SUPERMEMBRANES AND ELEVEN-DIMENSIONAL SUPERGRAVITY

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Received 6 February 1987

We construct an action for a supermembrane propagating in \( d = 11 \) supergravity background. Using the constraints of \( d = 11 \) curved superspace, we show that the action is invariant under Siegel-type transformations recently generalized by Hughes, Li and Polchinski. The transformation parameter is a world-volume scalar and \( \psi = 11 \) spacetime spinor. We also discuss the general problem of the coupling of \( n \)-dimensional extended objects to \( d \)-dimensional supergravity.

I. Now that we have become accustomed to the notion that strings should replace particles, it is natural to investigate the properties of higher-dimensional extended objects, in particular of membranes since they are the simplest extended objects, and they might describe strings in an appropriate limit.

In 1962 Dirac [1] put forward a theory of an extended electron based on the idea of a relativistic membrane. In 1976, Collins and Tucker [2] studied the classical and quantum mechanics of free relativistic membranes. A year later a locally supersymmetric and reparametrization-invariant action for a spinning membrane was constructed by Howe and Tucker [3]. The action describes anti de Sitter supergravity coupled to a number of scalar multiplets in three dimensions. It is the membrane analog of the Neveu-Schwarz-Ramond formulation of the spinning string theory.

More progress towards the construction of a membrane theory was made by Sugamoto [4] in 1983. More recently, Hughes, Li and Polchinski [5] have constructed a Green-Schwarz-type action for a three-extended object propagating in flat six-dimensional spacetime. The consistency of the action requires the existence of a closed superspace five-form, in analogy with the Henneaux and Mezincescu [6] construction for the Green-Schwarz superstring action [7], where a closed superspace three-form is required. The novel feature of the theory of Hughes et al. is that the parameter of the Siegel-type transformation [8] is a scalar rather than a vector on the world volume.

The generalization of the Hughes et al. model to \( n \)-extended objects propagating in flat \( d \)-dimensional superspace is evident. All that is required is the existence of a closer super \((n+2)\)-form given by

\[
H = E^a E^a E^a ... E^a (\gamma_{a1...a_n})_{0\theta},
\]

where \((E^a, E^a)\) are the basis one-forms in superspace. This form is closed provided that the following \(\Gamma\)-matrix identity holds:

\[
(\gamma^a)_{0\theta} (\gamma_{a1...a_{n-1}})_{\gamma_{0}} = 0.
\]

The purpose of this note is to construct Hughes et al. \( \psi \)-type actions describing the propagation of an \( n \)-extended object in \( d \)-dimensional \( \textit{curved} \) superspace. We give a general formula for the action and the transformation rules, whose consistency requires, among other things (see below), the existence of a closed \((n+2)\)-form in curved superspace. Thus we
expect that the \( n \)-extended objects under considera-
tion can consistently propagate only in \( d \leq 11 \) super-
gravities whose superspace formulation involves a closed \((n+2)\)-form. We further expect that such forms exist in supergravity theories in which a closed bosonic \((n+2)\)-form occurs. As far as we know, the following possibilites exist (we include the Yang–
Mills couplings whenever possible):

The dual formulation of \( d=10, N=1 \) supergravity
involves a closed seven-form. Its dimensional reduc-
tion on a \((10-d)\)-dimensional torus leads to real
closed \((d-3)\)-forms in \( d \)-dimensional supergravi-
ties. (These are \( N=1 \) supergravities in \( d=8, 9, 10;\)
\( N=2 \) in \( d=7 \) and \( N=2 \) or 4 in \( d=6 \)) [9]. Apart from
these, there is: (i) A real closed four-form in \( d=11,\)
\( N=1 \) supergravity, (ii) a real closed three-form in
non-chiral \( d=10, N=2 \) supergravity, (iii) a complex
closed three-form in chiral \( d=10, N=2 \) or 4 supergravity.

Excluding Yang–Mills couplings, as is well known,
closed super three-forms exist in \( d=3, 4, 6, \) and 10.

Considering the case of the membranes, from the
above list it follows that the candidate dimensions
are 7 and 11. Since the superspace formulation
of \( d=7, N=2 \) supergravity is not known at present, we
are led to consider the supermembrane propagating
in eleven-dimensional spacetime.

