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Stabilization and tracking for switching linear systems under unknown switching sequences

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\textbf{A B S T R A C T}

This paper describes recent progress in the study of switching linear systems i.e. linear systems whose dynamics can switch among a family of possible configurations/modes. The attention is focused on those classes of switching systems governed by unknown switching sequences. For this case, we address the problem of devising suitable control actions able to stabilize the plant and regulate its output about a desired reference trajectory. The approach considered in this paper consists in designing a suitable family of feedback controllers and a mode-estimator that at any time determines which candidate controller has to be placed in the feedback loop on the grounds of a real-time estimate of the current plant mode. It is shown that, even if a correct reconstruction of the plant mode from the measured data is not always possible, under certain conditions, exponential stability of the closed-loop can be guaranteed for any slow-on-the-average plant switching sequence.

\section{1. Introduction}

In recent years, switching systems have received considerable attention both in theory and applications, as they allow one to describe the behaviour of a large class of plants resulting from the interactions of continuous dynamics, discrete dynamics, and logic decisions \cite{1}. Switching systems represent a special class of hybrid systems, namely those systems whose dynamics can switch among a family of possible configurations/modes. From a theoretical viewpoint, the main contributions to the study of switching systems have been basically of a two-fold nature: on one side several studies have focused on state/mode observability, viz. on the possibility of reconstructing from measured data the continuous state, the discrete mode, or both \cite{2–8}; on the other side, the main interest has been devoted to stability and stabilization problems \cite{9–11}. Within this latter source of contribution, however, the major emphasis has been on basic issues, namely the characterization of the control laws which can ensure stability to the switching system under the assumption that an exact knowledge of the current process mode is available in real-time or with delay. However, in many control applications involving switching, the plant switching signal is neither known nor observed. Typical examples are found in connection with fault-tolerant control schemes where different plant modes are introduced to model possible faults in the process to be controlled \cite{12}.

As a matter of fact, the departure from the assumption that an exact knowledge of the process mode sequence is available still poses many challenges. This is because, in order to properly configure the control action, specific mechanisms have to be devised apt to estimate the current process mode on the grounds of the available data. To the best of our knowledge, there are only few contributions which address the case where the knowledge of the plant configuration is not available, neither in real time nor with delay \cite{13–16}. Moreover, much of the work in this area is concerned with the only case of zero-regulation. A notable exception is \cite{16}, which consider the use of periodic controllers for simultaneous stabilization and step-tracking under persistent plant changes. However, no analytical framework is available to approach more general reference-tracking problems in a systematic kind of way.

In this paper, we address stabilization and tracking problems for continuous-time multi-input multi-output (MIMO) systems governed by unknown switching sequences, with special emphasis on reference-tracking. We study this problem within the framework of generalized reference-tracking \cite{17}, by considering the class of reference trajectories consisting of bounded periodic sequences generated by autonomous linear time-invariant systems.

To be able to deal with possible persistent plant changes, we consider a mode estimator whose task is to infer on-line the active plant mode from measured data. The mode estimator is embedded in a supervisory unit that orchestrates the switching between a family of pre-designed candidate controllers, according to the current process mode estimate. The findings of this paper have their roots in \cite{18}, which considered the zero-regulation problem for switching linear systems under unknown switching signal. In that paper,
the estimate of current plant mode is carried out by evaluating the distance of the plant input/output data collected over a moving horizon from the subspaces associated to each possible mode. The key feature of \[18\] is to show that, under certain closed-loop observability conditions, it is possible to ensure finite-time reconstruction of the plant mode whenever the latter stays constant. At first glance, one might think that reference-tracking does not bring additional complexity with respect to zero-regulation, at least for stability purposes (roughly, one might be tempted to conclude that stability cannot be destroyed by the presence of bounded reference signals). It turns out that this is not the case. In fact, the reference signal may force the plant to evolve along trajectories which prevent, in sharp contrast with zero-regulation, exact plant mode reconstruction. The analysis carried out in this paper shows that, even if a correct reconstruction of the plant mode is not always possible, mode estimation schemes do exist which guarantee closed-loop exponential stability for any slow-on-the-average plant switching sequence. It is also shown that, under suitable design conditions, the proposed control scheme ensures asymptotic reference-tracking for any finitely convergent plant switching sequence.

The paper is organized as follows. Section 2 describes the problem of interest. Section 3 discusses conditions under which it is possible to uniquely reconstruct the plant mode from the measured data. The proposed control scheme is described in Section 4. The stability properties of the resulting closed-loop system are analysed in Section 5 along with simulation results. In Section 6, conclusions are drawn.

**Notations.** Given a matrix \(M, M^T\) is its transpose and \(\|M\| = \left[\lambda_{\max}(M^T M)\right]^{1/2}\) its norm, where \(\lambda_{\max}\) denotes the maximum eigenvalue. Given a measurable time function \(v : \mathbb{R}^+ \to \mathbb{R}^d\) and a time interval \(I \subseteq \mathbb{R}^+\), we denote its \(L_2\) and \(L_\infty\) norm on \(I\) as \(\|v\|_{2,I} = \sqrt{\int_I |v(t)|^2 dt}\) and \(\|v\|_{\infty,I} = \text{ess sup}_{t \in I} |v(t)|\) respectively. Finally, \(L_2(I)\) and \(L_\infty(I)\) denote the sets of square integrable time functions and, respectively, (essentially) bounded time functions on \(I\).

2. **Background**

Consider a plant described by a continuous-time switching linear system \(\mathcal{P}_p\) of the form

\[
\begin{cases}
\dot{x}(t) = A_{\rho(t)} x(t) + B_{\rho(t)} u(t) \\
y(t) = C_{\rho(t)} x(t)
\end{cases}
\]

where \(t_0 \in \mathbb{R}\) is the initial time instant; \(x \in \mathbb{R}^{n_x}\) is the plant state, \(u \in \mathbb{R}^{n_u}\) is the control input, \(y \in \mathbb{R}^{n_y}\) is the plant output and \(\rho \in \mathcal{N}\), \(\mathcal{N} := \{1, 2, \ldots, N\}\) is the set of switching modes. \(A_1, B_1, C_1\), \(i \in \mathcal{N}\), are constant matrices of appropriate dimensions. It is supposed that the unknown and unobserved switching signal \(\rho : \mathbb{R}^+ \to \mathcal{N}\) belongs to the class \(\mathcal{E}\) of all the functions that are piecewise constant, right continuous, and admit no Zeno behaviour, i.e. the number of switching instants is finite on every finite interval.

Along with \(1\), consider also a reference signal \(r\), which is assumed to be generated by a finite-dimensional linear time-invariant system \(\mathcal{E}\) of the form

\[
\begin{cases}
\dot{p}(t) = E p(t) \\
p(t_0) = p_0
\end{cases}
\]

where \(p \in \mathbb{R}^{n_p}\) is the state and \(r \in \mathbb{R}^{n_r}\). Throughout the paper, it is understood that \(\mathcal{E}\) is the observable subsystem, obtained via a Gilbert–Kalman observability decomposition, which generates \(r\). System \(\mathcal{E}\) will be referred to as the esystem.

The problem of interest is to devise, based on \(y\) and \(r\), suitable feedback controls \(u = f(y, r)\) such that: (i) the closed-loop is stable in the sense that all its internal state variables remain bounded in response to any bounded reference signal and (ii) the plant output is regulated about the desired reference trajectory \(r\).

