We construct an action for a supermembrane propagating in $d=11$ supergravity background. Using the constraints of $d=11$ curved superspace, we show that the action is invariant under Siegel-type transformations recently generalized by Hughes, Li and Polchinski. The transformation parameter is a world-volume scalar and $d=11$ spacetime spinor. We also discuss the general problem of the coupling of $n$-dimensional extended objects to $d$-dimensional supergravity.

1. Now that we have become accustomed to the notion that strings should replace particles, it is natural to investigate the properties of higher-dimensional extended objects, in particular of membranes since they are the simplest extended objects, and they might describe strings in an appropriate limit.

In 1962 Dirac [1] put forward a theory of an extended electron based on the idea of a relativistic membrane. In 1976, Collins and Tucker [2] studied the classical and quantum mechanics of free relativistic membranes. A year later a locally supersymmetric and reparametrization-invariant action for a spinning membrane was constructed by Howe and Tucker [3]. The action describes anti de Sitter supergravity coupled to a number of scalar multiplets in three dimensions. It is the membrane analog of the Neveu–Schwarz–Ramond formulation of the spinning string theory.


The generalization of the Hughes et al. model to $n$-extended objects propagating in flat $d$-dimensional superspace is evident. All that is required is the existence of a closed super $(n+2)$-form given by

$$H = E^a E^a E^a ... E^a (\gamma_{a_1 ... a_n},)_{a_\theta},$$

(1)

where $(E^a, E^a)$ are the basis one-forms in superspace. This form is closed provided that the following $\Gamma$-matrix identity holds:

$$(\gamma_{a_1 ... a_n})_{a_\theta} (\gamma_{a_1 ... a_n})_{a_\theta} = 0.$$  

(2)

The purpose of this note is to construct Hughes et al. -type actions describing the propagation of an $n$-extended object in $d$-dimensional curved superspace. We give a general formula for the action and the transformation rules, whose consistency requires, among other things (see below), the existence of a closed $(n+2)$-form in curved superspace. Thus we
expect that the \( n \)-extended objects under consideration can consistently propagate only in \( d \leq 11 \) supergravities whose superspace formulation involves a closed \((n+2)\)-form. We further expect that such forms exist in supergravity theories in which a closed bosonic \((n+2)\)-form occurs. As far as we know, the following possibilities exist (we include the Yang-Mills couplings whenever possible):

The dual formulation of \( d=10, N=1 \) supergravity involves a closed seven-form. Its dimensional reduction on a \((10-d)\)-dimensional torus leads to real closed \((d-3)\)-forms in \( d \)-dimensional supergravities. (These are \( N=1 \) supergravities in \( d=8, 9, 10 \); \( N=2 \) in \( d=7 \) and \( N=2 \) or \( 4 \) in \( d=6 \)) [9]. Apart from these, there is: (i) A real closed four-form in \( d=11, N=1 \) supergravity, (ii) a real closed three-form in non-chiral \( d=10, N=2 \) supergravity, (iii) a complex closed three-form in chiral \( d=10, N=2 \) or \( 4 \) supergravity.

Excluding Yang-Mills coupling, as is well known, closed super three-forms exist in \( d=3, 4, 6, \) and \( 10 \).

Considering the case of the membranes, from the above list it follows that the candidate dimensions are 7 and 11. Since the superspace formulation of \( d=7, N=2 \) supergravity is not known at present, we are led to consider the supermembrane propagating in eleven-dimensional spacetime.

Our main result is the construction of an action which describes a consistent coupling of \( d=11 \) supergravity to a supermembrane. In particular the Kalb-Ramond-like third rank antisymmetric field of \( d=11 \) supergravity [10] is needed for the superspace description of \( d=11 \) supergravity. In the following we focus our attention on the description of the supermembrane action in \( d=11 \). The extension to the case of \( n \)-extended objects is given in the appendix.

2. We propose the following action for a supermembrane coupled to \( d=11 \) supergravity:

\[
S = \int d^2 \xi \left( \frac{1}{2} \sqrt{-g} \, g^{ij} E_i^a E_j^b \eta_{ab} + \epsilon^{ijk} E_i^a E_j^b E_k^c B_{cba} - \frac{1}{2} \sqrt{-g} \right).
\]

Here \( i=0, 1, 2 \) labels the coordinates \( \xi^i = (\tau, \sigma, \rho) \) of the world volume with metric \( g_{ij} \) and signature \((-,-,+,-)\). The super three-form \( B \) is needed for the superspace description of \( d=11 \) supergravity [10]. For the Levi-Civita symbol \( \epsilon^{ijk} \) we use the same conventions as in ref. [11]. In (3) we have used the notation

\[
E_i^a = (\partial_i Z^M) E_M^a, \tag{4}
\]

where \( Z^M(\xi) \) are the superspace coordinates, and \( E_M^a(Z) \) is the supervielbein.

