We construct an action for a supermembrane propagating in $d=11$ supergravity background. Using the constraints of $d=11$ curved superspace, we show that the action is invariant under Siegel-type transformations recently generalized by Hughes, Li and Polchinski. The transformation parameter is a world-volume scalar and $d=11$ spacetime spinor. We also discuss the general problem of the coupling of $n$-dimensional extended objects to $d$-dimensional supergravity.

1. Now that we have become accustomed to the notion that strings should replace particles, it is natural to investigate the properties of higher-dimensional extended objects, in particular of membranes since they are the simplest extended objects, and they might describe strings in an appropriate limit.

In 1962 Dirac [1] put forward a theory of an extended electron based on the idea of a relativistic membrane. In 1976, Collins and Tucker [2] studied the classical and quantum mechanics of free relativistic membranes. A year later a locally supersymmetric and reparametrization-invariant action for a spinning membrane was constructed by Howe and Tucker [3]. The action describes anti de Sitter supergravity coupled to a number of scalar multiplets in three dimensions. It is the membrane analog of the Neveu–Schwarz–Ramond formulation of the spinning string theory.


The generalization of the Hughes et al. model to $n$-extended objects propagating in flat $d$-dimensional superspace is evident. All that is required is the existence of a closer super $(n+2)$-form given by

$$H = E^a E^{a_2} E^{a_3} \ldots E^{a_n} (\gamma_{a_1 \ldots a_n})_{\alpha \beta},$$  \hspace{1cm} (1)$$

where $(E^a, E^\alpha)$ are the basis one-forms in superspace. This form is closed provided that the following $\Gamma$-matrix identity holds:

$$\gamma^{a_{n}}_{\alpha \beta} (\gamma_{a_1 a_2 \ldots a_{n-1}})_{\gamma \delta} = 0.$$  \hspace{1cm} (2)$$

The purpose of this note is to construct Hughes et al. -type actions describing the propagation of an $n$-extended object in $d$-dimensional curved superspace. We give a general formula for the action and the transformation rules, whose consistency requires, among other things (see below), the existence of a closed $(n+2)$-form in curved superspace. Thus we
expect that the n-extended objects under consideration can consistently propagate only in $d \leqslant 11$ supergravities whose superspace formulation involves a closed $(n+2)$-form. We further expect that such forms exist in supergravity theories in which a closed bosonic $(n+2)$-form occurs. As far as we know, the following possibilities exist (we include the Yang-Mills couplings whenever possible):

The dual formulation of $d=10$, $N=1$ supergravity involves a closed seven-form. Its dimensional reduction on a $(10-d)$-dimensional torus leads to real closed $(d-3)$-forms in $d$-dimensional supergravities. (These are $N=1$ supergravities in $d=8$, 9, 10; $N=2$ in $d=7$ and $N=2$ or 4 in $d=6$) [9]. Apart from these, there is: (i) A real closed four-form in $d=11$, $N=1$ supergravity, (ii) a real closed three-form in non-chiral $d=10$, $N=2$ supergravity, (iii) a complex closed three-form in chiral $d=10$, $N=2$ supergravity.

Excluding Yang-Mills coupling, as is well known, closed super three-forms exist in $d=3, 4, 6$, and 10.

Considering the case of the membranes, from the above list it follows that the candidate dimensions are 7 and 11. Since the superspace formulation of $d=7$, $N=2$ supergravity is not known at present, we are led to consider the supermembrane propagating in eleven-dimensional spacetime.

Our main result is the construction of an action which describes a consistent coupling of $d=11$ supergravity to a supermembrane. In particular the Kalb-Ramond-like third rank antisymmetric tensor field of $d=11$ supergravity [10] is used.

$$S = \int d^2 \xi \left( \frac{1}{4} \sqrt{-g} g^{ij} E_i^a E_j^b \eta_{ab} + \epsilon^{ijk} E_i^a E_j^b E_k^c B_{cba} - \frac{1}{4} \sqrt{-g} \right).$$

(3)

Here $i=0, 1, 2$ labels the coordinates $\xi^i = (\tau, \sigma, \rho)$ of the world volume with metric $g_{ij}$ and signature $(-, +, +)$. The super three-form $B$ is needed for the superspace description of $d=11$ supergravity [10]. For the Levi-Civita symbol $\epsilon^{ijk}$ we use the same conventions as in ref. [11]. In (3) we have used the notation

$$E_i^A = (\partial_i Z^M) E_M^A,$$

(4)

where $Z^M(\xi)$ are the superspace coordinates, and $E_M^A(Z)$ is the supervielbein.