Our main result is the construction of an action
which describes a consistent coupling of \( d=11 \)
supergravity to a supermembrane. In particular the
Kalb–Ramond-like third rank asymmety tensor
field of \( d=11 \) supergravity [10]. In the following we focus our attention on the
description of the supermembrane action in \( d=11 \).

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description of the supermembrane action in \( d=11 \).
The extension to the case of \( n \)-extended objects is
given in the appendix.

2. We propose the following action for a super-
membrane coupled to \( d=11 \) supergravity:

\[
S = \int d^2 \xi \left( \frac{1}{2} \sqrt{-g} \, g^{\mu \nu} \partial_\mu \xi^\nu \partial_\lambda \eta_{\lambda \beta} + \epsilon^{012} \xi^\mu \partial_\mu \xi^\nu \partial_\lambda \eta_{\lambda \beta} \right).
\]

Here \( i = 0, 1, 2 \) labels the coordinates \( \xi^i = (\tau, \sigma, \rho) \) of
the world volume with metric \( g_{ij} \) and signature
(−, +, +). The super three-form \( B \) is needed for the
superspace description of \( d=11 \) supergravity [10].

For the Levi-Civita symbol \( \epsilon^{012} \) we use the same con-
ventions as in ref. [11]. In (3) we have used the notation

\[
E_i^A = (\partial_i Z^M) E_M^A,
\]

where \( Z^M(\xi) \) are the superspace coordinates, and
\( E_M^A(Z) \) is the supervielbein.

Note that the action has a cosmological constant
with a fixed magnitude. This is so that the field equa-
tion of the metric \( g_{ij} \) gives the embedding equation

\[
g_{ij} = E_i^a E_j^b \eta_{ab} = T_{ij}.
\]

We require that the action \( S \) is invariant under a
fermionic gauge transformation of the form [5]

\[
\delta E^a = 0, \quad \delta E^a = (1 + \Gamma^\alpha_\beta) \kappa^a,
\]

\[
\delta g_{ij} = 2 \left[ X_{ij} - g_{ij} \gamma^c \gamma^f / (n-1) \right]
\quad (n=2 \text{ for membrane}),
\]

where the transformation parameter \( \kappa^a(\xi) \) is a 32
component Majorana spinor, and a world-volume scalar, and

\[
\delta E^a = \delta Z^M E_M^A,
\]

\[
\Gamma^\alpha_\beta = (1/6 \sqrt{-g}) \epsilon^{012} E_i^a E_j^b E_k^c \gamma^{abc} \kappa^a_\beta.
\]

Here \( \gamma^a(a=0, 1, \ldots, 10) \) are the Dirac matrices in
eleven dimensions. \( X_{ij} \) is a function of \( E_i^a \) which will
be determined by the invariance of the action. The
choice of \( \delta g_{ij} \) is due to the fact that, given a variation
of the action of the form \( \delta S = \delta T_{ij} X^{ij} \), and writing this
variation as

\[
T_{ij} X^{ij} = g_{ij} X^{ij} + (T_{ij} - g_{ij}) X^{ij},
\]

the second term on the right-hand side cancels
(\( \delta S/\delta g_{ij} \)) \( \delta g_{ij} \). Thus we are left with the first term on
the right-hand side, which equals the left-hand side
upon the use of (5). Effectively, this means that
whenever we encounter a variation of the form \( T_{ij} X^{ij} \),
we can use eq. (5), provided that we add \( X^{ij} \) to \( \delta g_{ij} \)
as in (6).