Note that the action has a cosmological constant with a fixed magnitude. This is so that the field equation of the metric \( g_{ij} \) gives the embedding equation

\[
g_{ij} = E_i^a E_j^b \eta_{ab} \equiv T_{ij}. \tag{5}
\]

We require that the action \( S \) is invariant under a fermionic gauge transformation of the form [5]

\[
\delta E^a = 0, \quad \delta E^a = (1 + \Gamma)^\alpha_\beta \kappa^\beta, \]

\[
\delta g_{ij} = 2 \left[ X_{ij} - g_{ij} X^k X_k^{(n-1)} \right] \quad (n=2 \text{ for membrane}), \tag{6}
\]

where the transformation parameter \( \kappa^\alpha(\xi) \) is a 32 component Majorana spinor, and a world-volume scalar, and

\[
\delta E^a = \delta Z^M E_M^a, \tag{7}
\]

\[
\Gamma^\alpha_\beta = (1/6 \sqrt{-g}) \epsilon^{ijk} E_i^a E_j^b E_k^c (\gamma_{abc})^\alpha_\beta. \tag{8}
\]

Here \( \gamma^a(a=0, 1, \ldots, 10) \) are the Dirac matrices in eleven dimensions. \( X_{ij} \) is a function of \( E_i^a \) which will be determined by the invariance of the action. The choice of \( \delta g_{ij} \) is due to the fact that, given a variation of the action of the form \( \delta S = T_{ij} \delta X^i \), and writing this variation as

\[
T_{ij} X^i = g_{ij} X^i + (T_{ij} - g_{ij}) X^i, \tag{9}
\]

the second term on the right-hand side cancels \( \delta S = (\delta g_{ij}) \delta X^i \). Thus we are left with the first term on the right-hand side, which equals the left-hand side upon the use of (5). Effectively, this means that whenever we encounter a variation of the form \( T_{ij} X^i \), we can use eq. (5), provided that we add \( X^i \) to \( \delta X^i \) as in (6).

The matrix \( \Gamma^\alpha_\beta \) occurring in (8) satisfies the property

\[
\Gamma^\alpha_\beta \Gamma^\beta_\delta = (T^i_\tau T^i_\tau T^k_\eta) \delta^\alpha_\delta \equiv \Gamma^2 \delta^\alpha_\delta. \tag{10}
\]

The normalization in (8) is chosen such that upon the use of the equation \( T_{ij} = g_{ij} \), the matrix \( \Gamma^\alpha_\beta \) satisfies the relation \( \Gamma^\alpha_\beta \Gamma^\beta_\gamma = \delta^\alpha_\gamma \).
Now using (6) the variation of the action (3) is (we consider a closed supermembrane and therefore discard the surface terms)

\[ \delta S = \int \, d^2 \xi \left( \sqrt{-g} \, g^{ij} \left( -\delta E^\beta E_i \gamma T^a \gamma^\beta \right) E_{\mu} + \right. \\
+ \sqrt{-g} \, g^{ij} \left( -\delta E^\beta E_i \gamma T^a \gamma^\beta \right) E_{\mu} \\
+ \epsilon^{ijk} E_i \gamma^a E_j \gamma^b C \delta E^\alpha H_{\alpha C B A} \\
- \frac{1}{2} \sqrt{-g} \, g^{ij} \left( T^a - \frac{1}{2} g^{ij} T - \frac{1}{2} g^{ij} \right) \right]. \quad (11) \]

The torsion two-form \( T^A \) and the four-form field strength \( H \) are defined by (our superspace conventions are those of Howe [12])

\[ T^A = dE^A + E^B \Omega_B^A = \frac{1}{2} E^B E^C T_{CB} \, A, \]

\[ H = dB = \epsilon^{EDE} C ESEA HASCD. \quad (12) \]

We now organize the terms in (11) according to the number of one-forms \( E^a \) they contain. Those with three \( E^a \) and two \( E^a \) come only from the Wess-Zumino term. They must vanish separately, and this requires the constraints

\[ H_{a\beta \gamma \delta} = H_{a\beta \gamma \delta} = 0. \quad (13) \]

The cancellation of the terms linear in \( E^a \) lead to the constraints

\[ T^a_{\alpha \beta} = (\gamma^a)_{\alpha \beta}, \quad (14) \]

\[ H_{a\beta \alpha} = -\frac{1}{2} (\gamma_{ab})_{\alpha \beta}, \quad (15) \]

while the cancellation of the terms not containing \( E^a \) require the constraint

\[ \eta_{(a} T^c_{b) \alpha} = \eta_{ab} A_{\alpha}, \quad (16) \]

\[ H_{a\beta \alpha} = -\frac{1}{2} A_\beta (\gamma_{abc}) \beta^\alpha. \quad (17) \]

Here \( A_\alpha \) is an arbitrary spinor superfield which is vanishing in \( d=11 \) [10].