Note that the action has a cosmological constant with a fixed magnitude. This is so that the field equation of the metric $g_{ij}$ gives the embedding equation

$$g_{ij} = E_i^a E_j^b \eta_{ab} \equiv T_{ij}.$$  

(5)

We require that the action $S$ is invariant under a fermionic gauge transformation of the form [5]

$$\delta E^a = 0, \quad \delta E^a = (1 + \Gamma)^{a}_\beta \kappa^\beta,$$

$$\delta g_{ij} = 2 [X_{ij} - g_{ij} X_k X^k (n-1)]$$

$$= 2[n=2 \text{ for membrane}],$$

(6)

where the transformation parameter $\kappa^\alpha(\xi)$ is a 32 component Majorana spinor, and a world-volume scalar, and

$$\delta E^A = \delta Z^M E_M^A,$$

(7)

$$\Gamma^a_\beta = (1/6 \sqrt{-g}) \epsilon^{ijk} E_i^a E_j^b E_k^c (\gamma_{abc})^{\alpha}_\beta.$$

(8)

Here $\gamma^i(a=0, 1, \ldots, 10)$ are the Dirac matrices in eleven dimensions. $X_{ij}$ is a function of $E_i^A$ which will be determined by the invariance of the action. The choice of $\delta g_{ij}$ is due to the fact that, given a variation of the action of the form $\delta S = T_{ij} X^{ij}$, and writing this variation as

$$T_{ij} X^{ij} = g_{ij} X^{ij} + (T_{ij} - g_{ij}) X^{ij},$$

(9)

the second term on the right-hand side cancels $(\delta S / \delta g_{ij}) \delta g_{ij}$. Thus we are left with the first term on the right-hand side, which equals the left-hand side upon the use of (5). Effectively, this means that whenever we encounter a variation of the form $T_{ij} X^{ij}$, we can use eq. (5), provided that we add $X^{ij}$ to $\delta g_{ij}$ as in (6).

The matrix $\Gamma^a_\beta$ occurring in (8) satisfies the property

$$\Gamma^a_\beta \Gamma^\beta_\delta = (T^t_{ij} T^t_i T^k_j T^k_i | k_j) \delta^a_\delta \equiv \Gamma^a_\delta \delta^a_\delta.$$

(10)

The normalization in (8) is chosen such that upon the use of the equation $T_{ij} = g_{ij}$, the matrix $\Gamma^a_\beta$ satisfies the relation $\Gamma^a_\beta \Gamma^\beta_\gamma = \delta^a_\gamma$. 

Now using (6) the variation of the action (3) is
(we consider a closed supermembrane and therefore
discard the surface terms)
\[ \delta S = \int d^2 \xi \left[ \sqrt{-g} g^{ij} \left( - \delta E^\beta E_i \gamma T^a E_j \right) \right] + \sqrt{-g} g^{ij} \left( - \delta E^\beta E_i \gamma T^a E_j \right) \]
\[ + \epsilon^{ijk} E_i^a E_j^b E_k^c \delta E^\alpha H_{\alpha CBA} \]
\[ - \frac{1}{2} \sqrt{-g} \delta g^{ij} (T^{ij} - \frac{1}{2} g^{ij} T - \frac{1}{2} g^{ij}) . \] (11)
The torsion two-form \( T^A \) and the four-form field
strength \( H \) are defined by (our superspace conven-
tions are those of Howe [12])
\[ T^A = dE^a + E^b \Omega^A_b = \frac{1}{2} E^B E^C T^A_{BC} , \]
\[ H = dB = \frac{1}{2} E^D E^E E^F H_{ABCD} . \] (12)

We now organize the terms in (11) according to
the number of one-forms \( E^a \) they contain. Those with
three \( E^a \) and two \( E^a \) come only from the
Wess-Zumino term. They must vanish seperately,
and this requires the constants
\[ H_{a\beta \gamma \delta} = H_{a\beta \gamma \delta} = 0 . \] (13)

The cancellation of the terms linear in \( E^a \) lead to
the constraints
\[ T^a_{\alpha \beta} = (\gamma^a)_{\alpha \beta} , \] (14)
\[ H_{a\beta \gamma \delta} = - \frac{1}{2} (\gamma_{\alpha \beta})_{\alpha \beta} , \] (15)
while the cancellation of the terms not containing \( E^a \)
require the constraint
\[ \eta_{(a} T^{b)}_{\alpha \gamma} = \eta_{ab} A_{\alpha} \] (16)
\[ H_{a\beta \gamma \delta} = - \frac{1}{2} A_{\alpha} (\gamma_{\alpha \beta})_{\beta \gamma} \] (17)

Here \( A_{\alpha} \) is an arbitrary spinor superfield which is
vanishing in \( d=11 \) [10].