**Remark 1.** Before proceeding, it is to be pointed out that the “tracking” problem, if intended as the problem of achieving zero-offset despite plant variations, has in general no solution, no matter how the action is chosen. To maintain continuity, we defer a more detailed treatment of this point to Section 5. In order to avoid possible ambiguities, here we stress that by tracking we mean that, in each interval wherein the plant switching signal remains constant, the tracking error \(e := r - y\) is upper bounded by an exponentially converging function. Notice that such a property also implies that the tracking error will converge to zero for any finitely convergent plant switching sequence.

The control action used for the problem in question consists in a one degree-of-freedom linear switching controller \(C_\rho\) of the form

\[
\begin{cases}
\dot{q}(t) = F_{\rho(t)} q(t) + G_{\rho(t)} e(t) \\
u(t) = H_{\rho(t)} q(t) + K_{\rho(t)} e(t)
\end{cases}
\]

\(q(t_0) = q_0\)

where \(q \in \mathbb{R}^{n_q}\) is the controller state, \(e = r - y\) is the tracking error, and \(\sigma : \mathbb{R}^+ \to \mathcal{N}\) is the controller mode, which is supposed to be generated in the set \(\mathcal{E}; F_\sigma, G_\sigma, H_\sigma, K_\sigma, i \in \mathcal{N}\), are constant matrices of appropriate dimensions. Our aim is therefore to suitably design a family of linear time-invariant controllers along with a mechanism for deciding which of the candidate controllers has to be placed at any time in the feedback loop.

The overall scheme is depicted in Fig. 1. Hereafter, \(P_i\) and \(C_i\) will denote the LTI systems associated with the \(i\)-th plant and, respectively, controller mode. Throughout the paper, for simplicity of exposition, we shall at times resort to input/output model descriptions. Consider a left and a right polynomial matrix fraction description (PMFD) of the plant

\[
P_i(s) := U_i^{-1}(s) Q_i(s) = \mathcal{F}_i(s) \mathcal{G}^{-1}_i(s)
\]

where, for each index \(i \in \mathcal{N}\), \(U_i\) and \(Q_i, \mathcal{F}_i\) and \(\mathcal{G}_i\) are left [right] coprime polynomial matrices of appropriate dimensions with det(\(s I - A_i\)) = det \(U_i(s)\) = \(\det (\mathcal{F}_i(s)), \mathcal{G}_i(s) \in \mathbb{R} \setminus \{0\}\). Likewise, consider a left and a right PMFD of the controller

\[
C_i(s) := R_i^{-1}(s) S_i(s) = \mathcal{H}_i(s) \mathcal{S}^{-1}_i(s)
\]

As beforehand, for each \(j \in \mathcal{N}\), \(C_j\) is the transfer matrix of \(C_j\) and \(R_j, S_j, \mathcal{H}_j, \mathcal{S}_j\) are left [right] coprime polynomial matrices of appropriate dimensions with det(\(s I - F_j\)) = det \(R_j(s)\) = \(\det (\mathcal{H}_j(s)), h \in \mathbb{R} \setminus \{0\}\).

2.1. **Open-loop assumptions**

In order to render the exposition as simple as possible, we introduce from the very beginning a number of standing assumptions. As for the individual systems, we assume the following.
Assumption 1. For any plant mode \( i \in \mathcal{N} \) and any controller mode \( j \in \mathcal{N} \) the corresponding time-invariant systems \( \mathcal{P}_i \) and \( \mathcal{C}_j \) are controllable and observable. □

Assumption 2. All the eigenvalues of the matrix \( E \) have zero real part and multiplicity one in the minimal polynomial. □

Assumption 1 entails no loss of generality in that if \( \mathcal{P}_i \) or \( \mathcal{C}_j \) is stabilizable [detectable] but not controllable [observable], all subsequent developments could be suitably modified so as to obtain similar results. Assumption 2 guarantees that all trajectories of the exosystem are bounded. This assumption is quite common in the area of internal-model-based control, the resulting tracking problem being usually referred to as the generalized tracking problem [17]. We now need to discuss some general conditions for the solution of the stabilization and tracking problems here considered. In relation with the former issue, it is natural to assume that for each plant mode there exists a corresponding stabilizing controller. This can always be ensured under Assumption 1. The question of reference-tracking requires a bit more care. It is convenient to first introduce the relevant assumptions and then discuss them. Given a matrix \( M \), let \( \text{spec}(M) \) denote its spectrum. We assume the following.

Assumption 3. For any \( i \in \mathcal{N} \), the matrix \( Q_i(\lambda) \) is full-rank for every \( \lambda \in \text{spec}(E) \). □

Assumption 4. For any \( i \in \mathcal{N} \), the controller \( \mathcal{C}_j \) stabilizes the corresponding plant \( \mathcal{P}_i \). Furthermore, for any \( i \in \mathcal{N} \), the matrix \( R_i(\lambda) = 0 \) for every \( \lambda \in \text{spec}(E) \). □

Both these assumptions are quite common in the area of internal-model-based control and notably simplify the analysis with no excessive compromise in terms of generality. In particular, Assumption 3 is equivalent to the property that none of the eigenvalues of \( E \) is a transmission zero of \( \mathcal{P} \), thus ensuring well-posedness of the reference-tracking problem [17]. This assumption is necessary whenever each component of the vector \( r \) is a combination of all the modes of the exosystem. Assumption 4 expresses the property that for each possible plant configuration there exists a corresponding controller ensuring stability and offset-free behaviour. The latter prescription is achieved by letting each candidate controller to incorporate a model of the reference to be tracked. Note that this design constraint becomes necessary, in order to satisfy the internal-model principle, when asymptotic tracking cannot be taken care by the internal plant modes. Moreover, such a practice is also advisable in order to preserve output regulation under arbitrarily small perturbations of the plant parameters [19, Section 2.6]. Finally, notice that Assumption 4 has to be regarded as a design condition which, under Assumptions 1–3, can always be satisfied through a suitable choice of the controllers.

3. Distinguishability of feedback linear systems

Intuitively, the problem of selecting a suitable rule for the controller mode \( \sigma \) is closely related to the problem of suitably inferring the plant model \( \rho \) from the measured data. This section is entirely devoted to this issue.

Let \( w := (x^T \ q^T)^T \in \mathbb{R}^{n_x+n_q} \) denote the state of the closed-loop resulting from the feedback interconnection of (1) with (3). Furthermore, let \( z := (u^T \ e^T)^T \in \mathbb{R}^{2n_x} \) denote the corresponding output. We obtain

\[
\begin{align*}
\dot{w}(t) &= A_{i,j}^{t}w(t) + B_{i,j}^{t}r(t), \\
\dot{z}(t) &= C_{i,j}^{t}w(t) + D_{i,j}^{t}r(t), \quad w(0) = w_0,
\end{align*}
\]

where

\[
A_{i,j}^{t} := \begin{pmatrix} A_i - B_i K_j C_i \\ -C_i G_j \end{pmatrix}, \quad B_{i,j}^{t} := \begin{pmatrix} B_i K_j \\ G_j \end{pmatrix}, \quad C_{i,j}^{t} := \begin{pmatrix} -K_j C_i \\ H_j \end{pmatrix}, \quad D_{i,j}^{t} := \begin{pmatrix} K_j \end{pmatrix}, \quad i, j \in \mathcal{N}.
\]

Let now \( w_{ij}(t, t_0, w_0, p_0) \) and \( z_{ij}(t, t_0, w_0, p_0) \) denote the state and, respectively, output response of (6) at time \( t \) when the controller switching signal is \( \sigma(t) = j \) for any \( t \in [t_0, t] \), the plant switching signal is \( \rho(t) = i \) for any \( t \in [t_0, t] \), the closed-loop system initial state is \( w(t_0) = w_0 \), and the initial state of the exosystem is \( p(t_0) = p_0 \).