It is important to realize that in obtaining (14)–(17) we have used the identity

\[ \delta E^a = \Gamma^a_{\beta \gamma} \delta E^\beta + (1 - \Gamma^2) \kappa^a. \quad (18) \]

Using this identity in the variation of the kinetic term, the terms arising from \( \Gamma^a_{\beta \gamma} \) in (18) can be shown to cancel similar terms coming from the variation of the Wess-Zumino term, modulo terms which cancel by an appropriate variation of \( g_{ij} \). [Using the argument below (8) once.] In the remaining terms coming from \( (1 - \Gamma^2) \), we use the argument given below (8) repeatedly to compute further contributions to \( \delta g_{ij} \). Thus we find the result

\[ X_{ij} = -\frac{1}{4} \epsilon_{ijkl} E_k E_l E_i \gamma^a \delta E^\beta E_j^a \]

\[ + \frac{1}{2} \kappa^a E_n (\gamma^d)_{\alpha \beta} E^m g_{ij} (T^k \gamma^a T^l), \]

\[ + i \leftrightarrow j. \quad (19) \]

In summary, the action (3) is invariant under (6) provided that (13)–(17) hold, and \( X_{ij} \) is given by (19). In addition, the following Bianchi identities must hold:

\[ DT^A = -E^B \wedge R_B^A, \quad DH = 0. \quad (20) \]

The generalization of the results of this section to the general case of \( n \)-extended objects in \( d \)-dimensional supergravity is straightforward. The result is given in the appendix.

3. We observe that the superspace constraints of \( d=11 \) supergravity given by Cremmer and Ferrara [10] and Brink and Howe [10] do provide a solution to (13)–(17) and the Bianchi identities (20), with \( A_\alpha = 0 \).

In conclusion, we have shown that there exists a consistent coupling of a closed supermembrane to eleven-dimensional supergravity. (Note that it is natural to consider a closed supermembrane in eleven dimensions, since there are no matter multiplets in this dimension).

4. There are several directions in which the present work can be extended. We shall name a few.

Firstly, it is of interest to study the quantization of the supermembrane model in eleven dimensions. In particular, the question of whether massless gauge fields can possibly arise is a challenging one. Although usually one encounters difficulties in finding massless excitations of a membrane [13], it is encouraging that, here, we have a spacetime supersymmetric membrane action.

Secondly, it is natural to consider the dimensional reduction of our model from eleven- to ten-dimensional spacetime, and at the same time from three-dimensional world volume to a two-dimensional
world sheet. It would be interesting to see what kind of \( d = 10 \) string theories could possibly emerge in an infinitely thin membrane limit.

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Appendix. In this appendix we construct the action for an \( n \)-extended object propagating in \( d \)-dimensional supergravity background. We also give the transformation rules, and the constraints on the background.

The action is

\[
S = \int d^2 x \left[ \frac{1}{2} \sqrt{-g} g^{\mu \nu} E^a_{\nu} E^a_{\mu} \eta_{a b} \right. \\
+ \epsilon^{i_1 \ldots i_{n+1}} E_{i_1}^{a_1} \ldots E_{i_{n+1}}^{a_{n+1}} B_{a_1 \ldots a_{n+1}} \\
- \frac{1}{2} (n-1) \sqrt{-g} \right].
\]

The transformation rules are those in (6), where the matrix \( \Gamma^a_\beta \) is now given by

\[
\Gamma^a_\beta = \left[ \frac{\eta(n+1)!}{\eta} \right] \sqrt{-g} \\
\times \epsilon^{i_1 \ldots i_{n+1}} E_{i_1}^{a_1} \ldots E_{i_{n+1}}^{a_{n+1}} (\gamma_{a_1 \ldots a_{n+1}})^{a_\beta}
\]

where \( \eta \) is given by

\[
\eta = (-1)^{n+1} \eta_n (n-2)/4.
\]

Invariance of the action (A1) is ensured by imposing the following set of constraints:

\[
T^a_{\alpha \beta} = (\gamma^a)_{\alpha \beta}, \quad \eta_{(a} T^a_{b) \gamma} = \eta_{a b} \gamma_a
\]

\[
H_{a a_1 \ldots a_n} = (\eta(n+1)! A_\beta (\gamma_{a_1 \ldots a_n})_\alpha)_{\beta \alpha}
\]

\[
H_{a a_1 \ldots a_n} = (\eta(-1)^n(n+1)! (\gamma_{a_1 \ldots a_n})_{a \beta})_{\alpha \beta}
\]

\[
H_{a a_1 \ldots a_n} = 0,
\]

and by taking \( X_\mu \) occurring in (6) to be

\[
X_\mu = (-\eta/2n!)
\]

\[
\times \epsilon^{i_1 \ldots i_{n+1}} E_{i_1}^{a_1} \ldots E_{i_{n+1}}^{a_{n+1}} (\gamma_{a_1 \ldots a_{n+1}})_{a \beta} \delta E^\beta E^\alpha
\]

\[
+ \frac{1}{2} \kappa \delta [E_\alpha]_{(a} E_{a \beta)} E^{a a} + (n+1) A_\beta
\]

\[
\times g_{i j} (T^{k_1} \ldots T^{k_{n+1}} a_{k_1} T^{k_2} \ldots T^{k_{n+1}} a_{k_1})
\]

\[
+ \delta^{k_1} T^{k_2} \ldots \delta^{k_{n+1}} k_{n+1} T^{k_n} a_{k_1}
\]

\[
- \frac{1}{2} \delta E^\beta A_\gamma g_{i j} + (i \rightarrow j).
\]

References