The matrix \( \Gamma^\alpha_\beta \) occurring in (8) satisfies the
property

\[
\Gamma^\alpha_\beta \Gamma^\beta_\delta = (T_{ij} T_{ij} T_{jk} T_{kl}) \delta^\alpha_\delta \equiv \Gamma^2 \delta^\alpha_\delta.
\]

The normalization in (8) is chosen such that upon
the use of the equation \( T_{ij} \equiv \delta_{ij} \), the matrix \( \Gamma^\alpha_\beta \) satis-
ifies the relation \( \Gamma^\alpha_\beta \Gamma^\beta_\gamma = \delta^\alpha_\gamma \).
Now using (6) the variation of the action (3) is (we consider a closed supermembrane and therefore discard the surface terms)

\[ \delta S = \int \text{d}^2 \xi \left[ \sqrt{-\text{g}} \, g^{ij}(-\delta E^\beta E_i C T^a g_{ip} E_j a) \right. \]
\[ + \sqrt{-\text{g}} \, g^{ij}(-\delta E^\beta E_i T^{a g} g_{ip} E_j a) \]
\[ + \epsilon^{ijk} E_i E_j B_k C g^{ij} \delta E^\alpha H_{\alpha CBA} \]
\[ - \frac{1}{2} \delta g^{ij}(T^{ij} - \frac{1}{2} g^{ij}T - \frac{1}{4} g^{ij}) \] . \hspace{1cm} (11)

The torsion two-form \( T^A \) and the four-form field strength \( H \) are defined by (our superspace conventions are those of Howe [12])

\[ T^A = \delta E^A + E^B \Omega_{BA} = \frac{1}{2} E^B E^C T_{CB}^A , \]
\[ H = dB = \delta E^A E^B E^C E^D H_{ABCD} . \hspace{1cm} (12) \]

We now organize the terms in (11) according to the number of one-forms \( E^a \) they contain. Those with three \( E^a \) and two \( E^a \) come only from the Wess-Zumino term. They must vanish separately, and this requires the constraints

\[ H_{a b g l d} = H_{a b y f d} = 0 . \hspace{1cm} (13) \]

The cancellation of the terms linear in \( E^a \) lead to the constraints

\[ T_{a b} = (\gamma_{a b}) , \hspace{1cm} (14) \]
\[ H_{a b g a b} = - \frac{1}{2} (\gamma_{a b})_{a b} , \hspace{1cm} (15) \]

while the cancellation of the terms not containing \( E^a \) require the constraint

\[ \eta_{(a} T^{b)}_{a} = \eta_{a b} A_a , \hspace{1cm} (16) \]
\[ H_{a b c d e} = \frac{1}{2} A_B (\gamma_{a b c})_{a b c} . \hspace{1cm} (17) \]

Here \( A_a \) is an arbitrary spinor superfield which is vanishing in \( d=11 \) [10].

It is important to realize that in obtaining (14)–(17) we have used the identity

\[ \delta E^a = \Gamma^a_{\beta} \delta E^\beta + (1 - \Gamma^2) \kappa^a . \hspace{1cm} (18) \]

Using this identity in the variation of the kinetic term, the terms arising from \( \Gamma^a_{\beta} \) in (18) can be shown to cancel similar terms coming from the variation of the Wess-Zumino term, modulo terms which cancel by an appropriate variation of \( g_{ij} \). [Using the argument below (8) once.] In the remaining terms coming from \((1 - \Gamma^2)\), we use the argument given below (8) repeatedly to compute further contributions to \( \delta g_{ij} \). Thus we find the result

\[ X_{ij} = - \frac{1}{2} \epsilon^{ijkl} E_k a E_i \alpha \delta E^\beta E_j ^\alpha \]
\[ + \frac{1}{2} \kappa^g E_n \alpha \delta (\gamma^d)_{a b} E^m \delta g_{ij} (T^{k} T^l \delta g_{kl} + \delta g_{k} T^l \delta g_{l}) \]
\[ + i = j . \hspace{1cm} (19) \]

In summary, the action (3) is invariant under (6) provided that (13)–(17) hold, and \( X^{ij} \) is given by (19). In addition, the following Bianchi identities must hold:

\[ DT^A = - E^B \wedge R_B^A , \hspace{1cm} DH = 0 . \hspace{1cm} (20) \]

The generalization of the results of this section to the general case of \( n \)-extended objects in \( d \)-dimensional supergravity is straightforward. The result is given in the appendix.

3. We observe that the superspace constraints of \( d=11 \) supergravity given by Cremmer and Ferrara [10] and Brink and Howe [10] do provide a solution to (13)–(17) and the Bianchi identities (20), with \( A_a = 0 \).