The following notion of closed-loop distinguishability between plant modes is introduced.

Definition 1. For system (6), two plant modes \( i, \ell \in \mathcal{N} \) with \( i \neq \ell \) are said to be closed-loop distinguishable if

\[
z_{ij}(\cdot, t_0, w_0, p_0) \neq z_{ij}(\cdot, t_0, \hat{w}_0, p_0) \quad \text{a.e. on } [t_0, t],
\]

for any \( t_0, t \) with \( t > t_0 \). Indeed, any non-zero vector \( (w_0^T \ \hat{w}_0^T \ p_0^T)^T \in \mathbb{R}^{2n_x+n_q+n_y} \).

In Definition 1, by a.e. we mean everywhere except on a set whose Lebesgue measure is zero. Roughly speaking, two plant modes are closed-loop distinguishable when, over any finite interval, they always lead to different data provided that their state trajectories are not jointly null. Our aim is therefore to address the problem of discerning which plant modes \( i \in \mathcal{N} \) could have produced the data \( z \) collected with the controller \( \mathcal{C}_j \) in the feedback loop.

In the remainder of this section, we shall devote the attention to the following aspects: (i) there exist plant-dependent conditions under which it is not possible to ensure closed-loop distinguishability; (ii) there exist controller-dependent conditions under which the existence of trajectories leading to closed-loop indistinguishability does not compromise the fulfilment of the control goals. We point out that, the “distinguishability” problem is not new in the literature of switching systems (e.g. [2,3,7]). Nonetheless, the framework considered here is different (hence item (i) above) due to the presence of the feedback interconnection as well as of the exosystem. For this reason, it is convenient to address this issue afresh, and in a way that is suited for the specific problems of interest.

Definition 2. The feedback loop \( \mathcal{P}_i/\mathcal{C}_j \) admits a steady-state response if there exists a matrix \( W_{ij} \) such that, for every \( p_0 \), there exists a \( w_0^i \) such that

\[
w_{ij}(\cdot, t_0, w_0^i, p_0) = W_{ij} e^{(t-t_0)p_0}, \quad t \geq t_0
\]

for any \( t \in [t_0, t] \) with \( t > t_0 \). Furthermore, provided that \( W_{ij} \) exists, the feedback loop \( \mathcal{P}_i/\mathcal{C}_j \) is said to be in steady-state on \([t_0, t] \) with \( t \geq t_0 \) if \( w_0 = W_{ij} p_0 \).

Notice that, according to Definition 2, the steady-state response can be defined irrespective of the stability of \( \mathcal{P}_i/\mathcal{C}_j \). In fact, with the term steady-state response, we simply refer to the condition where the response contains only the modes of the exosystem. It is not difficult to verify that the existence of a steady-state response for \( \mathcal{P}_i/\mathcal{C}_j \) is equivalent to the existence of a matrix \( W_{ij} \) that solves the Sylvester equation [20, Section 4.4.1]

\[
A_{i,j}^{t} W_{ij} + B_{i,j}^{t} L = W_{ij} E.
\]
With this in mind, we can now draw some general conclusions regarding (7) and (8). Let \( \psi_{ij}(\lambda) \) denote the characteristic polynomial of the closed-loop \((\mathcal{P}_i/\mathcal{E}_j)\). Then, for any \( \lambda \in \text{spec}[E] \) we have

\[
\psi_{ij}(\lambda) = \det \begin{pmatrix} Q(s) & U_i(s) \\ -R_i(s) & S_j(s) \end{pmatrix} \bigg|_{s=\lambda} = \det \begin{pmatrix} Q(\lambda) & U_i(\lambda) \\ 0 & S_j(\lambda) \end{pmatrix} = \det(Q(\lambda)) \det(S_j(\lambda)),
\]

where the second equality follows from Assumption 4. Thus we conclude that \( \psi_{ij}(\lambda) \neq 0 \) for every \( i,j \in \mathcal{N} \) and every \( \lambda \in \text{spec}[E] \). This property is indeed a direct consequence of Assumption 3 and the fact that \( R_i \) and \( S_j \) are coprime. From (9) it follows that

\[
\text{spec}(A^{cl}_{ij}) \cap \text{spec}[E] = \emptyset, \quad \forall i,j \in \mathcal{N}.
\]

Theoretically, the main implication of (10) is that for each feedback loop \((\mathcal{P}_i/\mathcal{E}_j)\) the following properties hold: (i) there always exists a steady-state solution; (ii) the steady-state solution is unique. These properties are indeed a direct consequence of the fact that (10) implies existence and uniqueness of the solution \( W_{ij} \) of (8) [20, Section 2.5.3]. Notice also that in this case the initial state \( W_{ij}^{0} \) in (7) associated with the steady-state response takes the form \( W_{ij}^{0} = W_{ij} \). From a practical viewpoint, the main implication of (10) is that it allows to rewrite the closed-loop output data \( z_{ij}(t, t_0, W_{ij}, p_0) \) in a form well-suited for analysis purposes.

In fact, simple calculations yield

\[
z_{ij}(t, t_0, W_{ij}, p_0) = C_{ij}^{cl} e^{A_{ij}^{cl}(t-t_0)} W_{ij} + C_{ij}^{cl} \int_{t_0}^{t} e^{A_{ij}^{cl}(t-t)} B_{ij}^{cl} L e^{E_{\tau}} p_0 d\tau + D_{ij}^{cl} e^{E_{t}} p_0 = C_{ij}^{cl} e^{A_{ij}^{cl}(t-t_0)} (W_{ij} - W_{ij} p_0) + Z_{ij}(t, t_0, W_{ij}, p_0),
\]

where

\[
Z_{ij} := C_{ij}^{cl} W_{ij} + D_{ij}^{cl} L.
\]

In words, for any possible feedback interconnection \((\mathcal{P}_i/\mathcal{E}_j)\) it is possible to decompose the output in terms of a “transient” response, namely \( C_{ij}^{cl} e^{A_{ij}^{cl}(t-t_0)} (W_{ij} - W_{ij} p_0) \), plus a “steady-state” response, namely \( Z_{ij}(t, t_0, W_{ij}, p_0) \).

In the next two subsections, we enter into the detail of the comments made right before Definition 3. In particular, we will see that, while it may be impossible to uniquely recover the plant mode when the system evolves along its steady-state trajectory, suitable design conditions do exist under which it is possible to uniquely recover the plant mode when the system evolves along its non steady-state trajectories. We will also discuss why this latter property is sufficient to successfully address both the problems of stabilization and tracking.

3.1. Steady-state distinguishability

**Definition 3.** Two plant modes \( i, \ell \in \mathcal{N} \) with \( i \neq \ell \) are said to be closed-loop steady-state distinguishable if they are distinguishable along their steady-state output trajectories. \( \square \)

Recall now that the steady-state output response of \((\mathcal{P}_i/\mathcal{E}_j)\), obtained by letting \( W_{ij} = W_{ij} p_0 \), can be written as

\[
z_{ij}(t, t_0, W_{ij} p_0, p_0) = Z_{ij}(t, t_0, W_{ij} p_0, p_0) = Z_{ij}(t, t_0, W_{ij} p_0, p_0).
\]

Then, a first characterization of steady-state distinguishability can be readily obtained as follows.

