5 PATTERN FORMATION IN PRESTRESSED POLYGONAL FILMS

In this chapter we explore the spontaneous formation of micropatterns in thin pre-stressed polygonal films using finite element simulations. We study films with different size, thickness and shape, including square, rectangular, pentagonal and hexagonal films. Patterns form when the films release the internal eigenstrain by buckling-up, after which the films bond-back to the substrate. After an initial symmetric evolution of the buckling profile, the symmetry of the deflection pattern breaks when the wavelength of wriggles near the film edges decreases. During bond back the deflection morphology converges to a four-fold, five-fold and six-fold ridging pattern for the square, pentagonal and hexagonal films, respectively, showing a close resemblance with experimental film systems of similar size and shape. Rectangular films of large length to width ratio go through a transition in buckling shapes from the initial Euler mode, through the varicose mode into the anti-symmetric telephone-cord mode. For all film shapes, the film height scales with the square-root of the film area and eigenstrain and is independent of thickness. The bond-back mechanism determines the final wrinkle morphology and is governed by the eigenstrain value at the end of the buckling-up stage and the dimensionless parameter \( \Gamma / (E W_{eq}) (W_{eq}/t)^3 \), relating the interface energy to the strain energy in the film.

5.1 Introduction

Wrinkled surfaces are omnipresent in many systems in nature such as vegetables, fruits, leaves and human skin. Such patterns/wrinkles of thin surfaces with controllable dimensions at the micro- and nanometer scale are promising tools for modern technological applications as e.g., microelectromechanical systems (MEMS). In recent times it has been successfully demonstrated that these patterned surfaces can be used as tunable optical gratings [Edmondson and Huck, 2004; Harrison et al., 2004; Xia et al., 1996], replica for microfluidic channels [Edmondson et al., 2006], flexible electronics [Baca et al., 2008; Jiang et al., 2007b; Kim et al., 2008; Lacour et al., 2003; Sun et al., 2006], microfluidic channels [Malachias et al., 2008; Mei et al., 2007], microreactors [Watts and Wiles, 2007], particle separators [Efimenko
Figure 5.1: Patterns formed by the release and bond back of pre-stressed thin films. (a, b) Optical micrographs of fourfold and fivefold ridge patterns formed in square (a) and pentagonal (b) polymer films released by pulse electrolysis (reproduced with permission from Edmondson et al. [2006]). (c) SEM image of a nanochannel network formed by the controlled under-etching of semi-conductor films (reproduced with permission from Malachias et al. [2008]). (d) A zoomed view of one unit-cell of (c).

While nature controls the shape and size of patterns observed in natural systems (vegetables, leaves etc.,) through complex biochemical processes along with mechanical forces, the microsystems mentioned above need sophisticated processes to be developed for better control and reproducibility. In the last two decades, there was considerable effort to develop fabrication methods and the most successful of them are based on the three-step process of conventional lithography, thin film deposition and etching [Haneveld et al., 2006, 2003; Hoang et al., 2009; Stern et al., 1997; Tas et al., 2002]. However, these lithography based
methods are time consuming and they become expensive as the feature size reduces. In the search for alternatives a method based on the self-organization of surface patterns and channels through the buckling of pre-stressed thin films was found to be promising for it does not require pre-patterning at small length scales [Bowden et al., 1998]. Since then, thin film buckling has been explored in a wide range of applications [Cerda and Mahadevan, 2003; Cerda et al., 2002; Fedorchenko et al., 2005; Huang et al., 2007; Moon et al., 2004; Tarasovs and Andersons, 2008; Vella et al., 2009] describing how the -otherwise unwanted- phenomenon of buckling could be used to develop micro- and nanomechanical systems for various technological applications.

Recently, it has been demonstrated that an additional level of complexity can be introduced based on the release and bond-back of pre-stressed thin films [Edmondson et al., 2006; Malachias et al., 2008; Mei et al., 2007], leading to patterns with different morphologies (Fig. 5.1). Figures 5.1(a) and 5.1(b) show optical micrographs of fourfold and fivefold ridge patterns on square and pentagonal films formed by pulse electrolysis proposed by Edmondson and Huck [2004]. In their method, thin polymer (PGMA) brushes are grown and cross-linked on a gold substrate. The film growth process induces eigenstrains in the film resulting in compressive stresses. Then, the film-substrate system is placed in an electrolytic cell and a short electric pulse is given to break the bonds between the film and the substrate, resulting in buckling-driven relaxation of the pre-stressed polymer film followed by bond-back. The final pattern morphology depends strongly on the film shape (lines, square, pentagons, etc., see Edmondson et al. [2006]). Figure 5.1(c) shows a SEM image of a nanochannel network fabricated by Malachias et al. [2008]. In their method semiconductor films are epitaxially grown on sacrificial-layer-covered substrates. Eigenstrains are induced in the film due to lattice mismatch between film and substrate. Circular holes are introduced in the film as etch pits and placed in an etchant solution, leading to a gradual underetching starting from the periphery of the holes. The film releases its eigenstrains by buckling when the sacrificial layer is removed. When the system is left for drying the film bonds back to the substrate due to cohesive film-substrate attraction. For a square pattern of circular etch pits a diagonal buckling pattern develops (Figs. 5.1(c) and 5.1(d)) that shows a clear resemblance with the fourfold ridging pattern for the square polymer films (Fig. 5.1(a)). The objective of this chapter is to understand the mechanics of pattern formation in these pre-stressed thin film systems and study how the film geometry (thickness, size, and shape) and film and interface properties (prestrain, stiffness and interface energy) relate to the final pattern morphology. To do so, we will perform three-dimensional finite element calculations to simulate the buckling, post-buckling and bond back of pre-stressed thin film systems following closely the experimental set up of Edmondson et al. [2006].

