n)

$$L^A_I \to (L^I_{1j}, L^I_{2a}), \quad i, j = 1, \ldots, 4, \quad a = 1, \ldots, n,$$

where $L^I_{1j}$ is antisymmetric in SU(4) indices and subject to an SU(4) covariant constant

$$L^I_{1j} = (L_{1j})^* = \frac{1}{2} e^{ijkl} L^j_{kl}.$$

In terms of the fields defined by (1) the orthogonality condition for the SO(n,6) matrix $L^A_I$ reads:

$$-L^I_{1j} L_{1j} + L^I_{2a} L_{2a} = \eta_{Ij},$$

where $\eta_{Ij} = \text{diag}(-, +, +, +, +, +, +, +, +, +, +, +, +, +, +)$. The components of the inverse matrix $L^A_I$ are defined by

$$L^A_{ij} L^I_{jk} = 0, \quad L^I_{ij} L^I_{jk} = -\frac{1}{2} (\delta^A_{ij} \delta^I_{jk} - \delta^I_{ij} \delta^A_{jk}),$$

$$L^I_{1j} L^I_{2a} = \delta^I_{ai} \delta^I_{aj}.$$

The matrix $L^A_I$ transforms under global SO(n,6) from the left and local composite SO(n) $\times$ SU(4) from the right. The latter transformations are given in table 1.

Similarly, the complex scalar $z$ of the supergravity multiplet can be described by two constrained complex scalars $\phi$ and $\psi$:

$$\phi^* \psi + \psi^* \phi = 2.$$

In terms of the unconstrained variable $z$ they are given by

$$\phi = (1-z^*)(1-zz^*)^{1/2}, \quad \psi = (1+z^*)(1-zz^*)^{1/2}.$$

The transformations of $\phi$ and $\psi$ under the composite $U(1)$ are given in table 1. The SU(1,1) invariance is realized only in the equations of motion [4] We find that after gauging the global SU(1,1) is broken.

We now require that all the remaining fields of the supergravity and vector multiplets transform in definite representations of the global SO(n,6) and the local composite SO(n) $\times$ SU(4) $\times$ U(1) symmetries of the coset space $[SO(n,6)/SO(n) \times SU(4)]$ $\times$ (SU(1,1)/U(1)). The $n+6$ vector fields $A^I_\mu (I = 1, \ldots, n+6)$ transform according to the defining representation of SO(n,6) [7]. The transformation properties of the remaining fields under the composite SO(n) $\times$ SU(4) $\times$ U(1) are given in table 1. The vector fields $A^I_\mu$ are now used to gauge an $(n+6)$ parameter subgroup of SO(n,6) with structure constant $f^I_{jk}$. The central role in the construction of an action and supersymmetry transformation rules is played by the gauge covariant Maurer–Cartan form $L^{-1}(\partial + A)L$, which has the following components [11,12,8].

$$P^I_{a\mu} = L^I_a (\partial^a_\mu \delta^K_I + f_{ijk}^K A^I_\mu) L^I_{Kj},$$

$$Q_{a\mu} = L^I_a (\partial^a_\mu \delta^K_I + f_{ijk}^K A^I_\mu) L^I_{Kb},$$

$$Q^I_{a\mu} = L^I_{(k} (\partial^a_\mu \delta^K_I + f_{ijk}^K A^I_\mu) L^I_{Kb)}.$$

Note that we have absorbed the gauge coupling constant into the structure constants $f^I_{jk}$.

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\(^3\) We use the chiral notation, i.e. $\gamma_5 \dot{\psi}_\mu^I = +\psi_\mu^I$ and $\gamma_5 \psi_\mu = \gamma_5 C(\psi_\mu)^I_T = -\psi_\mu^I$. We use the signature $(++,+,,+)$, and the Ricci tensor $R^a_\mu = e_b^a(\partial^a_\mu \omega^b_\nu - ...)$. 

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Table 1

<table>
<thead>
<tr>
<th>Field</th>
<th>SO(n) $\times$ SU(4) $\times$ U(1)</th>
<th>Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^m_\mu$</td>
<td>(1, 1, 0)</td>
<td></td>
</tr>
<tr>
<td>$\psi_\mu^I$</td>
<td>(1, 4, $-1/2$)</td>
<td>$\gamma_5 \psi_\mu^I = +\psi_\mu^I$</td>
</tr>
<tr>
<td>$A^I_\mu$</td>
<td>(1, 1, 0)</td>
<td></td>
</tr>
<tr>
<td>$L^I_{1j}$</td>
<td>(n, 1, 0)</td>
<td></td>
</tr>
<tr>
<td>$L^I_{1j}$</td>
<td>(1, 6, 0)</td>
<td>$L^I_{1j} = e^{ijkl} L^j_{kl}$</td>
</tr>
<tr>
<td>$\chi^I$</td>
<td>(1, 4, 3/2)</td>
<td>$\gamma_5 \chi^I = -\chi^I$</td>
</tr>
<tr>
<td>$\chi^I$</td>
<td>(n, 4, 1/2)</td>
<td>$\gamma_5 \chi^I = +\chi^I$</td>
</tr>
<tr>
<td>$\phi, \psi$</td>
<td>(1, 1, 1)</td>
<td>$\phi^* \psi + \psi^* \phi = 2$</td>
</tr>
</tbody>
</table>