**Proposition 1.** Two plant modes \( i, \ell \in \mathcal{N} \) with \( i \neq \ell \) are steady-state distinguishable if and only if the pair \((Z_{ij} - Z_{ij}, E)\) is observable for any \( j \in \mathcal{N} \). \( \square \)

An alternative, more intuitive, condition for steady-state distinguishability can be obtained by resorting to modal analysis. To this end, consider that by Assumption 2 there exists a similarity transformation \( T \in \mathbb{C}^{n_p \times n_p} \) of the form

\[
T := \begin{pmatrix} v_{1} & v_{2} & \cdots & v_{n_p} \end{pmatrix}
\]

such that \( T^{-1} E T = A_{k}, A_{k} := \text{diag}\{\lambda_{1}, \ldots, \lambda_{n_p}\} \), where the columns of \( T \) are eigenvectors of \( E \). Here and in the sequel it is understood that if \( v_{i} \) is an eigenvector of \( E \) corresponding to the complex eigenvalue \( \lambda_{i} \), then \( v_{i} = v_{i}^{*} \) for some \( j \in \{1, \ldots, n_p\} \), where \( v_{i}^{*} \) denotes complex conjugate. By letting \( \xi_{0} := T^{-1} p_{0} = (\xi_{10} \cdots \xi_{n_p0})^{T} \), we get

\[
r(t) = 1 \sum_{k=1}^{n_p} \xi_{0k} e^{\lambda_{k}(t-t_0)} v_{k}.
\]

Thus, one easily verifies that the steady-state output response of \((\mathcal{P}_i/\mathcal{E}_j)\) can be written as

\[
z_{ij}(t, t_0, W_{ij} p_0, p_0) = Z_{ij}(t, t_0, W_{ij} p_0, p_0) = \sum_{k=1}^{n_p} \xi_{0k} g_{ij}(\lambda_{k}) L e^{\lambda_{k}(t-t_0)} v_{k},
\]

where

\[
g_{ij}(s) := C_{ij}^{cl} (s I - A_{ij}^{cl})^{-1} B_{ij}^{cl} + D_{ij}^{cl}, \quad i,j \in \mathcal{N},
\]

stands for the transfer matrix of the system \((\mathcal{P}_i/\mathcal{E}_j)\) mapping \( r \) into \( z \). One can therefore state the following result.

**Proposition 2.** Two plant modes \( i, \ell \in \mathcal{N} \) with \( i \neq \ell \) are steady-state distinguishable if and only if

\[
v_{k} \not\in \ker \left\{ (g_{ij}(\lambda_{k}) - g_{ij}(\lambda_{\ell})) L \right\},
\]

for every \( k = 1, \ldots, n_p \) and every \( j \in \mathcal{N} \), where \( \ker \{ \cdot \} \) denotes kernel. \( \square \)

Although the steady-state responses \( z_{ij}(t, t_0, W_{ij} p_0, p_0) \) as well as the conditions of Propositions 1 and 2 seemingly depend on both the indices \( i \) and \( j \), actually, steady-state distinguishability does only depend on the open-loop plant features. Let Assumptions 3 and 4 hold and notice that, even if Assumption 4 is controller-dependent, it may become a necessary prerequisite for the control arrangement when asymptotic tracking cannot be taken care by the internal plant modes. Consider now that the steady-state output response (12) satisfies

\[
z_{ij}(t, t_0, W_{ij} p_0, p_0) = \sum_{k=1}^{n_p} \xi_{0k} e^{\lambda_{k}(t-t_0)} \begin{pmatrix} u_{k}^{i} \\ e_{k}^{ij} \end{pmatrix},
\]

where \( u_{k}^{i} \) and \( e_{k}^{ij} \) satisfy

\[
\begin{pmatrix} Q_{i}(\lambda_{k}) & U_{i}(\lambda_{k}) \\ -R_{i}(\lambda_{k}) & S_{j}(\lambda_{k}) \end{pmatrix} \begin{pmatrix} u_{k}^{i} \\ e_{k}^{ij} \end{pmatrix} = \begin{pmatrix} U_{i}(\lambda_{k}) \\ 0 \end{pmatrix} L v_{k}.
\]

Consider next that, for any feedback interconnection \((\mathcal{P}_i/\mathcal{E}_j)\) we can write (dropping the argument \( s \))

\[
\begin{pmatrix} Q_{i} & U_{i} \\ -R_{i} & S_{j} \end{pmatrix} \begin{pmatrix} \lambda_{i} - \gamma_{i} \\ \gamma_{ij} \end{pmatrix} = \begin{pmatrix} V_{ij} \\ 0 \end{pmatrix} \begin{pmatrix} \gamma_{ij} \end{pmatrix},
\]
where \( V_{ij} := Q_i \otimes U_i \otimes \alpha_i \) and \( \alpha_{ij} := R_j \otimes S_j \otimes \beta_i \). Since both \( V_{ij} \) and \( \alpha_{ij} \) are invertible for \( s = \lambda, \lambda \in \text{spec}(E) \), we have that
\[
\begin{align*}
\left( \begin{array}{c}
\mathbf{u}_{ij}^k \\
\mathbf{e}_{ij}^k
\end{array} \right) &= \left( \begin{array}{c}
\alpha_{ij}(\lambda_k) \\
\beta_{ij}(\lambda_k)
\end{array} \right) V_{ij}^{-1}(\lambda_k) \mathbf{u}_i(\lambda_k) L v_k.
\end{align*}
\]
(17)

Hence, by virtue of Assumption 4, one has
\[
\begin{align*}
\mathbf{e}_{ij}^k &= 0, \\
\mathbf{u}_{ij}^k &= \mathbf{u}_i(\lambda_k)^{-1} U_i(\lambda_k) L v_k.
\end{align*}
\]
(18)

In view of this fact, the following considerations can be made:

(i) Two plant modes \( i \) and \( \ell \) are indistinguishable in steady-state if and only if there exists at least one index \( k \in \{1, \ldots, n_{p_i}\} \) for which
\[
Q_i(\lambda_k)^{-1} U_i(\lambda_k) L v_k = Q_\ell(\lambda_k)^{-1} U_\ell(\lambda_k) L v_k.
\]
Under such circumstances, it is impossible to determine whether a steady-state output response is generated by \( \mathcal{P}_i \) or \( \mathcal{P}_\ell \).

(ii) Indistinguishability along steady-state trajectories does not influence the control goals since a steady-state is always associated to offset-free tracking.

As will be shown in Section 5, this second property ensures that, in order to achieve the desired control objectives, it is only required that the closed-loop (6) be distinguishable along its non steady-state trajectories.

**Remark 2.** Throughout the remainder of the paper, we emphasize the fact that the steady-state output responses of the systems (7) and (8) are NSS distinguishable if, for any \( j \in \mathcal{N} \), the closed-loop characteristic polynomials \( \varphi_{ij}(s) \) and \( \varphi_{\ell j}(s) \) are coprime.

3.2. Non steady-state distinguishability

**Definition 4.** Two plant modes \( i, \ell \in \mathcal{N} \) with \( i \neq \ell \) are said to be closed-loop distinguishable in non steady-state (NSS distinguishable, for short) if they are closed-loop distinguishable along their non steady-state output trajectories. Further, (6) is said to be mode-observable in non steady-state (NSS mode-observable, for short) if any two different plant modes \( i, \ell \in \mathcal{N} \) are NSS distinguishable.