The chapter is organized as follows. In section 5.2 the boundary value problem is described followed by a dimensional analysis (see Appendix I) and a detailed discussion of the numerical ingredients. In section 5.3 we present the results of our simulations, in which we subsequently analyze buckling-up and bond-back of square (Sec. 5.3.1), rectangular (Sec. 5.3.2), pentagonal and hexagonal films (Sec. 5.3.3). Finally, in section 5.4, we summarize and conclude the chapter.
5. Pattern formation in prestressed polygonal films

5.2 Problem definition and method

In this section, we define the mechanical boundary value problem to describe the above mentioned mechanism for pattern formation. Figure 5.2 shows the schematic of various stages in the process, which, for the polymer film system [Edmondson and Huck, 2004], can be divided into the following three steps: (i) Film growth and cross linking of the polymer brushes, during which the film is pre-strained with an eigenstrain $\varepsilon^*$ (Fig. 5.2(a)); (ii) release of eigenstrains by buckling of the film into a dome-like shape due to the electrolytic pulse (reducing the interface strength between film and substrate) attaining a central amplitude $H$ (Fig. 5.2(b)) and (iii) bond-back of the buckled-up film (due to cohesive interaction between the film and the substrate) forming a final ridge shape (Fig. 5.2(c)). In the present study we combine the first two steps by loading the film with eigenstrain in the absence of any interface cohesion. In doing so, we neglect all details connected to buckling-driven interface delamination due to the gradual reduction of interface strength, as studied in a two-dimensional setting in chapter 3 [Annabattula et al., 2010a]. In that study it has been shown that the driving force for delamination at the edge of the film drastically reduces, so that full delamination only occurs in the presence of large eigenstrains. We incorporate this observation in the present analysis by simply constraining the edges of the film. We study a film of width $W$ and length $L$ in case of rectangular films, thickness $t$, Young’s modulus $E$, and Poisson’s ratio $\nu$ on a rigid substrate. Also pentagonal and hexagonal films will be analyzed in Sec. 5.3.3, following the same approach as explained here for square and rectangular films. The film is loaded by eigenstrains (from $\varepsilon^* = 0$ to 5%), during which the film first buckles at its characteristic buckling load (or buckling strain) after which it enters into the postbuckling regime with a further increase of the buckling amplitude from almost 0 to $H$ (see Fig. 5.2(b)). Once the eigenstrain loading is completed, the film will bond back to the substrate due to the cohesive attraction between the film and the substrate, based on a cohesive interface law [Xu and Needleman, 1993]. As a result of the film/substrate interaction, the film geometry changes and its height $H$ gradually reduces, reaching a final configuration as shown schematically in Fig. 5.2(c). The von Karman non-linear plate theory as discussed in Ch. 4 (see Sec. 4.2.1) is used with the same governing equations for kinematics and constitutive behaviour for the film and the interface.

To investigate which dimensionless parameters govern the deformation of the film during the eigenstrain loading step, we carry out a dimensional analysis based on the principle of virtual work. From the dimensional analysis (see Appendix I) it turns out that for square films of size $W$ the solution depends solely on the parameter $\varepsilon^*(W/t)^2$ for a given set of boundary conditions and Poisson’s ratio $\nu$.

5.3 Results and discussion

In this section, we analyze the strain-driven pattern evolution in films of different shapes, i.e., square, rectangular, pentagonal and hexagonal films, followed by bond back. To solve the boundary value problem defined in Fig. 5.2 we use the finite-element method [Zienkiewicz and Taylor, 2000]. The film is discretized by four-noded shell elements (S4) [ABAQUS,
5.3. Results and discussion

Figure 5.2: Schematic of the pattern formation mechanism (a) Initial stage, (b) Buckled-up configuration and (c) Configuration after bond-back.
with all boundaries of the film fully constrained and the substrate is modelled as a rigid surface (R3D4).* We assume that the material of the film is linear elastic with an elastic modulus $E = 7 \text{ GPa}$ and Poisson’s ratio $\nu = 0.32$. To trigger buckling the initial configuration of the film is perturbed in the out-of-plane direction with sufficiently small imperfections. A static solution procedure is used during the buckling-up stage (eigenstrain loading) with a stabilization procedure to overcome small local instabilities [ABAQUS, 2007]. The maximum allowable dissipated energy due to stabilization is limited to 3% of the total strain energy of the film. During the buckling-up stage a normal contact condition is enforced between the film and the substrate, which ensures the film not to penetrate into the substrate. The bond-back stage is carried out using an explicit dynamic solution procedure so that the inertia of the film can be used to overcome local instabilities; the loading rate however is kept small to mimic quasi-static loading. We also incorporate a “no-separation” contact condition with rough friction during bond-back so that the nodes of the film that come in contact with the substrate will effectively be anchored from that instant onwards. Furthermore, we use a mass-proportional damping (Rayleigh damping) during bond-back, such that the total energy dissipated is a small fraction (5%) of the total strain energy in the film. During bond-back the film is acted upon by a non-uniformly distributed normal traction based on the cohesive interface law proposed by Xu and Needleman [1993] (see Fig. 2.3(a) and Eq. 4.6). To accomplish this we have implemented a user subroutine VDLOAD [ABAQUS, 2007] to apply the non-uniform traction during the bond-back stage.

### 5.3.1 Square films

In this section, we study various aspects of the pattern evolution during eigenstrain loading followed by bond-back in square films. The film thickness $t$ and film size $W$ will be varied in the range of 20 nm - 50 nm and 2 $\mu$m - 5 $\mu$m, respectively. Figure 5.3 shows the results for a square film of size 5 $\mu$m and thickness 30 nm (reference case). The top row shows the pattern evolution during the buckling-up stage and the bottom row shows the bond-back stage. The contour plots correspond to the vertical position $z$ of the film (see Fig. 5.2). During eigenstrain loading, the film accommodates the strain by in-plane compression. When the eigenstrain reaches a critical strain, the film buckles. The critical buckling load $P_{cr}$ for square films clamped along the edges has been obtained using analytical techniques [Sezawa, 1931; Taylor, 1933] and is given by

$$P_{cr} = 5.3 \frac{D\pi^2}{W^2},$$

(5.1)

with $D = Et^3/12(1-\nu^2)$. By using $\sigma_{cr} = P_{cr}/t$ and $\varepsilon_{cr}^* = \sigma_{cr}(1-\nu)/E$ it follows that

$$\varepsilon_{cr}^* = \frac{5.3\pi^2}{12(1+\nu)} \left(\frac{t}{W}\right)^2.$$

(5.2)

*A gap of 20 nm is maintained between the film and the substrate to avoid instabilities in the static solution procedure during the buckling-up stage.
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Figure 5.3: Evolution of deflection patterns of a square film of width $W = 5 \mu m$ and thickness $t = 30 \text{ nm}$ during the buckling-up (a-c) and bond-back stages (d-f). The maximum eigenstrain in the film is 5% (c) and the bond-back process is started from the configuration at 1.9% eigenstrain (d). The corresponding eigenstrains during the buckling-up stage are (a) 0.3%, (b) 1.78% and (c) 5.0%. The corresponding normalized interface strength ($\sigma_{\text{max}}/E$) during bond back is (d) 0.0, (e) $0.3 \times 10^{-3}$ and (f) $0.5 \times 10^{-3}$.

