SUPERMEMBRANES AND ELEVEN-DIMENSIONAL SUPERGRAVITY

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We construct an action for a supermembrane propagating in $d=11$ supergravity background. Using the constraints of $d=11$ curved superspace, we show that the action is invariant under Siegel-type transformations recently generalized by Hughes, Li and Polchinski. The transformation parameter is a world-volume scalar and $d=11$ spacetime spinor. We also discuss the general problem of the coupling of $n$-dimensional extended objects to $d$-dimensional supergravity.

1. Now that we have become accustomed to the notion that strings should replace particles, it is natural to investigate the properties of higher-dimensional extended objects, in particular of membranes since they are the simplest extended objects, and they might describe strings in an appropriate limit.

In 1962 Dirac [1] put forward a theory of an extended electron based on the idea of a relativistic membrane. In 1976, Collins and Tucker [2] studied the classical and quantum mechanics of free relativistic membranes. A year later a locally supersymmetric and reparametrization-invariant action for a spinning membrane was constructed by Howe and Tucker [3]. The action describes anti de Sitter supergravity coupled to a number of scalar multiplets in three dimensions. It is the membrane analog of the Neveu–Schwarz–Ramond formulation of the spinning string theory.


The generalization of the Hughes et al. model to $n$-extended objects propagating in flat $d$-dimensional superspace is evident. All that is required is the existence of a closer super $(n+2)$-form given by

$$H = E^a E^a E^a ... E^a (y_{a_1 ... a_n})_{a_0},$$

where $(E^a, E_\alpha)$ are the basis one-forms in superspace. This form is closed provided that the following $\Gamma$-matrix identity holds:

$$(y^a)_{a_0} (y^{a_1 ... a_{n-1}})_{a_0} = 0.$$ (2)

The purpose of this note is to construct Hughes et al. -type actions describing the propagation of an $n$-extended object in $d$-dimensional curved superspace. We give a general formula for the action and the transformation rules, whose consistency requires, among other things (see below), the existence of a closed $(n+2)$-form in curved superspace. Thus we...
expect that the $n$-extended objects under consideration can consistently propagate only in $d \leq 11$ supergravities whose superspace formulation involves a closed $(n+2)$-form. We further expect that such forms exist in supergravity theories in which a closed bosonic $(n+2)$-form occurs. As far as we know, the following possibilities exist (we include the Yang-Mills couplings whenever possible):

The dual formulation of $d=10, N=1$ supergravity involves a closed seven-form. Its dimensional reduction on a $(10-d)$-dimensional torus leads to real closed $(d-3)$-forms in $d$-dimensional supergravities. (These are $N=1$ supergravities in $d=8, 9, 10$; $N=2$ in $d=7$ and $N=2$ or $4$ in $d=6$) [9]. Apart from these, there is: (i) A real closed four-form in $d=11$, $N=1$ supergravity, (ii) a real closed three-form in non-chiral $d=10, N=2$ supergravity, (iii) a complex closed three-form in chiral $d=10, N=2$ supergravity.

Excluding Yang-Mills couplings, as is well known, closed super three-forms exist in $d=3, 4, 6,$ and $10$.

Considering the case of the membranes, from the above list it follows that the candidate dimensions are 7 and 11. Since the superspace formulation of $d=7, N=2$ supergravity is not known at present, we are led to consider the supermembrane propagating in eleven-dimensional spacetime.

Our main result is the construction of an action which describes a consistent coupling of $d=11$ supergravity to a supermembrane. In particular the Kalb-Ramond-like third rank antisymmetric tensor field of $d=11$ supergravity is needed for the superspace description of $d=11$ supergravity [10].

In the following we focus our attention on the description of the supermembrane action in $d=11$. The extension to the case of $n$-extended objects is given in the appendix.

2. We propose the following action for a supermembrane coupled to $d=11$ supergravity:

$$S = \int d^2 \xi \left( \frac{1}{2} \sqrt{-g} \, g^{ij} E_i^a E_j^b \eta_{ab} \right.$$

$$+ \epsilon^{ijk} E_i^a E_j^b E_k^c B C_{RBA} - \frac{1}{2} \sqrt{-g} \right).$$

Here $i=0, 1, 2$ labels the coordinates $\xi^i = (\tau, \sigma, \rho)$ of the world volume with metric $g_{ij}$ and signature $(-, +, +)$. The super three-form $B$ is needed for the superspace description of $d=11$ supergravity [10]. For the Levi-Civita symbol $\epsilon^{ijk}$ we use the same conventions as in ref. [11]. In (3) we have used the notation

$$E_i^a = (\partial_i Z^M) E_M^a,$$

where $Z^M(\xi)$ are the superspace coordinates, and $E_M^a(Z)$ is the supervielbein.