In words, NSS mode-observability corresponds to the invertibility of the mapping from any non steady-state output trajectory to the plant switching signal \( \rho \). Hereafter, necessary and sufficient conditions for NSS mode-observability of (6) are given. For a given pair \((M, N), M \in \mathbb{R}^{h \times h}, N \in \mathbb{R}^{l \times l}\), let
\[
\Theta_{(M,N)} := (M^T (M N)^T \cdots (M (N)^{-1})^T)^T
\]
(19)
denote its observability matrix. It is worth pointing out that, in view of Assumption 1, the pair \((C_{ij}^d; A_{ij}^d)\) turns out to be observable, i.e. the observability matrix \(\Theta_{(C_{ij}^d; A_{ij}^d)}\) is full-rank for any \( i \) and \( j \). The following proposition unveils that the joint observability matrix
\[
\Theta_{i,f_{ij}} := \left( \Theta_{(C_{ij}^d; A_{ij}^d)} \ , \ 
\Theta_{(C_{i,j}^d; A_{i,j}^d)} \right)
\]
(20)
plays a key role in determining NSS distinguishability of two plant modes \( i \) and \( \ell \).

**Proposition 3.** Two plant modes \( i, \ell \in \mathcal{N} \) with \( i \neq \ell \) are NSS distinguishable if and only if their joint observability matrix is full-rank, i.e.,
\[
\text{rank } \Theta_{i,f_{ij}} = 2(n_u + n_{p_i}), \quad \forall j \in \mathcal{N}.
\]
As a consequence, the feedback system (6) is NSS mode-observable if and only if (21) holds for any pair of different plant modes \( i, \ell \in \mathcal{N} \).

The results of Proposition 3 are closely connected to those derived in [2,3] for autonomous switching linear systems. In fact, a closer examination of this result reveals that (20) coincides with the observability matrix of the autonomous system
\[
\begin{align*}
\dot{x}(t) &= \left( A_{ij}^d \ \ 0 \right) \chi(t) \\
\upsilon(t) &= \left( C_{ij}^d \right)^T \chi(t)
\end{align*}
\]
obtained from the parallel connection of \((\mathcal{P}_j/\mathcal{E}_j)\) and \((\mathcal{P}_\ell/\mathcal{E}_\ell)\). Thanks to such equivalent interpretation of the joint observability matrix (20), we can invoke classical results on the observability properties of composite systems [21] and derive sufficient conditions for the distinguishability of two plant modes (see also [18] for a more detailed discussion on this point).

**Proposition 4.** Under Assumption 1, two plant modes \( i, \ell \in \mathcal{N} \) with \( i \neq \ell \) are NSS distinguishable if, for any \( j \in \mathcal{N} \), the closed-loop characteristic polynomials \( \varphi_{ij}(s) \) and \( \varphi_{\ell j}(s) \) have no common roots. It therefore provides guidelines on how each controller mode has to be designed in order to make it possible to uniquely reconstruct the plant model when the system evolves along non steady-state trajectories. With this in mind, we introduce the following assumption, which, based on the previous arguments, has to be regarded as a design condition for the controller.

**Assumption 5.** The feedback system (6) is NSS mode-observable.

4. Mode estimator and controller switching signal

In this section, we describe the mechanism, called mode estimator, apt to generate the estimate of the current plant mode based on the measured data \( z \). Such an estimate is used to orchestrate the controller switching. More precisely, the mode estimator updates the estimate of the plant mode \( \rho \) at discrete-time instants of the type \( kT \), where \( k \in \mathbb{Z}_+ \), and \( T > 0 \) is the so called controller dwell-time. This implies that the controller switching signal stays constant over each time interval \( I_k := [kT, (k+1)T) \),
\[
\sigma(t) = \sigma_k, \quad t \in I_k,
\]
(22)
with \( \sigma_k \in \mathcal{N} \). In order to generate \( \sigma_{k+1} \), the mode estimator evaluates the distance of the plant input/output data collected over \( I_k \) from the subspaces associated to each possible mode. In particular, the mechanism in question relies on the following observation: if, in addition to the controller mode, also the plant mode takes on a constant value, say \( i \), over \( I_k \), then the evolution of the plant input/output data on \( I_k \) becomes
\[
z(t) = z_{i_0}(t, kT, \upsilon(kT), p(kT)), \quad t \in I_k.
\]
(23)
Thus, the set \( \delta_{i_0/k}(I_k) \) of all possible measured data on \( I_k \) associated with a plant mode \( i \) and a controller mode \( \sigma_k \) corresponds to the linear variety
\[
\delta_{i_0/k}(I_k) := \left\{ \hat{z} \in \mathcal{L}_2(I_k) : \hat{z}(\cdot) = z_{i_0}(\cdot, kT, \hat{\upsilon}, p(kT)) \right\}
\]
on \( I_k \), for some \( \hat{\upsilon} \in \mathbb{R}^{n_u+n_{p_i}} \).
(24)

Next proposition descends directly from the definition of NSS mode-observability.
Proposition 5. Under Assumptions 3–5, for any two different plant modes \( i, \ell \in \mathcal{N} \) and any controller mode \( \alpha_k \in \mathcal{N} \),
\[
\delta_t/\delta_t(\delta_t) \cap \delta_t/\delta_t(\delta_t) = \begin{cases} 
Z_t e^{(i-\ell)t} p(t), & \text{if } p(t) \in \ker \{\Theta_t(\delta_t - Z_t, \ell)\} \\
\emptyset, & \text{otherwise},
\end{cases}
\]
with \( t \in I_k \).
\[\square\]

In agreement with the comments made in Section 3.1, disjointness of any two sets \( \delta_t(\delta_t) \) and \( \delta_t(\delta_t) \) is a controller-independent feature. Based on Proposition 5, a convenient criterion represents the natural (unforced) output response of the loop denotethe output transition matrix of the closed-loop system evolution in Appendix A. A convenient approach for estimating the plant mode on the interval \( I_k \) consists in selecting the index \( i \) for which the distance between the observed data \( z \) on \( I_k \) and \( \delta_t(\delta_t) \) is minimal. Let
\[
\Psi_t(t, kT) \triangleq C_{t/t}^i e^{(i-\ell)t} (t-kT) \tag{25}
\]
denote the output transition matrix of the closed-loop \((P_t/C_t)\). Then, \( \sigma_{k+1} \) can be obtained according to the minimum-distance criterion
\[
\delta_t(\delta_t) \triangleq \min_{\delta_t(\delta_t)} \| z_t - \delta_t(\delta_t)(, kT, t, \delta_t, p(t)) \|_{2, I_k} = \min_{\delta_t(\delta_t)} \| \xi_{t/t}(\delta_t, t, kT) - \Psi_t(t, kT) \delta_t(t) \|_{2, I_k}, \tag{26}
\]
where
\[
\xi_{t/t}(\delta_t, t, kT) \triangleq z_t - C_{t/t}^i e^{(i-\ell)t} b_{t/t}^i r(t) - D_{t/t}^i r(t) \tag{27}
\]
represents the natural (unforced) output response of the closed-loop \((P_t/C_t)\) on \([kT, t]\).