Figure 5.3(a) shows the out-of-plane position of the film at 0.3% eigenstrain after initial buckling, which happened at $\varepsilon_{\text{cr}}^* \approx 0.011\%$, with a similar buckling profile as in Fig. 5.3(a). It can be observed that the profile is symmetric with respect to a horizontal and vertical axes through the center. At a certain strain level the symmetry is broken, associated with a reduction in wavelength near the boundary, which further decreases with strain (see Fig. 5.3(b) and 5.3(c)). Figure 5.3(d) shows the profile at the beginning of the bond-back process at 1.9% eigenstrain. When the interface strength is increased the film starts to bond back at the center of the four edges (Fig. 5.3(e)), ultimately “freezing” in a diagonal ridge pattern (Fig. 5.3(f)). It can be observed that the final pattern evolved after bond back has a very close resemblance with the experimentally-obtained patterns shown in Fig. 5.1(a).

Figure 5.4 shows the contour plots of the normalized out-of-plane position ($\bar{z} = z/H^*$) during the buckling-up stage for $W = 4 \mu m$ (first row), $3 \mu m$ (second row), and $2 \mu m$ (third row) for a film thickness $t = 30 \text{ nm}$. The contour plots in the first column correspond to an eigenstrain of 1%, the second column corresponds to the strain at the onset of rotation (or break-up of symmetry at the top boundary) and the third column corresponds to an eigenstrain of 5% (maximum strain applied). The break-up of symmetry occurs at $\varepsilon^* = 2.1\%$, 2.1%, and 4.7% for films of size $W = 4$, 3, and 2 $\mu m$, respectively. Clearly, when the film width
Figure 5.4: Evolution of deflection patterns of square films of different size $W$ during the buckling-up stage. The figures (a-c) corresponds to $W = 4 \, \mu m$, (d-f) to $W = 3 \, \mu m$ and (g-i) to $W = 2 \, \mu m$. The thickness of the film in all the cases is 30 nm. The eigenstrains are (a, d, g) 1.0%, (b, e) 2.1%, (h) 4.7% and (c, f, i) 5%. The contours show the out-of plane position of the film ($z$) normalized with $H^* = 500$ nm and $z_{min} = 0.04$. The (maximum) buckling amplitude at 5% eigenstrain is 650 nm, 450 nm, and 320 nm for $W = 4 \, \mu m$, $3 \, \mu m$, and $2 \, \mu m$, respectively.
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reduces, the breaking of symmetry is postponed to larger strain values resulting in more regular patterns at $\varepsilon^* = 5\%$. Figure 5.5 shows the contour plots of the normalized out-of-plane position ($\bar{z} = z/H^*$) during the bond-back stage for the above mentioned cases. The contour plots in the first column corresponds to the configuration at the beginning of the bond-back process (i.e., at the end of 2% eigenstrain during buckling-up). The contour plots in the second column are shown at $\sigma_{\text{max}}/E = 0.3 \times 10^{-3}$ and the third column corresponds to $\sigma_{\text{max}}/E = 0.5 \times 10^{-3}$, the final point of the bond-back simulation. Note that the configuration at 2% strain is symmetric for all widths. It turns out that for $W = 4$ and 3 $\mu$m, the onset of symmetry breaking happens right after 2% (at 2.1%, see Fig. 5.4). As a result, during bond back the system is stable, yielding regular square patterns.

Figure 5.6(a) shows the evolution of buckling amplitude $H$ as a function of normalized eigenstrain $\bar{\varepsilon}^*$ during buckling-up (left hand frame) and as a function of $\sigma_{\text{max}}/E$ during bond-back (right hand frame) for films of size $W = 2$, 3, 4, and 5 $\mu$m with a film thickness of 30 nm. The normalized eigenstrain $\bar{\varepsilon}^*$ is the ratio of eigenstrain $\varepsilon^*$ to the maximum eigenstrain applied which is 1.9% for $W = 5$ $\mu$m film and 2% for remaining cases. As mentioned before, the buckling-up configuration at 2% eigenstrain is the starting point for the bond-back simulation (except for the 5 $\mu$m film for which bond-back is started from the configuration at 1.9% eigenstrain). Clearly, the buckling height $H$ increases with increasing film size $W$, during both stages of the process (buckling-up and bond-back). Next, we take films of size $W = 5$ $\mu$m and change the film thickness ($t = 20, 30$ and 40 nm, see Fig. 5.6(b)). The buckling amplitude during the buckling-up stage appears to be independent of the thickness. During bond-back, however, the effect of the film thickness on the final height can be clearly seen, leading to increasing buckle heights with increasing thickness. Figure 5.7 shows the corresponding deflection patterns during the bond-back stage for films of thickness 20 nm (first row), 30 nm (second row) and 40 nm (third row). The first column shows the buckled-up configuration at 2% eigenstrain (1.9% for Fig. 5.7(d)), which is the initial configuration for the bond-back process. Although the buckling amplitude $H$ for all cases is approximately the same (see Fig. 5.6(b)), the effect of thickness comes in through the breaking of symmetry triggered by a reduction of wavelength near the edges (see Fig. 5.7(a), (d) and (g)). Only for $t = 40$ nm the buckling-up process retains symmetry of film shape, while for the smaller thicknesses, the breaking of symmetry during buckling up causes a rotational asymmetry of the final bond-back configuration, characteristic for the patterns found in square films (see Figs. 5.7(c), 5.7(f) and Fig. 5.1). For thicker films the rotation is not observed, thus resulting in a regular (non-rotated) fourfold pattern (see Fig. 5.7(i)). There are two characteristic features in Fig. 5.7(c) and 5.7(f) that are also observed in the experimental systems shown in Fig. 5.1: (i) the splitting of the microchannels in the corners of the square films (see the upper right corner of the square films in Fig. 5.1(a)) and (ii) the breaking of symmetry leading to a rotated configuration of the fourfold ridging patterns (see Figs. 5.1(a) and 5.1(d)). Clearly, the morphology of the final bond-back configuration depends on the onset of symmetry breaking during buckling up. The onset of asymmetry can be traced back to the buckling profile near the edges. In Fig. 5.8(a) we plot the out-of-plane ($z$) displacement along the length of the film (see $X-X$ in Fig. 5.7(a)) for a film of thickness 20 nm at a distance of $W/8$ from the top boundary. The profile is plotted at different eigenstrain values, showing a clear decrease in wavelength with an increase in eigenstrain. The change of wavelength is a result of “mode-
Figure 5.5: Evolution of deflection patterns of square films of different size $W$ during bond-back. The figures (a-c) corresponds to $W = 4 \ \mu m$, (d-f) to $W = 3 \ \mu m$, and (g-i) to $W = 2 \ \mu m$. The thickness of the film in all cases is 30 nm. The first column corresponds to the initial configuration of the bond back (i.e., the configuration at 2% eigenstrain during eigenstrain loading). The second column corresponds to a value of $\sigma_{max}/E = 0.3 \times 10^{-3}$ and the third column corresponds to $\sigma_{max}/E = 0.5 \times 10^{-3}$. The contours show the out-of-plane position of the film ($z$) normalized with $H^* = 300$ nm and $z_{min} = 0.07$. The maximum buckling amplitude (i.e., at the beginning of bond-back stage) is 350 nm, 300 nm, and 200 nm for $W = 4 \ \mu m$, 3 \ \mu m, and 2 \ \mu m, respectively.
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Figure 5.6: Buckling height $H$ at the central node during buckling-up as a function of normalized eigenstrain $\bar{\varepsilon}^*$ (left frame) and during bond-back as a function of $\sigma_{\text{max}}/E$ (right frame) for films of widths 5, 4, 3, and 2 $\mu$m for a film thickness of 30 nm (a) and for a film of 5 $\mu$m for different film thicknesses of 20, 30 and 40 nm (b). The black dots in Fig. 5.6(a) correspond to Fig. 5.3(e), and the second column of Fig. 5.5 and in Fig. 5.6(b) to the second column of Fig. 5.7.