Note that the action has a cosmological constant with a fixed magnitude. This is so that the field equation of the metric $g_{ij}$ gives the embedding equation

$$g_{ij} = \epsilon^{ij} E_i^a E_j^b \eta_{ab} \equiv T_{ij}.$$  

We require that the action $S$ is invariant under a fermionic gauge transformation of the form [5]

$$\delta E^a = 0, \quad \delta g^\alpha = (1 + \Gamma^\alpha_\beta) \kappa^\beta,$$

$$\delta g_{ij} = 2 [X_{ij} - g_{ij} X^{kl} (n-1)]$$

$$(n=2 \text{ for membrane}),$$

where the transformation parameter $\kappa^\alpha(\xi)$ is a 32 component Majorana spinor, and a world-volume scalar, and

$$\delta E^a = \delta Z^M E_M^a,$$

$$\Gamma^\alpha_\beta = (1/6 \sqrt{-g}) \epsilon^{ijk} E_i^a E_j^b E_k^c (\gamma_{abc})^\alpha_\beta - \frac{1}{2} \delta^\alpha_\beta.$$ (8)

Here $\gamma^\alpha(a=0, 1, ..., 10)$ are the Dirac matrices in eleven dimensions. $X_{ij}$ is a function of $E_i^a$ which will be determined by the invariance of the action. The choice of $\delta g_{ij}$ is due to the fact that, given a variation of the action of the form $\delta S = T_{ij} X^{ij}$, and writing this variation as

$$T_{ij} X^{ij} = g_{ij} X^{ij} + (T_{ij} - g_{ij}) X^{ij},$$

the second term on the right-hand side cancels $(\delta S/\delta g_{ij}) \delta g_{ij}$. Thus we are left with the first term on the right-hand side, which equals the left-hand side upon the use of (5). Effectively, this means that whenever we encounter a variation of the form $T_{ij} X^{ij}$, we can use eq. (5), provided that we add $X^{ij}$ to $\delta g_{ij}$ as in (6).

The matrix $\Gamma^\alpha_\beta$ occurring in (8) satisfies the property

$$\Gamma^\alpha_\beta \Gamma^\beta_\delta = (1 + \Gamma^\alpha_\beta) \kappa^\beta \\ \delta^\alpha_\delta = \Gamma^2 \delta^\alpha_\delta.$$ (10)

The normalization in (8) is chosen such that upon the use of the equation $T_{ij} = g_{ij}$, the matrix $\Gamma^\alpha_\beta$ satisfies the relation $\Gamma^\alpha_\beta \Gamma^\beta_\gamma = \delta^\alpha_\gamma$. 

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Now using (6) the variation of the action (3) is (we consider a closed supermembrane and therefore discard the surface terms)

$$\delta S = \int d^2 \xi \left[ \sqrt{-g} g^{ij} \left( - \delta E^\beta E_i^c T^a_{\alpha \beta} \right) E_{ja} 
+ \sqrt{-g} g^{ij} \left( - \delta E^\beta E_i^c T^a_{\alpha \beta} \right) E_{ja} 
+ \epsilon^{jk} E_i^A E_j^B E_k^C \delta E^\alpha H_{\alpha CB} 
- \frac{1}{2} \sqrt{-g} \delta g^{ij} (T^{ij} - \frac{1}{2} g^{ij} T - \frac{1}{2} g^{ij}) \right]. \quad (11)$$

The torsion two-form $T^A$ and the four-form field strength $H$ are defined by (our superspace conventions are those of Howe [12])

$$T^A = dE^\alpha + E^B \Omega^A_B = \frac{1}{2} E^\alpha E^C T^A_{BCD},$$

$$H = dB = E^D E^C E^E H_{ABCD} \cdot (12)$$

We now organize the terms in (11) according to the number of one-forms $E^\alpha$ they contain. Those with three $E^\alpha$ and two $E^\alpha$ come only from the Wess-Zumino term. They must vanish separately, and this requires the constraints

$$H_{\alpha \beta \gamma \delta} = H_{\alpha \beta \gamma \delta} = 0. \quad (13)$$

The cancellation of the terms linear in $E^\alpha$ lead to the constraints

$$T^a_{\alpha \beta} = (\gamma^a)_{\alpha \beta}, \quad (14)$$

$$H_{ab} = - \frac{1}{2} (\gamma_{ab})_{\alpha \beta}, \quad (15)$$

while the cancellation of the terms not containing $E^\alpha$ require the constraint

$$\eta_{(a} T^b)_{\alpha \beta} = \eta_{ab} A_{\alpha \beta}, \quad (16)$$

$$H_{abc} = - \frac{1}{2} A_B (\gamma_{abc})_{\gamma \delta}, \quad (17)$$

Here $A_\alpha$ is an arbitrary spinor superfield which is vanishing in $d=11$ [10].