Effectiveness of (26) stems from the following observations:

1. first, being the pair \((C_{t/t}^i, b_{t/t}^i)\) completely observable by assumption, the minimization in (26) has always an unique solution for any \( i, j \) and \( k \); second, when the plant mode \( \rho(t) \) is constant on \( I_k \) and equal, say, to \( \rho_k \), one has
\[
\xi_{t/t}(\delta_t, t, kT) = 0, \tag{28}
\]
and, hence,
\[
\delta_t(\delta_t)(z, I_k) = 0. \tag{29}
\]

Based on (28), the controller mode can be updated according to the following logic: at any time instant \( kT \) one first selects, arbitrarily, a value \( i_k \in \mathcal{N} \) among those which achieve the minimum, i.e. \( i_k \in \arg \min_{\delta_t(\delta_t)} \delta_t(\delta_t)(z, I_k) \); then, if \( \delta_t(\delta_t)(z, I_k) \) is smaller than \( \delta_t(\delta_t)(z, I_k) \), then \( \sigma_{k+1} \) is set equal to \( i_k \), otherwise the controller mode is left unchanged. For brevity, the considered logic will be referred to as dwell-time switching logic (DTS). The main implication stemming from Proposition 4 and (28) is that whenever the system evolves along non steady-state trajectories and the plant mode does not vary over \( I_k \), the proposed mechanism ensures that the plant mode can be exactly reconstructed, i.e.
\[
\sigma_{k+1} = \rho_k. \tag{30}
\]

To maintain continuity, we now proceed to analyse the stability properties deriving from the use of the proposed minimum-distance criterion. Comments regarding the practical implementation of the mode-estimator are deferred to Appendix A.

5. Closed-loop stability and tracking properties

In this section, we analyse the stability properties deriving from the use of the minimum-distance criterion (26). To begin with, observe that Assumption 4 implies the existence of positive reals \( \lambda \) and \( \mu \) such that
\[
\| e^{\rho \cdot t} \| \leq \mu e^{-\lambda t}, \quad \forall t \in \mathbb{R}_+, \forall i \in \mathcal{N}. \tag{31}
\]
Further, since the set \( \mathcal{N} \) is finite, there also exist positive reals \( \eta \) and \( \theta \) such that
\[
\| e^{\rho \cdot t} \| \leq \theta e^{\eta t}, \quad \forall t \in \mathbb{R}_+, \forall i, j \in \mathcal{N}. \tag{32}
\]
In order to prove Theorem 1, we avail of the following result.

Lemma 1. Let Assumptions 1–5 hold and let the controller switching signal be selected in accordance with the minimum-distance criterion (26) and the DTS. Further assume that the plant mode is constant on the time interval \([t_0, t_1]\). Then, there exist finite positive reals \( \beta \) and \( \gamma \) such that
### Remark 3.
In relation to item (ii) of Theorem 1, it can be seen from the proof that the decaying rate of $e^0(\cdot)$ is actually exponential in the sense that, in each interval wherein the plant switching signal is constant, the norm of $e^0(\cdot)$ is upper bounded by an exponentially converging function.

In connection with the results of Theorem 1 and the comments made in Remark 3, it should be clear that it is in general impossible to ensure asymptotic tracking (item (ii)) under persistent variations of the plant to be controlled, and that this sort of limitation is not due to the specific control architecture considered. Just to give an example, recall that Assumption 3 is equivalent to having [12, Section 3.11]

$$\det \begin{bmatrix} A_i - \lambda_k I & B_i \\ C_i & 0 \end{bmatrix} \neq 0,$$

for every $i \in \mathcal{N}$ and every $\lambda_k \in \text{spec}(E_i)$. Assume now that the closed-loop is given by $(\mathcal{P}_i/E_i)$ and that the loop is in steady-state from a certain time $t_0$. Let

$$W_{ij} := \begin{bmatrix} \Pi_{ij} \\ \Sigma_{ij} \end{bmatrix},$$

where $\Pi_{ij} \in \mathbb{R}^{n_i \times n_p}$ and $\Sigma_{ij} \in \mathbb{R}^{n_i \times n_p}$. From (8) it is simple to verify that we have

$$\Pi_{ij} E = A_i \Pi_{ij} + B_i \mathcal{E}_i, \quad L = C_i \Pi_{ij},$$

where $\mathcal{E}_i := H_i \Sigma_i$, $\Pi_{ij}$, $\Sigma_{ij}$ and $\mathcal{E}_i$ being the matrices corresponding to the plant steady-state, the controller steady-state and, respectively, the controller steady-output [17]. Consider now $\Pi_{ij} := \Pi_{ij} \frac{1}{C_i} \mathcal{E}_i \Sigma_{ij} := \mathcal{E}_{ij} \frac{1}{C_i} \mathcal{E}_j \Sigma_{ij}$ where by resorting to modal analysis, the plant steady-state and the controller steady-output satisfy

$$\begin{aligned}
\xi(t) &= \Pi_{ij} e^{(t-t_0)} \mathcal{P}_0 \xi_0 + \mathcal{E}_{ij} e^{(t-t_0)} \mathcal{P}_0 \xi_0 \\
&= \sum_{k=1}^{n_p} \xi_0 e^{(t-t_0)} \mathcal{P}_0 \xi_0 \\
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}(t) &= \mathcal{E}_i e^{(t-t_0)} \mathcal{P}_0 \xi_0 + \mathcal{E}_{ij} e^{(t-t_0)} \mathcal{P}_0 \xi_0 \\
&= \sum_{k=1}^{n_p} \xi_0 e^{(t-t_0)} \mathcal{P}_0 \xi_0 \\
\end{aligned}$$

where $\mathcal{P}_0$ represents: longitudinal velocity and the pitch angle of the helicopter are regulated around desired set-points. Note that the velocity represents a deviation from trim condition rather than an absolute value [24]. The reader is referred to [24] and the references therein for a detailed discussion of the problem.

A simple way for synthesizing a switching controller ensuring offset-free tracking is to resort to the polynomial matrix formulation [25]. To this end, consider the following model for the exosystem

$$E = \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

having PMFD $\Delta(s) = \text{diag}[s, s]$ and satisfying $\Delta(D) \mathcal{F}(t) = 0$ where $D = (s, s)$ is the differential operator. Consider next a left PMFD of (39) $\mathcal{P}_i(s) = U_i^{-1}(s) Q_i(s)$. Note that $Q_i(0)$ is full-rank for

### Table 1
Model parameters for the airspeed ranges [24].

<table>
<thead>
<tr>
<th>Airspeed (knots)</th>
<th>Regime $i$</th>
<th>$a_i$</th>
<th>$b_i$</th>
<th>$y_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50–75</td>
<td>1</td>
<td>0.06365</td>
<td>0.1198</td>
<td>0.9775</td>
</tr>
<tr>
<td>75–100</td>
<td>2</td>
<td>0.2</td>
<td>0.5</td>
<td>1.9</td>
</tr>
<tr>
<td>100–125</td>
<td>3</td>
<td>0.285</td>
<td>0.9</td>
<td>2.73</td>
</tr>
<tr>
<td>125–150</td>
<td>4</td>
<td>0.3681</td>
<td>1.42</td>
<td>3.5446</td>
</tr>
<tr>
<td>150–175</td>
<td>5</td>
<td>0.5045</td>
<td>2.526</td>
<td>5.112</td>
</tr>
</tbody>
</table>

satisfied. Assume that $W_{ij} = W$ and $Z_{ij} = Z$ for all $i, j \in \mathcal{N}$. Under such conditions, the switched system (6) admits the state-coordinate transformation $\mathcal{W}(t) := w(t) - Wp(t)$. Accordingly, by means of simple manipulations, it is not difficult to see that the closed-loop dynamics can be rewritten as

$$\begin{bmatrix} \mathcal{W}(t) = A_{\phi(t)}(s) \mathcal{W}(t) \\ \mathcal{E}(t) = C_{\phi(t)}(s) \mathcal{W}(t) \end{bmatrix},$$

where $\mathcal{E}(t) := \Delta(t) - \Delta(0)$. Asymptotic stabilization of (38) implies asymptotic stabilization of (6) along with the offset-free tracking property.