The simulations show that the breaking of symmetry happens much sooner (i.e., at smaller eigenstrains) for thinner films and that the wavelength is larger (but the amplitude smaller) for thicker films for a given eigenstrain.

Next, we quantify how the wavelength at the boundary evolves with eigenstrain for films with different thickness and width. Figure 5.8(b) shows the normalized wavelength ($\lambda_{ts}/W$), i.e., the inverse of the number of waves, at the top strip (along $X-X'$) as a function of eigenstrain. The jumps in wavelength are discrete as the change in wavelength occurs at distinct eigenstrains due to mode-jumping in the post-buckling regime [Everall and Hunt, 2000]. The trends that we observe here are similar to those found for linear nanochannel networks, showing a decrease of wavelength with strain, with wavelengths being larger for thicker films [Annabattula et al., 2010b]. Figure 5.8(b) shows that the reduction of wavelength sets in at large eigenstrains when the film is thicker relative to its width. To explore this further, we plot the onset-strain $\varepsilon_{\lambda}^*$ (i.e., the strain at which the buckling profile changes from being symmetric to asymmetric) as a function of normalized film thickness $t/W$ in the inset of Fig. 5.8(b). Similar to the buckling strain this onset strain has an approximate square dependence on the slenderness $t/W$, see Eq. 5.2.

A dimensional analysis of the system under consideration (see Sec. 5.2 and the Appendix)
5. Pattern formation in prestressed polygonal films

Figure 5.7: Evolution of deflection patterns of square films during the bond-back stage for different thicknesses. All the films are of size $W = 5 \mu m$. The figures (a-c) corresponds to films of thickness $t = 20$ nm, (d-f) 30 nm and (g-i) 40 nm, respectively. The first column corresponds to the configuration at 2% eigenstrain (1.9% for Fig.(d)) at the end of the buckling-up stages. The second column corresponds to a value of $\sigma_{\text{max}}/E = 0.3 \times 10^{-3}$ and the third column corresponds to $\sigma_{\text{max}}/E = 0.5 \times 10^{-3}$. The out-of-plane deflection $z$ is normalized with $H^* = 400$ nm and $z_{\text{min}} = 0.05$. 
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Figure 5.8: (a) Out-of-plane position ($z$) of the film along the section ($X-X$) shown in Fig. 5.7(a) for a film of width $W = 5 \mu m$ and thickness $t = 20$ nm at different eigenstrain values. (b) Normalized wavelength ($\lambda_{ts}/W$) at the top cut-section $X-X$ as a function of eigenstrain $\varepsilon^*$. (Inset of b) Strain ($\varepsilon_X^*$) at the onset of change in wavelength along $X-X$ as a function of normalized film thickness ($t/W$). The symbols in figure (b) corresponds to different thickness and width combinations. Square symbols correspond to $W = 5 \mu m$, $t = 20$ nm, diamonds correspond to $W = 3 \mu m$, $t = 30$ nm and circles correspond to $W = 2.5 \mu m$ and $t = 10$ nm. The numbers shown in parenthesis of the inset of figure (b) represent ($W$, $t$), with $W$ in $\mu m$ and $t$ in nm.

shows that during eigenstrain loading the solution only depends on the parameter $\varepsilon^* (W/t)^2$. In Fig. 5.9(a) we plot the normalized channel height $H/t$ versus $\varepsilon^* (W/t)^2$ for all cases analyzed. The results follow a unique relation which can be accurately captured by a square-root dependence, $H/t \propto \sqrt{\varepsilon^* W/t}$. This suggests that the thickness-dependence is small. Indeed, by plotting $(H/W)^2$ versus $\varepsilon^*$ (inset of Fig. 5.9(a)), it follows that the thickness-dependence can be neglected and the results can be well fitted to $(H/W)^2 = 0.6 \varepsilon^*$. The final pattern/morphology after bond back is governed by the interface energy, film thickness, film size and eigenstrain in the film. This final configuration is determined by the balance between the interface energy in the rebonded region and the strain energy of the film. By neglecting the strain energy due to stretching, Annabattula et al. [2010a,b] (see also Sec. 3.4 and 4.3.4) have shown that the bond-back process is governed by the non-dimensional parameter $(\Gamma/WE)(W/t)^3$, where $\Gamma = \sigma_{\text{max}} \delta_n \exp(1)$. Figure 5.9(b) shows the normalized buckling amplitude $(H/W)$ at the center of the film as a function of the above non-dimensional parameter for all cases analyzed in Figs. 5.3 - 5.7. After an initial scatter, all the curves collapse on a single curve for increasing values of $(\Gamma/WE)(W/t)^3$. Two cases can be seen to
deviate from the main trend in the initial stages of bond-back, \((W', t) = (5 \mu m, 20 nm)\) and \((5 \mu m, 30 nm)\). This is believed to be related to the breaking of symmetry during buckling-up (see Fig. 5.7); all other cases have a symmetric buckled-up profile (see Figs. 5.5 and 5.7), facilitating bond back compared to an asymmetric profile. One additional case, \((W, t) = (2 \mu m, 30 nm)\) has a larger initial height, which is known to affect the scaling [Annabattula et al., 2010a]. For some cases appreciable membrane strains develop in the film, which were not accounted for in the non-dimensional parameter. These data points (at large interface energies) are excluded from Fig. 5.9(b).