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In order to ensure the composite local SO(n) X SU(4) transformation properties of P and Q, the structure constants must satisfy

$$ f_{IJ} L \eta_{KL} = f_{IJ} L \eta_{KL} L. \tag{8} $$

This condition can be satisfied by taking \( \eta_{IJ} \) to be the Cartan–Killing metric of the gauged algebra. Since \( \eta_{IJ} \) has 6 minus, and n plus signature it follows that the gauge group is either (i) SU(2) X SU(2) X H (dim H = n), (ii) SO(4,1) X H (dim H = n - 4) or (iii) SO(6,1) X H (dim H = n - 15). In case (i) the coupling constants of the two SU(2)'s need not be equal. Furthermore, it is convenient to introduce the following projections of the structure constants \( f_{IJ} \): \( C_{ij} \), \( C_{ai} \), \( C_{ab} \).

The \( P \) and \( Q \) defined in eq. (7) satisfy the following Maurer–Cartan equations \[8,10\]:

$$ D_{[\mu} P_{\nu]} = -i F_{\mu \nu}(A) L_{i}^{k} C_{ai}^{k} + \frac{1}{2} F_{\mu \nu}(A) L_{i}^{k} \gamma_{\mu \nu} h.c. \tag{10} $$

where \( F_{\mu \nu}(A) \) and \( Q_{\mu \nu} \) are the field strengths of \( A_{\mu} \) and \( Q_{\mu} \), respectively and h.c. denotes hermitian conjugation. In products of \( C \) functions one can make use of the Jacoby identity

$$ \sum_{i} C_{ij} C_{ik} - \sum_{i} C_{ij} C_{ik} = 0. \tag{11a} $$

$$ C_{ab} C_{ij} = \frac{-4}{9} C_{ik} C_{kj} \quad \text{trace} = 0. \tag{11b} $$

We remark that similar identities play an important role in \( N = 8, d = 4 \) [12], and in \( N = 8, d = 5 \) [13] supergravities. Derivatives of \( C \) functions yield again \( C \) functions. For instance,

$$ D_{\mu} C_{ai} = i P_{\mu} b_{ik} C_{abk} - \frac{2}{3} i P_{\mu} a_{ik} C_{kj} + h.c. \quad \text{trace}, \tag{12} $$

$$ D_{\mu} C_{ij} = -3 i P_{\mu} (ik) C_{ai}^{k} \quad \text{trace}. $$

Similarly, we define for the SU(1,1)/U(1) coset space

$$ a_{\mu} = -\frac{1}{2} (\phi_{\mu} \psi^{+} + \psi \partial_{\mu} \phi^{+}), \tag{13} $$

Their Maurer–Cartan equations are given by

$$ D_{[\mu} P_{\nu]} = 0, \quad P_{[\mu} P_{\nu]} = -\partial_{[\mu} a_{\nu]}. \tag{14} $$

Another useful identity is

$$ D_{\mu} \phi = -P_{\mu} \psi^{+}, \quad D_{\mu} \psi = P_{\mu} \phi^{+}. \tag{15} $$

We are now able to derive the action and supersymmetry transformation rules for the coupled system. Using all the symmetries discussed above we can immediately write down an ansatz for the action and transformation rules up to some undetermined coefficients. These coefficients can easily be determined by requiring closure of the commutation algebra on the bosonic fields and supersymmetry of the action. The resulting lagrangian is given by

$$ e^{-1} \mathcal{L} = \frac{1}{2} R - \frac{1}{2} a_{ij} F_{\mu}^{i} L_{ij} + \frac{1}{2} a_{ij} F_{\mu}^{i} F_{\mu}^{j} + \frac{1}{2} a_{ij} F_{\mu}^{i} + F_{\mu}^{j} $$

$$ - \frac{1}{2} \sqrt{2} D_{\mu} \psi_{1} \gamma^{\mu} \psi_{2}^{+} P_{\mu} P_{\nu} + 2 \sqrt{2} D_{\mu} \gamma^{\mu} \psi_{1}^{+} P_{\mu} P_{\nu} + 2 \sqrt{2} D_{\mu} \gamma^{\mu} \psi_{2}^{+} P_{\mu} P_{\nu} $$

$$ - (1/\xi) F_{\mu}^{i} \left( \frac{1}{2} \psi_{1} \psi_{2} \gamma^{i} \gamma^{j} \gamma^{l} L_{ij} L_{lj} \right) $$

$$ + \frac{1}{2} \psi_{1} \gamma^{\mu} \psi_{2}^{+} \gamma^{\nu} \gamma^{\lambda} \chi^{l} L_{ij} L_{lj} $$

$$ + \frac{1}{2} \frac{1}{2} \sqrt{2} \gamma^{\mu} \gamma^{\nu} \gamma^{\lambda} \chi^{l} L_{ij} L_{lj} $$

$$ + \frac{1}{2} \frac{1}{2} \sqrt{2} \gamma^{\mu} \gamma^{\nu} \gamma^{\lambda} \chi^{l} L_{ij} L_{lj} $$

$$ - \frac{1}{2} \phi^{+} \phi (A_{ai}^{i} C_{ai}^{i} - \frac{4}{6} C_{ij} C_{ij}) + h.c. $$