### 5.1. An example
As an illustrative example, we consider a simplified model for the longitudinal and vertical dynamics, linearized about the trim condition, of a helicopter moving at different longitudinal airspeeds [24]. The model has five regimes, described by the linear systems

$$\begin{bmatrix} A_i \\ B_i \\ C_i \end{bmatrix} = \begin{bmatrix} \alpha_i & -0.707 \\ 0 & 1 \end{bmatrix}$$

where the values of the parameters $\{a_i, b_i, y_i\}$ for the five airspeed ranges corresponding to regimes are reported in Table 1.

The model state $x = (x_1 x_2 x_3 x_4)^T$ is made up by four variables representing: longitudinal velocity $x_1$ (kt units); vertical velocity $x_2$ (kt units); pitch rate $x_3$ (deg/s units); pitch angle $x_4$ (deg units). The vector of control inputs $u = (u_1 u_2)^T$ is made up by two variables representing: collective pitch of the rotor blades $u_1$ (deg units); longitudinal cyclic pitch of the rotor blades $u_2$ (deg units). The problem of interest is to synthesize a control action so as to stabilize the helicopter model and ensure that the longitudinal velocity and the pitch angle of the helicopter are regulated around desired set-points. Note that the velocity represents a deviation from trim condition rather than an absolute value [24]. The reader is referred to [24] and the references therein for a detailed discussion of the problem.
all $i \in \mathcal{N}$, that is Assumption 3 is satisfied. Using the differential operator notation, the plant input–output relationship corresponding to the $i$-th regime takes the form $U_i(D)\nu(t) = Q_i(D)u(t)$. Now, pre-multiplying the equation of the exosystem by $U_i(s)$ and the equation of the model by $\Delta(s)$ we obtain the following incremental representation for the model corresponding to $i$-th regime

$$U_i(D)\Delta(D)e(t) = Q_i(D)\nu(t) \quad(41)$$

where $\nu(t) := \Delta(D)u(t)$. Thus, any stabilizing compensator for (41) ensures that $e(t)$ and $\nu(t)$ converge to zero as $t \to \infty$. More precisely, since $Q_i(s)$ and $\Delta(s)$ are left coprime for all $i$ (for each regime $Q_i(0)$ is full-rank) there exist right coprime polynomial matrices $\mathcal{A}_i^{(s)}$ and $\mathcal{R}_i^{(s)}$ such that

$$\text{det} \left( U_i(s)\Delta(s)\mathcal{A}_i^{(s)} + Q_i(s)\mathcal{R}_i^{(s)} \right)$$

is strictly Hurwitz. The resulting compensator has right PMFD $\mathcal{R}_i^{(s)}\mathcal{A}_i^{(s)}$. The resulting compensator has right PMFD $\mathcal{R}_i^{(s)}\mathcal{A}_i^{(s)}$ with $\mathcal{A}_i^{(s)} = \Delta(s)\mathcal{A}_i^{(s)}$ and, according to the internal model principle, incorporates the model of the reference to be tracked (cf. Assumption 4). In the present case, the matrices $\mathcal{A}_i^{(s)}$ and $\mathcal{R}_i^{(s)}$ have been selected so as to minimize the quadratic performance index for (41)

$$\int_0^{\infty} \left[ e^\top(t)\Psi_e e(t) + \nu^\top(t)\Psi_{\nu} \nu(t) \right] dt$$

with weighting matrices: $\Psi_e := 10$ for regime 1; $\Psi_e := 1$ for regime 2; $\Psi_e := 0.1$ for regimes 3, 4 and 5; $\Psi_{\nu} := 0.1$ for all regimes. The weighting matrices have been selected on a trial and error basis.

The controllers have been obtained using the MATLAB Polynomial Toolbox, their stabilization ranges being reported in Table 2. The resulting switching controller, as in (3), has minimal state-space realization of order $n_k = 8$. Although one could adopt a single controller for regimes 1–4, we use several controllers so as to maintain optimal LQ trackers.

The mode estimator has been implemented as detailed in Appendix A in order to detect transitions from one regime to another, and orchestrate the controller switching. The dwell-time in the DTSL was set equal to $T = 2$ s. Fig. 2 depicts the behaviour of the regulated variables in response to a step input of magnitude 1 for the longitudinal velocity and zero pitch angle reference, with $\rho(t) := \begin{cases} 1. & t < 100 \\ 2. & 100 \leq t < 150 \\ 3. & 150 \leq t < 300 \\ 4. & 300 \leq t < 350 \\ 5. & t \geq 350 \end{cases}$ (42)

which represents an increasing airspeed scenario. Some observations are in order. There are two factors responsible for the observed spikes. The first one is due to the presence of transitions between regimes. An examination of plant equations in (39) reveals, indeed, that it is impossible to maintain zero tracking error upon plant changes, the steady-state matrices $M_i$ in (36) relative to the plant modes being different one from another. The second factor responsible for the observed spikes is due to controller switching. In fact, the problem of achieving bumpless transfer between controllers [24], is complementary to the problem of deciding controller scheduling policies, and of interest in its own right. A study of bumpless transfer techniques to be used within the proposed control scheme constitutes an important area for future investigation. Fig. 3 depicts plant and controller switching signals. One sees that $\sigma$ follows quite rapidly $\rho$. Indeed, according to the analysis, when the plant mode is constant, at most $2T = 4$ s are required to set $\sigma = \rho$ (if a plant variation occurs within $k_i$, the plant mode, provided it stays constant, is reconstructed in the interval $I_{k_i+1}$). In this example, the importance of having small values for $T$ comes from the fact that no single controller ensures stability for all 5 regimes. Hence decreasing $T$ reduces the time duration of possibly unstable feedback interconnections. Fig. 4 depicts the time behaviour of the functionals (26) used by the mode estimator for controller selection.

Simulations have been performed using SIMULINK. We solved the differential equations with a variable step ode45 solver under plant and controller zero initial conditions. The initial condition of the exosystem was set equal to (1 0)$. The reader is referred to Appendix A for the description of the equations used for implementing the mode-estimator.

### 6. Conclusions

In this paper, we described recent progress in the study of switching linear systems governed by unknown switching sequences. We addressed the problem of stabilization and tracking for continuous-time plants, modelled by a MIMO linear system that may switch among different modes taken from a finite set. It was shown that suitable control schemes do exist which ensure global exponential stability for any slow-on-the-average plant mode switching sequence, even when it is conceptually impossible to reconstruct the plant mode from the measured data. Effectiveness of the proposed approach has been confirmed through a simulation example. The results here presented lend
themselves to be extended in various directions. Most notably, the problem of achieving robustness against disturbances and/or process unmodelled dynamics constitutes an important area for future investigation.

Appendix A. A note on the mode-estimator

In this section, we describe a simple way for implementing the mode-estimator and provide a number of suitable references to the subject. The switching methodology developed in Section 4 hinges upon the computation of the functionals

$$\delta_{ij}(z(\cdot), I_k) := \min_{\hat{w} \in \mathbb{R}^{n \times m}} \left\| z(\cdot) - z_{ij}(\cdot, kT, \hat{w}, p(kT)) \right\|_{2,I_k}$$

$$= \min_{\hat{w} \in \mathbb{R}^{n \times m}} \left\| \zeta_{ij}(\cdot, kT) - \Psi_{ij}(\cdot, kT) \hat{w} \right\|_{2,I_k},$$

where $\Psi_{ij}(\cdot, kT)$ and $\zeta_{ij}(\cdot, kT)$ are as in (25) and (27), respectively. The signal $\zeta_{ij}(\cdot, kT)$ can be obtained by computing the forced response of $(\mathcal{P}_i / \mathcal{E}_i)$, i.e. by running, on the interval $I_k$, the loop $(\mathcal{P}_i / \mathcal{E}_i)$ driven by $r$ under zero initial conditions.