### 5.3.2 Rectangular films

In this section we investigate the evolution of the film deflection with an increase in eigenstrain for rectangular films. Figure 5.10 shows the contour plots of the normalized out-of-plane position \(\bar{z}\) of a rectangular film of length \(L = 5 \mu m\), width \(W = 2 \mu m\) and thickness \(t = 30 nm\) (see Fig. 5.2(a)). During the eigenstrain loading, we can observe that the film goes through three stages. Figure 5.10(a) shows the initial buckling mode, referred to as the Euler buckling mode. With further increase in eigenstrain, the film enters the varicose mode (Fig. 5.10(b)) and finally symmetry is broken when entering into a telephone-cord mode (Fig. 5.10(c)-5.10(f)). A similar evolution of buckling modes with film stress in films
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Figure 5.10: Evolution of deflection patterns of a rectangular film during buckling-up. The length of the film $L = 5 \, \mu m$, the width $W = 2 \, \mu m$ and the thickness $t = 30 \, nm$. The contour plots correspond to eigenstrain $\varepsilon^*$ of (a) 0.08%, (b) 0.53%, (c) 0.56%, (d) 0.57%, (e) 0.80%, and (f) 1.4%, respectively. The maximum amplitude (i.e., at 1.4% eigenstrain) at the central node in the above case is 200 nm while the amplitude $z$ is normalized with $H^* = 300 \, nm$ and $z_{\text{min}} = 0.07$. We can observe a transition of modes with increase in eigenstrain. The film initially buckles in the Euler mode (a), and then it enters into the varicose mode (b) and with further increase in strain it enters into a telephone-cord mode (c), as also observed by Moon et al. [2004].

Figure 5.11 shows the contour plots of the normalized out-of-plane position of rectangular plates of length $L = 5 \, \mu m$ and widths $W = 2, 3.5$ and $4.5 \, \mu m$ with a thickness $t = 30 \, nm$, through the entire process of channel formation. The first column in all the figures correspond to the initial buckled-up profile at small eigenstrain ($\varepsilon^* = 0.19\%$), the second column corresponds to the fully buckled-up configuration at the end of eigenstrain loading (i.e., at 1.4% eigenstrain) and the third column corresponds to the configuration at the end of the bond-back process. It can be clearly seen that the final bond-back configuration strongly depends on the configuration at the beginning of bond-back; we see a non-symmetric pattern formation for films of large length ($L$) to width ($W$) ratio and the pattern transforms to the well-known fourfold ridge pattern (Fig. 5.11(i)) for square films (see Sec. 5.3.1).

The switching of modes in the case of a confined rectangular film may be explained as follows. A pre-stressed rectangular film relaxes the stresses by buckling in the transverse direction (i.e., in the $X$-direction, see Fig. 5.2) at small strains, called the Euler mode. With further increase in strain the film starts to relax in the longitudinal direction ($Y$-direction).
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Figure 5.11: Evolution of normalized out-of-plane position during buckling-up and bond-back for rectangular film of size (a-c) $L = 5 \, \mu m$, $W = 2 \, \mu m$, (d-f) $L = 5 \, \mu m$, $W = 3.5 \, \mu m$, and (g-h) $L = 5 \, \mu m$, $W = 4.5 \, \mu m$. The thickness of the film in all the cases is 30 nm. The first column shows the initial buckling configurations at 0.19% eigenstrain, the second column corresponds to the final buckled-up configuration (or beginning of the bond-back process) at 1.4% eigenstrain and the third column shows the final bond-back configuration (at $\sigma_{max}/E = 0.142 \times 10^{-3}$) showing a channel formation. The out-of-plane deflection $z$ is normalized with $H^* = 300 \, nm$ and $z_{min} = 0.07$. 
as well, resulting in a bifurcation of the Euler mode into a symmetric or anti-symmetric undulating profile, termed varicose and telephone-cord mode, respectively. It has been shown by Audoly [1999] that for an infinitely long rectangular film with a width $W$ the bifurcation mode depends solely on Poisson’s ratio of the film. If Poisson’s ratio of the film is less than 0.25, the secondary mode will be symmetric (varicose mode) and if it is higher the mode will be anti-symmetric (telephone-cord). In the case of a rectangular film of finite length and width it has been shown [Nakamura and Uetani, 1979] that the the ratio of secondary buckling load to primary buckling load also depends on the film aspect ratio and thickness. Indeed, our results show that for relatively large aspect-ratio films the Euler mode is followed by the varicose mode, but that the telephone-cord mode is attained at relatively small eigenstrains (see Fig. 5.10). Increasing the aspect ratio, postpones the change from varicose to telephone-cord mode to larger eigenstrains, as shown in the second column of Fig. 5.11, where $W = 2 \mu m$ and $3.5 \mu m$ correspond to the telephone-cord mode, and $W = 4.5 \mu m$ to the varicose mode at 1.4% eigenstrain. Clearly, the buckling morphology (symmetric versus asymmetric) is directly reflected in the final configuration after bond-back (see right column of Fig. 5.11). Increasing the film thickness also postpones the varicose-to-telephone-cord bifurcation to larger strains (results not shown), also leading to more regular channel morphologies.

We now investigate the evolution of the buckling height $H$ as a function of eigenstrain for films of different length ($L$) to width ($W$) ratios. Figure 5.12(a) shows the evolution of the buckling height ($H$) at the central node (i.e., the intersection of the two diagonals) as a function of eigenstrain for films having different widths, but for a fixed length $L = 5 \mu m$ and thickness $t = 30$ nm. It can be observed that the buckling amplitude decreases with a decrease in width $W$. The film obtains a larger bending stiffness when one dimension is reduced ($W$ in this case) increasing the critical buckling strain (see Eq. 5.2) which can also be seen in Fig. 5.12(a). The sudden jumps in buckling amplitude (at $\varepsilon^* = 0.005, 0.0085$ for $W = 2 \mu m$ and at $\varepsilon^* = 0.012$ for $W = 3.5 \mu m$) correspond to mode-changes (from Euler to varicose and from varicose to telephone-cord mode, see Figs. 5.10, and 5.11). The amplitude $H$ reported in Fig. 5.12(a) corresponds to the central node, which was the highest node until the onset of mode change. When the mode change occurs, the highest point shifts and the amplitude of the central node reduces. Next, we will develop an analytical solution to the postbuckling problem using minimization of energy.