+ quartic fermions. \tag{16}
Here we have used the following definitions

$$a_{IJ} = (\psi^* / \phi^*) L^U_{IJ} L^U_{JI} + (\psi / \phi) L^U_{IJ} L^U_{JI},$$

$$D_\mu \lambda^I_a = \partial_\mu \lambda^I_a + \frac{1}{2} \omega^m_{\mu n} (e) \gamma^m \lambda^I_a$$

$$+ Q^a_{\mu b} \lambda^I_b + Q^a_{\mu I} \lambda^I_a - \frac{1}{2} a_{\mu} \lambda^I_a,$$  \hspace{1cm} (17)

and similar definitions for $D_\mu \psi^I_b$ and $D_\mu \lambda^I_b$. The action defined by eq. (16) is invariant under the following supersymmetry transformation rules:

$$\delta \phi^I = \bar{\epsilon} \gamma^m \psi^I + h.c.,$$

$$\delta \psi^I = 2D_\mu \phi^I + (1/2\phi) \gamma^{mn} F^I_{mn} L^U_{IJ} e^I + \frac{1}{3} \bar{\epsilon} \gamma^m \mu C^U e^I$$

$$+ \text{bilinear fermions},$$

$$\delta A^I_\mu = -2\phi^I L^U_{IJ} (\bar{\epsilon} \gamma^m \phi^I_a \mu C^U a^I)$$

$$+ \text{bilinear fermions},$$

$$\delta L^I_a = 2iL^U_{IJ} \bar{\epsilon} \gamma^m \phi^I + h.c.,$$

$$\delta L^I_a = -2i \bar{\epsilon} \gamma^m \phi^I + \text{dual},$$

$$\delta \lambda^I_a = - (1/2\phi^*) \gamma^{mn} F^I_{mn} L^U_{IJ} e^I + 2\gamma^m P_m e^I + \frac{4}{3} \phi C^U e^I$$

$$+ \text{bilinear fermions},$$

$$\delta \phi^I = - \bar{\epsilon} i \chi^I + h.c.,$$

$$\delta \psi^I = + \bar{\epsilon} i \chi^I + \psi^I,$$  \hspace{1cm} (18)

where $D e^I$ is defined as in eq. (17). Both in the action and transformation rules the terms proportional to $C$ breaks the global SU(1,1) invariance. Note also the absence of $\bar{\chi} \chi$ terms in the Pauli and mass terms [4]. Furthermore, we observe an interesting interference between the SU(1,1)/U(1) and SO(6,1)/[SO(6) x SO(n)] coset structures in the definition of $a_{IJ}$ (see eq. (17)). For pure supergravity we expect that our results reduce to those of ref. [14].

Let us finally comment on the potential that emerges from our gauging. In terms of the unconstrained variable $z$ of the SU(1,1)/U(1) coset space (see eq. (6)) it is given by

$$V(z, L) = \frac{1 - z^*}{1 - z^*} (1 - z^*) (C^U e^I + \frac{4}{3} C^U e^I).$$  \hspace{1cm} (19)

Minimization of this potential with respect to the variables $z$ (0 $\leq |z| < 1$) and $L$ leads to the following equations, respectively

$$C^U e^I = \frac{4}{3} C^U e^I = 0, \quad C^U e^I = 0.$$

Using eqs (11) one can show that these equations lead to the following integrability conditions

$$\{ C^U e^I = \frac{4}{3} C^U e^I \} e^I = 0, \quad C^U e^I = 0.$$  \hspace{1cm} (22)

This shows that the minimization condition, eq (20), for the potential is sufficient for the fulfillment of the integrability conditions, eq. (22). The first condition in eq. (20) can only have a non-trivial solution if both $C^U e^I$ and $C^U e^I$ are non zero. It turns out that this cannot be achieved for the gauge group SU(2) x SU(2) x H (dim H = n) and SO(6,1) x H (dim H = n - 15). It would be interesting to see whether the $n+6$ vector fields, $A^I_\mu$ can be used to gauge groups other than the ones mentioned above and to see whether they can lead to an interesting super-Higgs effect that breaks the $N = 4$ supersymmetry down to a lower one.

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Note added After completing this work we became aware of the fact that de Roo [15] has also constructed the gauged matter couplings of $N = 4, d = 4$ supergravity.

References