It is important to realize that in obtaining
(14)–(17) we have used the identity
\[ \delta E^\alpha = \Gamma^\alpha_{\beta} \delta E^\beta + (1 - \Gamma^2) \kappa^\alpha . \] (18)

Using this identity in the variation of the kinetic term,
the terms arising from \( \Gamma^\alpha_{\beta} \) in (18) can be shown to
cancel similar terms coming from the variation of the
Wess-Zumino term, modulo terms which cancel by
an appropriate variation of \( g_{\mu \nu} \). [Using the argument
below (8) once.] In the remaining terms coming from
(1–\( I^2 \)), we use the argument given below (8)
repeatedly to compute further contributions to \( \delta g_{\mu \nu} \).
Thus we find the result
\[ X_{ij} = - \frac{1}{4} \epsilon^{ijkl} E_k^a E_l^b (\gamma_{ab})_{\alpha \beta} \delta E^\beta E_j^\alpha \]
\[ + \frac{1}{2} \kappa^\beta E_n \alpha (\gamma^d)_{\alpha \beta} E^m d g_{ij} (T^k E^l T^l) + \delta k^k T^l T^l \]
\[ + i \leftrightarrow j . \] (19)

In summary, the action (3) is invariant under (6)
provided that (13)–(17) hold, and \( X^{ij} \) is given by
(19). In addition, the following Bianchi identities
must hold:
\[ DT^A = - E^B \wedge R_B^A , \] (20)
\[ DH = 0 . \]

The generalization of the results of this section to
the general case of \( n \)-extended objects in \( d \)-dimen-
sional supergravity is straightforward. The result is
given in the appendix.

3. We observe that the superspace constraints of
\( d=11 \) supergravity given by Cremmer and Ferrara
[10] and Brink and Howe [10] do provide a solu-
tion to (13)–(17) and the Bianchi identities (20),
with \( A_{\alpha} = 0 \).

In conclusion, we have shown that there exists a
consistent coupling of a closed supermembrane to
eleven-dimensional supergravity. (Note that it is
natural to consider a closed supermembrane in eleven
dimensions, since there are no matter multiplets in
this dimension).

4. There are several directions in which the present
work can be extended. We shall name a few.
Firstly, it is of interest to study the quantization of
the supermembrane model in eleven dimensions. In
particular, the question of whether massless gauge
fields can possibly arise is a challenging one. Although
usually one encounters difficulties in finding mass-
less excitations of a membrane [13], it is encourag-
ing that, here, we have a spacetime supersymmetric
membrane action.

Secondly, it is natural to consider the dimensional
reduction of our model from eleven- to ten-dimen-
sional spacetime, and at the same time from three-
dimensional world volume to a two-dimensional...
world sheet. It would be interesting to see what kind of $d=10$ string theories could possibly emerge in an infinitely thin membrane limit.

P.K.T. would like to thank Professor Abdus Salam for his kind hospitality and ICTP in Trieste where this work was carried out.

Appendix. In this appendix we construct the action for an $n$-extended object propagating in $d$-dimensional supergravity background. We also give the transformation rules, and the constraints on the background.

The action is

$$S = \int d^2 \xi \left[ \frac{1}{2} \sqrt{-g} \mathcal{G}^{\mu\nu} E_\mu E_\nu + \ldots \right]$$

$$+ \epsilon^{i_1 \ldots i_n} E_{i_1}^{a_1} \ldots E_{i_{n+1}}^{a_{n+1}} B_{a_{n+1}}$$

$$- \frac{1}{2} (n-1) \sqrt{-g} \right]. \quad (A1)$$

The transformation rules are those in (6), where the matrix $\Gamma^\alpha_\beta$ is now given by

$$\Gamma^\alpha_\beta = \left[ \eta'/(n+1)! \right] \sqrt{-g}$$

$$\times \epsilon^{i_1 \ldots i_n} E_{i_1}^{a_1} \ldots E_{i_{n+1}}^{a_{n+1}} (\gamma_{\alpha_1 \ldots \alpha_{n+1}})^{\alpha_\beta}, \quad (A2)$$

where $\eta$ is given by

$$\eta = (-1)^{(n+1)(n-2)/4}. \quad (A3)$$

Invariance of the action (A1) is ensured by imposing the following set of constraints:

$$T^a_{\alpha \beta} = (\gamma^a)_{\alpha \beta}, \quad \eta_{(a} T^b_{\beta) \alpha} = \eta_{\alpha \beta} A_{\alpha}, \quad (A4)$$

$$H_{\alpha \beta \gamma \ldots \ldots \gamma_{\alpha \beta \gamma}} = 0, \quad (A5)$$

and by taking $X_i$ occurring in (6) to be

$$X_i = (-\eta/2n!)$$

$$\times \epsilon_{k_1 \ldots k_n} \mathcal{G}_{a_1 \ldots a_n} \delta E^\beta E_j^\alpha$$

$$+ \frac{1}{2} \kappa^{\beta} [E_{k_1} (\gamma^a)_{\alpha \beta} E_{m}^{\alpha} + (n+1) A_{\beta}]$$

$$\times g_{ij} (T^{k_1} T^{k_2} \ldots T^{k_n}) + \delta^{k_1} k_1 T^{k_2} k_2 \ldots T^{k_n} k_n$$

$$+ \delta^{k_1} k_1 \delta^{k_2} k_2 \ldots \delta^{k_n} k_n T^{k_1} k_1 \ldots T^{k_n} k_n$$

$$- \frac{1}{2} \delta E^\alpha A_{\alpha} g_{ij} + (i \leftrightarrow j). \quad (A6)$$

References