The solution for the boundary value problem (for a rectangular film constrained all around) can be approximated by

$$w(x, y) = (H/4) \left[ 1 - \cos(2\pi x/L) \right] \left[ 1 - \cos(2\pi y/W) \right], \quad (5.3)$$

where $L$ and $W$ are the length and width of the film, respectively (see Fig. 5.2). The bending strain energy density ($W_b$) and membrane strain energy density ($W_m$) are given by

$$W_b = \frac{Et^3}{24} \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w}{\partial y^2} + 2(1 - \nu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right], \quad (5.4)$$

$$W_m = \frac{t}{2} \left[ \sigma_{11} \varepsilon_{11}^{el} + 2\sigma_{12} \varepsilon_{12}^{el} + \sigma_{22} \varepsilon_{22}^{el} \right], \quad (5.5)$$
Figure 5.12: (a) Buckling amplitude as a function of eigenstrain for \( L = 5 \, \mu m \) films of different width \( W \) for a film thickness of 30 nm. (b) Analytical solution of buckling amplitude \( (H) \) for films of length \( L = 5 \, \mu m \) with two different widths \( W = 2 \) and 5 \( \mu m \) plotted as a function of eigenstrain \( \varepsilon^* \) for three different film thicknesses \( t = 10, 30 \) and 50 nm) in each case. The inset of Fig. (b) shows a zoomed view at small strains depicting the effect of film thickness. (c) Simulation results for the normalized buckling amplitude \( (H/W)^2 \) plotted as a function of eigenstrain for the results shown in Fig. 5.12(a). (d) Analytical solution of normalized buckling amplitude \( (H/W)^2 \) as a function of eigenstrain \( (\varepsilon^*) \) for films of different aspect ratio \( (L/W) \). Each line type corresponds to a specific aspect ratio of the film as shown in Fig. 5.12(c). The inset of figure (d) shows the slope of the lines as a function of the aspect ratio.
with $\sigma_{ij}$ defined in Eq. 4.5, $\bar{E} = E/(1 - \nu^2)$ and the elastic strains in the film (neglecting the contribution due to axial displacements) given by (see also Sec. 4.2.1)

$$\varepsilon_{11}^{\text{el}} = -\varepsilon^* + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2,$$

$$\varepsilon_{22}^{\text{el}} = -\varepsilon^* + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2,$$

$$\varepsilon_{12}^{\text{el}} = \frac{1}{2} \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right).$$

The total bending ($U_b$) and membrane ($U_m$) strain energy are given by

$$U_b = \int_0^L \int_0^W W_b \, dx \, dy,$$

$$U_m = \int_0^L \int_0^W W_m \, dx \, dy,$$

and hence the total strain energy in the film is given by $U = U_b + U_m$. The amplitude of the film is obtained by minimizing the total strain energy

$$\frac{\partial U}{\partial H}(E, H, L, W, t, \nu, \varepsilon^*) = \frac{\partial U^*}{\partial H}(H, L, W, t, \nu, \varepsilon^*) = 0.$$

This results in an implicit equation for $H$, which can be solved for $H$ and written as

$$\frac{H}{W} = f \left( \frac{t}{L}, \frac{L}{W}, \varepsilon^*, \nu \right).$$

In Fig. 5.12(b) we plot the analytical solution for $H$ for films of length $L = 5 \, \mu m$ and widths $W$ being 2 and 5 $\mu m$ with $\nu = 0.3$, for three different thicknesses $t = 10$ nm, 30 nm, and 50 nm as a function of $\varepsilon^*$. The results show a rather small effect of thickness. Only for small strains a thickness-dependence can be observed (see the inset of Fig. 5.12(b)). A similar observation is also reported in Ch. 3, see Fig. 3.6 [Annabattula et al., 2010a]. With the assumption that $t/L << \varepsilon^*$, Eq. 5.11 can be simplified to

$$\frac{H}{W} = \sqrt{\varepsilon^*} f(L/W, \nu),$$

showing the characteristic $\sqrt{\varepsilon^*}$ dependence of the height $H$ (see the inset of Fig. 5.9(a)). Equation 5.12 is plotted in Fig. 5.12(d) for different aspect ratios of the film, indicating that for the aspect ratios analyzed, its effect on $H/W$ is small. This can also be deduced from the inset of Fig. 5.12(d), clearly showing that the dependence of the slope of Fig. 5.12(d) on $L/W$ is weak for $L/W > 1$. These results are in close agreement with the numerical results shown in Fig. 5.12(c), demonstrating the square root dependence on $\varepsilon^*$ (Eq. 5.12) and the weak
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![Contour plots of the normalized out-of-plane position for pentagonal films](image)

**Figure 5.13**: Contour plots of the normalized out-of-plane ($\bar{z}$) deflection during buckling-up (a-c) and bond-back (d-f) for a pentagonal film of side 5 $\mu$m and thickness 30nm. During the buckling-up stage the corresponding eigenstrains are (a) 0.7%, (b) 1.1% and (c) 5.0%. During the bond-back stage the corresponding values of $\sigma_{\text{max}}/E$ are (d) 0.0 (i.e. the configuration at 2% eigenstrain during buckling up), (e) $1.25 \times 10^{-3}$, and (f) $1.25 \times 10^{-3}$ with a higher damping factor than in (e). The value of $H^*$ for the above case is 1200 nm and $z_{\text{min}} = 0.017$.

dependence on $L/W$. Finally, it is interesting to compare the slope of the $(H/W)^2$ versus $\varepsilon^*$ curves with the numerical results (equal to $\bar{f}^2$ in Eq. 5.12). For a square film ($L/W = 1$) the slope is approximately equal to 0.375 (see inset of Fig. 5.12(d)). Note that the analytical solution and simulation results do not match one-to-one as the slope of the $(H/W)^2$ versus $\varepsilon^*$ curve is found to be equal to 0.5 (see Fig. 5.12(c)). The difference in slopes may be due to the assumed functional form of the buckled-up profile in the analytical solution (Eq. 5.3) which does not exactly represent the profile in the simulations.

### 5.3.3 Pentagonal and hexagonal films

Next, we study the patterns formed in films with a pentagonal and hexagonal geometry. Figure 5.13 shows the contour plots of the normalized out-of-plane position for pentagonal films...
5.3. Results and discussion

Figure 5.14: Contour plots of the normalized out-of-plane ($\bar{z}$) deflection during buckling-up (a-c) and bond-back (d-f) for a hexagonal film of side 5 $\mu$m. During the buckling-up stage the corresponding eigenstrains are (a) 0.7%, (b) 1.07% and (c) 5.0% and the film thickness is 30 nm. During the bond-back stage the corresponding values of $\sigma_{\text{max}}/E$ are (d) 0.0 (i.e., the configuration at $\varepsilon^* = 2\%$ during buckling-up), (e) $0.375 \times 10^{-3}$ for a film thickness of 30 nm and (f) $0.375 \times 10^{-3}$ for a film of thickness 50 nm. The damping factor for both the thicknesses is taken to be same as in Fig. 5.13(f). The value of $H^*$ for the above case is 1200 nm and $z_{\text{min}} = 0.017$.