It is important to realize that in obtaining (14)–(17) we have used the identity

$$\delta E^\alpha = \Gamma^\alpha_{\beta} \delta E^\beta + (1 - \Gamma^2) \kappa^\alpha. \quad (18)$$

Using this identity in the variation of the kinetic term, the terms arising from $\Gamma^\alpha_{\beta}$ in (18) can be shown to cancel similar terms coming from the variation of the Wess-Zumino term, modulo terms which cancel by an appropriate variation of $g_{ij}$. [Using the argument below (8) once.] In the remaining terms coming from $(1 - \Gamma^2)$, we use the argument given below (8) repeatedly to compute further contributions to $\delta g_{ij}$. Thus we find the result

$$X_{ij} = - \frac{1}{4} \epsilon_i^{kl} E_k^a E_l^b (\gamma_{ab})_{\alpha \beta} \delta E^\alpha E_j^\alpha$$

$$+ \frac{1}{2} \kappa^\beta E_n (\gamma^d)_{\alpha \beta} E_m g_{ij} (T^k T^l + \delta^k T^l),$$

$$+ i \leftrightarrow j. \quad (19)$$

In summary, the action (3) is invariant under (6) provided that (13)–(17) hold, and $X_{ij}$ is given by (19). In addition, the following Bianchi identities must hold:

$$DT^A = - E^B \wedge R_B A, \quad DH = 0. \quad (20)$$

The generalization of the results of this section to the general case of $n$-extended objects in $d$-dimensional supergravity is straightforward. The result is given in the appendix.

3. We observe that the superspace constraints of $d=11$ supergravity given by Cremmer and Ferrara [10] and Brink and Howe [10] do provide a solution to (13)–(17) and the Bianchi identities (20), with $A_\alpha = 0$.

In conclusion, we have shown that there exists a consistent coupling of a closed supermembrane to eleven-dimensional supergravity. (Note that it is natural to consider a closed supermembrane in eleven dimensions, since there are no matter multiplets in this dimension).

4. There are several directions in which the present work can be extended. We shall name a few.

Firstly, it is of interest to study the quantization of the supermembrane model in eleven dimensions. In particular, the question of whether massless gauge fields can possibly arise is a challenging one. Although usually one encounters difficulties in finding massless excitations of a membrane [13], it is encouraging that, here, we have a spacetime supersymmetric membrane action.

Secondly, it is natural to consider the dimensional reduction of our model from eleven- to ten-dimensional spacetime, and at the same time from three-dimensional world volume to a two-dimensional.
world sheet. It would be interesting to see what kind of $d=10$ string theories could possibly emerge in an infinitely thin membrane limit.

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Appendix. In this appendix we construct the action for an $n$-extended object propagating in $d$-dimensional supergravity background. We also give the transformation rules, and the constraints on the background.

The action is

$$S = \int d^2 \xi \left[ \frac{1}{2} \sqrt{-g} \left( \mathcal{L} + \mathcal{L}_{\text{int}} \right) \right]$$

where

$$\mathcal{L} = \frac{1}{2} \sqrt{-g} \left( \frac{\partial \phi}{\partial x^a} \right)^2 - \frac{1}{16} F_{ab} F^{ab} - \frac{1}{4} \sqrt{-g} \left( \frac{\partial \phi}{\partial x^a} \right)^2$$

and

$$\mathcal{L}_{\text{int}} = \int d^4 x \sqrt{-g} \left( \frac{1}{2} \left( \frac{\partial \phi}{\partial x^a} \right)^2 - \frac{1}{16} F_{ab} F^{ab} - \frac{1}{4} \sqrt{-g} \left( \frac{\partial \phi}{\partial x^a} \right)^2 \right)$$

The transformation rules are those in (6), where the matrix $\Gamma^a_{\beta}$ is now given by

$$\Gamma^a_{\beta} = \frac{\eta \sqrt{-g}}{(n+1)!}$$

where $\eta$ is given by

$$\eta = \frac{(-1)^{(n+1)(n-2)/4}}{\sqrt{-g}}$$

Invariance of the action (A1) is ensured by imposing the following set of constraints:

$$T^a_{\alpha\beta} = \left( \frac{\partial}{\partial x^a} \right)^b \eta_{(a} T^c_{b \alpha \beta} = \eta_{ab} A_{\alpha}$$

$$H_{\alpha\beta\gamma\delta} = \left( \frac{\partial}{\partial x^a} \right)^b \eta_{(a} H_{\delta\gamma\beta\delta} = \eta_{ab} A_{\alpha}$$

$$H_{\alpha\beta\gamma\delta\epsilon} = \left( \frac{\partial}{\partial x^a} \right)^b \eta_{(a} H_{\delta\gamma\beta\delta\epsilon} = \eta_{ab} A_{\alpha}$$

References