In conclusion, we have shown that there exists a consistent coupling of a closed supermembrane to eleven-dimensional supergravity. (Note that it is natural to consider a closed supermembrane in eleven dimensions, since there are no matter multiplets in this dimension).

4. There are several directions in which the present work can be extended. We shall name a few.

Firstly, it is of interest to study the quantization of the supermembrane model in eleven dimensions. In particular, the question of whether massless gauge fields can possibly arise is a challenging one. Although usually one encounters difficulties in finding massless excitations of a membrane [13], it is encouraging that, here, we have a spacetime supersymmetric membrane action.

Secondly, it is natural to consider the dimensional reduction of our model from eleven- to ten-dimensional spacetime, and at the same time from three-dimensional world volume to a two-dimensional
world sheet. It would be interesting to see what kind of $d=10$ string theories could possibly emerge in an infinitely thin membrane limit.

P.K.T. would like to thank Professor Abdus Salam for his kind hospitality and ICTP in Trieste where this work was carried out.

Appendix. In this appendix we construct the action for an n-extended object propagating in d-dimensional supergravity background. We also give the transformation rules, and the constraints on the background.

The action is

$$S = \int d^2 \xi \left[ \frac{1}{2} \sqrt{-g} \right. g^{i\dot{i}} e^i_\alpha e^\dot{i}_\beta \eta_{\alpha\beta}$$

$$+ \epsilon^{i_1 \ldots i_n} E_{i_1}^{A_1} \ldots E_{i_{n+1}}^{A_{n+1}} B_{A_{n+1}}^{A_1 \ldots A_1}$$

$$- \frac{1}{4} (n-1) \sqrt{-g} \left. \right] . \tag{A1}$$

The transformation rules are those in (6), where the matrix $\Gamma^a_\beta$ is now given by

$$\Gamma^a_\beta = \left[ \eta/(n+1)! \right] \sqrt{-g}$$

$$\times \epsilon^{i_1 \ldots i_{n+1}} E_{i_1}^{a_1} \ldots E_{i_{n+1}}^{a_{n+1}} (\gamma_{a_1 \ldots a_{n+1}})_{\alpha}^\beta \tag{A2}$$

where $\eta$ is given by

$$\eta = (-1)^{(n+1)(n-2)/4} . \tag{A3}$$

Invariance of the action (A1) is ensured by imposing the following set of constraints:

$$T^a_{\alpha\beta} = (\gamma^a)_{\alpha\beta}$$

$$\eta_{(a} T^e_{\beta)\alpha} = \eta_{\alpha\beta} A_{\alpha} \tag{A4}$$

$$H_{\alpha\beta \gamma a_1 \ldots a_{n+1}} = (\eta/n!) A_{\beta} (\gamma_{a_1 \ldots a_{n+1}})_{\alpha \gamma}$$

$$H_{\alpha\beta a_n+1 \ldots a_1} = [\eta (-1)^n/(n+1)!] (\gamma_{a_1 \ldots a_{n+1}})_{\alpha \beta} \tag{A5}$$

$$H_{\alpha\beta \gamma a_1 \ldots a_{n+1}} = 0 , \tag{A5cont'd}$$

and by taking $X_{ij}$ occurring in (6) to be

$$X_{ij} = (-\eta/2n!)$$

$$\times \epsilon_{i_1 k_1 \ldots k_n} E_{i_1}^{a_1} \ldots E_{i_n}^{a_n} (\gamma_{a_1 \ldots a_n})_{\alpha\beta} \delta E^\beta E^\alpha$$

$$+ \frac{1}{2} \kappa^\beta \left[ E_n^a (\gamma^a)_{\alpha\beta} E^{ma} + (n+1) A_{\beta} \right]$$

$$\times g_{ij} (T^{k_1} \ldots T^{k_n})_{a_{n+1}} + \delta^{k_1} \delta^{k_2} \ldots T^{k_n}$$

$$+ \delta^{k_1} \delta^{k_2} \ldots \delta_{k_{n+1}} \ldots (T^{k_n}$$

$$- \frac{1}{4} \delta E^\beta A_{\beta} g_{ij} + (i=j) . \tag{A6}$$

References

[13] K. Kikkawa and M. Yamasaki, Can the membrane be a uni-

ification model?, Osaka University preprint (August 1986).