with a side length of 5 $\mu$m and a thickness of 30 nm through the buckling-up and bond-back stages. Figure 5.13(a) shows the configuration at $\varepsilon^* = 0.7\%$ with a symmetric profile while the symmetry is lost at $\varepsilon^* = 1.1\%$ (Fig. 5.13(b)) through a change of wavelength along the boundary and an associated rotation at the center. Figure 5.13(c) shows the configuration at the end of eigenstrain loading ($\varepsilon^* = 5\%$) showing a highly wriggled profile along the constrained boundary. The bond back is started from the configuration of the film at 2$\%$ eigenstrain (Fig. 5.13(d)) similar to the square films. Figure 5.13(e) shows the bond-back
configuration at $\sigma_{\text{max}}/E = 1.25 \times 10^{-3}$, i.e., at an interface energy of 0.27 J/m$^2$ (for $\delta_n = 20$ nm) and Fig. 5.13(f) shows the configuration at the same interface energy but with a higher damping factor†. From the figures it can be seen that the final pattern configuration with higher damping is more symmetric and has more narrow channels. This is due to the lower kinetic energy in the system during bond-back, which does not create any disturbances to break the symmetry (or even helps the system to suppress potential instabilities) as in the system with lower damping shown in Fig. 5.13(e). Note that both configurations can be observed in the optical micrographs of the experimental system (see Fig. 5.1(b)). Figure 5.14 shows the results for a hexagonal film of size 5 $\mu$m through the buckling-up and bond-back stages. Figures 5.14(a)-5.14(e) correspond to a film of thickness 30 nm. Figure 5.14(d) shows that the initial configuration of the bond-back stage (i.e., the configuration at the end of 2% eigenstrain during buckling-up process) is still symmetric in contrast to the pentagonal film in Fig. 5.13(d). It seems that the level of symmetry of the initial film shape affects the susceptibility to symmetry breaking of the buckling profile. Squares and hexagons (two axes of symmetry) break symmetry at eigenstrains of 1.78% and 2%, while pentagons (one axis of symmetry) break symmetry at much smaller eigenstrain 1.1%. Figure 5.14(e) shows the bond-back configuration at $\sigma_{\text{max}}/E = 0.375 \times 10^{-3}$ corresponding to an interface energy of 0.265 J/m$^2$. Figure 5.14(f) shows the configuration during bond back at the same interface energy, but with a higher thickness ($t = 50$ nm). The film with a lower thickness bonds back with less symmetry than the thicker film, while the channels are more narrow. The observation is in correspondence to those of the square films (see Fig. 5.7). Note that for both thicknesses the buckled-up configurations are identical (conform Fig. 5.6(b)).

Finally, we investigate the film height $H$ during the buckling-up stage of the pentagonal, hexagonal and square films. It has been shown that for square films the thickness dependence on buckling amplitude is rather small (see Fig. 5.6(b)). Indeed, by plotting $(H/W)^2$ versus $\varepsilon^*$ (see the inset of Fig. 5.9(a)) a very small thickness dependence was observed, while the curve can be well fitted to $(H/W)^2 = 0.6\varepsilon^*$. In other words, $H^2/A$, where $A = W^2$ is the area of the square film scales with $\varepsilon^*$. It turns out that such a scaling can also be extended to other geometries as well. Figure 5.15(a) shows the film height $H$ normalized with an equivalent film width ($W_{eq}$) plotted as a function of eigenstrain ($\varepsilon^*$). The equivalent film width $W_{eq}$ is the length of the side of a square of the same area as the polygonal films under consideration (i.e., square root of the film area). Thus $W_{eq} = W$ for square, $W_{eq} = 1.31W$ for pentagonal, and $W_{eq} = 1.61W$ for hexagonal films. It can be observed that the normalized buckling amplitude is independent of film shape, film size and film thickness when normalized with $W_{eq}$. The final pattern/morphology after bond back is governed by the interface energy, film thickness, film size and eigenstrain. The final configuration is determined by the balance between the interface energy in the rebonded region and the strain energy of the film. By neglecting the strain energy due to stretching of the film, it has been shown in Sec. 4.3.4 that the bond-back process is governed by the non-dimensional parameter $(\Gamma/EW)(W/t)^3$, where $\Gamma = \sigma_{\text{max}}\delta_n\exp(1)$. However, for polygonal films we replace $W$ by $W_{eq}$ in the above non-dimensional parameter as already mentioned before. Figure 5.15(b)

†The dissipated energy for the damping factor of Fig. 5.13(e) is 0.5% of the total energy, while with the higher damping factor of Fig. 5.13(f) it is 5% of the total energy in the system.
Figure 5.15: (a) Normalized buckling amplitude \((H/W_{eq})\) as a function of eigenstrain \((\varepsilon^*)\) (b) Normalized buckling amplitude \((H/W_{eq})\) as a function of the non-dimensional parameter \((\Gamma/(EW_{eq})(W_{eq}/t)^3\) for films of square (S), Pentagonal (P) and Hexagonal (H) geometry. The first number in the legend corresponds to the side length of the film in ‘\(\mu\)m’ and the second number corresponds to the film thickness in ‘nm’.

shows the evolution of the normalized amplitude \((H/W_{eq})\) as a function of normalized interface strength \((\Gamma/(EW_{eq})(W_{eq}/t)^3\) during the bond-back stage for square, pentagonal and hexagonal films of different size and thickness. All the curves collapse to a single curve. The normalized Figs. 5.15(a) and 5.15(b) can be seen as master curves for the buckling-up and bond-back behaviour for thin films of different shape, size and thickness.

5.4 Summary and conclusions

We have developed a finite element model to describe the mechanism of pattern formation as a result of buckling-up followed by bond-back of prestressed thin films on a rigid substrate. We studied films with different size, thickness and shape, including square, rectangular, pentagonal and hexagonal films. During eigenstrain loading, square films of width \(W\) first buckle in their initial (Euler) buckling mode at the critical buckling strain \(\varepsilon^*_c\). After an initial symmetric evolution of the deflection profile, the symmetry of the deflection pattern breaks when the wavelength of the wriggles near the fixed boundary starts to decrease. The strain at which wavelength reduction initiates and symmetry breaking commences depends on the normalized thickness \(t/W\). The normalized buckling height \(H/W\), however, is more-or-less independent of thickness and is observed to scale linearly with \(\sqrt{\varepsilon^*}\). During
bond back the deflection morphology converges to a four-fold ridging pattern having features that show a close resemblance with the experimental film systems of similar size and shape. Rectangular films of large length to width ratio and relatively small thickness go through a transition in buckling shapes from the initial Euler mode, through the varicose mode into the anti-symmetric telephone-cord mode. Increasing the thickness postpones the telephone-cord mode to occur at larger strains, while the ridging pattern becomes more regular with a decrease in length to width ratio, converging to the fourfold ridging pattern for square films.

We have also performed analytical calculations based on the minimization of strain energy in the film to study the evolution of buckling height $H$. We found that $H/W$ only depends weakly on normalized film thickness $t/W$ and aspect ratio $L/W$ and we confirmed the linear correlation between $H/W$ and $\sqrt{\varepsilon^*}$.

Finally, we studied the effect of film shape on pattern formation for pentagonal and hexagonal films. We found that the evolution of height $H$ for all film shapes can be captured by one master curve which scales linearly with $\sqrt{\varepsilon^*}$ if normalized by $W_{eq}$, being the square-root of the film area. In addition, the bond-back mechanism is governed by the eigenstrain value at the end of the buckling-up stage and the dimensionless parameter $\left(\Gamma/(EW_{eq})\right)(W_{eq}/t)^3$, relating the film dimensions to the interface energy $\Gamma = \sigma_{\text{max}} \delta_n \exp (1)$. The Young’s modulus of the film does not enter the buckling-up process during the initiation of buckling or during post-buckling. It does enter, however, during bond-back in which it contributes to the competition between the interface energy and the strain energy of the film, ultimately dictating the final wrinkle morphology.
Appendix

I Dimensional analysis

The principle of virtual work during eigenstrain loading is given by

\[ \delta W_{\text{in}} = \delta W_{\text{ex}} = 0, \quad (5.\text{A}.1) \]

where \( \delta W_{\text{ex}} \) is the external virtual work (being zero in the present case) and \( \delta W_{\text{in}} \) is the internal virtual work. The internal virtual work can further be written as

\[ \delta W_{\text{in}} = \int_V \sigma : \delta \varepsilon \, dV = \int_V \sigma_{ij} \delta \varepsilon_{ij} \, dV, \quad (5.\text{A}.2) \]

with \( \delta \varepsilon_{ij} \) the virtual strain. We neglect the contribution of in-plane stretching, \( \varepsilon^{\text{str}} \), to the total strain, so that

\[ \varepsilon_{ij}^{\text{cl}} = \varepsilon_{ij}^{\text{rot}} - z \chi_{ij} - \varepsilon^{\star} \delta_{ij}, \quad (5.\text{A}.3) \]
\[ \delta \varepsilon_{ij} = \delta \varepsilon_{ij}^{\text{rot}} - z \delta \chi_{ij}, \quad (5.\text{A}.4) \]

which upon substitution in Eq. 5.A.2 together with Eq. 4.5 yields

\[ \delta W_{\text{in}} = \frac{E}{1 + \nu} \int_0^W \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[ \varepsilon_{ij}^{\text{rot}} - z \chi_{ij} - \varepsilon^{\star} \delta_{ij} + \frac{\nu}{1 - \nu} \left( \varepsilon_{kk}^{\text{rot}} - z \chi_{kk} - 2 \varepsilon^{\star} \right) \delta_{ij} \right] \delta \varepsilon_{ij}^{\text{rot}} - z \delta \chi_{ij} \, dz \, dx \, dy. \quad (5.\text{A}.5) \]

We now integrate the above integral over \( z \), which gives

\[ \delta W_{\text{in}} = \delta W_{\text{rot}} + \delta W_{\text{bend}} + \delta W^{\star}, \]

with

\[ \delta W_{\text{rot}} = \frac{Et}{1 + \nu} \int_0^W \int_0^L \left( \varepsilon_{ij}^{\text{rot}} \delta \varepsilon_{ij}^{\text{rot}} + \frac{\nu}{1 - \nu} \varepsilon_{kk}^{\text{rot}} \delta \varepsilon_{kk}^{\text{rot}} \right) \, dx \, dy, \quad (5.\text{A}.6) \]
\[ \delta W_{\text{bend}} = \frac{Et^3}{12(1 + \nu)} \int_0^W \int_0^L \left( \chi_{ij} \delta \chi_{ij} + \frac{\nu}{1 - \nu} \chi_{kk} \delta \chi_{kk} \right) \, dx \, dy, \quad (5.\text{A}.7) \]
\[ \delta W^{\star} = \frac{-Et \varepsilon^{\star}}{1 - \nu} \int_0^W \int_0^L \delta \varepsilon_{kk}^{\text{rot}} \, dx \, dy. \quad (5.\text{A}.8) \]
Next, we dimensionalize these integrals by introducing dimensionless lengths $X = x/L$, $Y = y/W$ and $W = w/t$. Using this in the expressions for the strains yields

$$
\varepsilon_{ij}^{\text{rot}} = \frac{t^2}{LW} Q_{ijkl} \varepsilon_{kl}^{\text{rot}} \\
\chi_{ij} = \frac{W}{LW} Q_{ijkl} \chi_{kl}^{'}
$$

(5.A.9)

with $Q_{ijkl} = 0$, except for $Q_{1111} = W/L$, $Q_{1212} = Q_{1221} = Q_{2112} = Q_{2121} = 1/2$, $Q_{2222} = L/W$ and $\varepsilon_{kl}^{\text{rot}}$ and $\chi_{ij}^{'}$ being dimensionless. The expressions for $\delta \varepsilon_{ij}^{\text{rot}}$ and $\delta \chi_{ij}$ are similar. Substituting these relations in Eq. 5.A.6–5.A.8 yields

$$
\delta W_{\text{rot}} = \frac{Et^5}{(1 + \nu)(LW)} \int_0^1 \int_0^1 \left( Q_{ijmn} \varepsilon_{mn}^{\text{rot}} Q_{ijop} \delta \varepsilon_{op}^{\text{rot}} + \frac{\nu}{1 - \nu} Q_{kkqr} \varepsilon_{qr}^{\text{rot}} Q_{kkst} \delta \varepsilon_{st}^{\text{rot}} \right) dX dY,
$$

$$
\delta W^{*} = \frac{-Et^3 \varepsilon^{*}}{(1 - \nu)} \int_0^1 \int_0^1 Q_{kkmn} \delta \varepsilon_{mn}^{\text{rot}} dX dY,
$$

$$
\delta W_{\text{bend}} = \frac{Et^5}{12(1 + \nu)(LW)} \int_0^1 \int_0^1 \left( Q_{ijmn} \chi_{mn}^{'} Q_{ijop} \delta \chi_{op}^{'} + \frac{\nu}{1 - \nu} Q_{kkqr} \chi_{qr}^{'} Q_{kkst} \delta \chi_{st}^{'} \right) dX dY.
$$

(5.A.10)

The above three integrals are dimensionless and they only depend on $\nu$, the boundary conditions and the ratio $L/W$. By dividing all the terms with $Et^5/(1 + \nu)LW$, it follows that for plates with the same $\nu$, $L/W$ and similar boundary conditions, the solution only depends on the unique factor $\varepsilon^{*}LW/t^2$, independent of the stiffness.