The above problem is a standard non-redundant least-squares problem. Taking the first derivative with respect to $\hat{w}$ we get

$$\delta_{ij}(z(\cdot), I_k) = \left( \int_{I_k} (\zeta_{ij}(\tau, kT) - \Psi_{ij}(\tau, kT) \hat{w}) d\tau \right)_{1/2}$$

where $\Psi_{ij}(T)$ and $\hat{w}_n$ denote the observability Gramian of $(\mathcal{P}_i / \mathcal{E}_i)$ on $I_k$ and the optimal solution of the minimization problem, respectively. The Gramian does only depend on the controller dwell-time $T$, and can be computed (off-line) via the differential equation

$$\frac{d}{dt} \Psi_{ij}(t) = (A_{ij}^T)^T \Psi_{ij}(t) + (C_{ij}^T)^T \zeta_{ij}(t, kT),$$

$$\Psi_{ij}(0) \equiv 0.$$

The optimal solution $\hat{\omega}_n$ is given by

$$\hat{\omega}_n = \Psi_{ij}^{-1}(T) \int_{I_k} \Psi_{ij}(\tau, kT)^T \zeta_{ij}(\tau, kT) d\tau.$$

The integral term in the above expression, say $\mathcal{P}_{ij}((k + 1)T)$, can be obtained by solving the differential equation

$$\frac{d}{dt} \mathcal{P}_{ij}(t) = -(A_{ij}^T)^T \mathcal{P}_{ij}(t) + (C_{ij}^T)^T \zeta_{ij}(t, kT),$$

$$\mathcal{P}_{ij}(kT) \equiv 0.$$

Some observations are in order. In the computation of $\hat{\omega}_n$, inversion of $\mathcal{P}_{ij}(T)$ can be avoided by resorting to $\mathcal{P}_{ij}(T)$-balanced realizations. In particular, since $(\mathcal{P}_i / \mathcal{E}_i)$ is observable by hypothesis, it is always possible to choose a suitable similarity transformation for the closed-loop $(\mathcal{P}_i / \mathcal{E}_i)$ yielding $\mathcal{P}_{ij}(T) = I$ [26]. Further, instead of computing the relevant integrals on the whole interval $I_k$ (which would require real-time operations), one can collect data on subintervals of the type $[kT, kT + r]$ with $r \in (0, T)$, reserving the subinterval $[kT + r, (k + 1)T]$ for all the necessary computations. It is simple to verify that all developments continue to hold also in this more general setting.

The considered mode-estimation scheme is the simplest possible approach to least-square estimation (as found in several textbooks [27, Section 9.2], [20, Section 2.6]), and was motivated here by simplicity of analysis. The literature on least-square estimation/finite-memory filters is, nonetheless, quite vast. Concerning continuous-time approaches, the time delay operator approach [26,28] as well as the FIR filters-based schemes [29], for instance, certainly prove relevant in this regard. To speed up computations and simplify the mode-estimation scheme, a suitable approach is that of replacing continuous-time integrations with finite summations, which can always be done provided that the sampling instants are adequately chosen. For discrete-time approaches to observability of continuous-time systems, see, for instance, [30,31].

Appendix B. Proofs

Proof of Proposition 3. Consider (11). Since the spectra of $A_{ij}^T$ and $E$ are disjoint for all $i, j \in \mathcal{N}$, the functions $C_{ij}^T e^{A_{ij}(\tau - w)} (w_0 - W_{ij} p_0)$ and $Z_{ij} e^{(\tau - t_0)} p_0$ are linearly independent on every interval $[t_0, t]$. Therefore, the NSS distinguishability condition amounts to requiring

$$C_{ij}^T e^{A_{ij}(\tau - w)} w_0 \neq C_{ij}^T e^{A_{ij}(\tau - w)} w_0, \quad \text{a.e. on } [t_0, t],$$

for any non-zero vector $(w_0^T w_0^T)^T$, where $w_0 := w_0 - W_{ij} p_0$ and $w_0 := w_0 - W_{ij} p_0$. Note that the vectors $w_0$ and $\tilde{w}_0$ are arbitrary since both $w_0$ and $\tilde{w}_0$ are such. Then, (45) is equivalent to the observability of the pair

$$\left( \begin{array}{c} C_{ij}^T \\ C_{ij}^T \\ A_{ij}^T \\ 0 \end{array} \right),$$

which leads to (21). □

Proof of Lemma 1. Let $\rho_0$ denote the value taken on by the plant mode on $[t_0, t_1)$. Consider now the state response of the closed-loop $(\mathcal{P}_{\rho_0} / \mathcal{E}_{\sigma(t)})$ on the interval $[t_0, t_1)$.

$$w(t) = \Phi(t, t_0) w(t_0) + \int_{t_0}^t \Phi(\xi, t_0) \rho^T_{\rho_0 / \sigma(\xi)}(\xi) d\xi,$$

where $\Phi(t, t_0)$ denotes the state transition matrix of the loop $(\mathcal{P}_{\rho_0} / \mathcal{E}_{\sigma(t)})$ on the interval $[t_0, t_1)$. Two possible cases arise: if $t_1 - t_0 \leq 2\tau$, taking (31) into account and picking $\theta$ greater than one if necessary, we have $\| \Phi(t, t_0) \| \leq \theta e^{2\theta T - 1}$. In fact, being $t_1 - t_0 \leq 2\tau$ there can be at most two switching in the controller mode over $[t_0, t_1)$. This, combined with (46), yields

$$\| w(t) \| \leq \theta e^{2\theta T - 1} \| w(t_0) \| + k_\theta (\theta^2 / 2n) e^{2\theta T} \| r \| \infty, [t_0, t_1),$$

forall $t \in [t_0, t_1)$.
where $k_B = \max_{i,j,k} \|B_{ij}^k\|$. Further, picking $\theta$ and $\mu$ greater than one if necessary, the first term in the right-hand side of (47), can be upper bounded by $\theta^2 \mu^e e^{(q+\lambda)T} e^{-\lambda(t-h)} |w(t)|$.

Assume next $t_{h} - t_{h} > 2T$, and let $k_{h}$ denote the least integer such that $k_{h}T > t_{h}$. Then, the interval $[t_{h}, t_{h})$ can be decomposed into subintervals $[t_{h}, k_{h}T), (k_{h}T, (k_{h} + 1)T)$, and $(k_{h}T, (k_{h} + 1)T)$. Of course, over the interval $[t_{h}, t_{h})$, $C_{p_h}$ (the controller tuned on $s_{p_h}$) is switched-on or it is not. For clarity, we will address these two cases separately. In the negative case, by Assumption 5, the closed-loop is in steady state at and after time $k_{h}T$. In particular, (47) holds true over the interval $[t_{h}, (k_{h} + 1)T]$ because $(k_{h} + 1)T - t_{h} < 2T$,

$$|w(t)| \leq \theta^2 \mu^e e^{(q+\lambda)T} e^{-\lambda(t-h)} |w(t_{h})| + \kappa \theta (\theta^2 / 2)e^{2T} \|r\|_{\infty,[t_{h},t_{h})} \quad \forall t \in [t_{h}, (k_{h} + 1)T). \quad (48)$$

Moreover,

$$|w(t)| = |W_{p_{h}/\sigma}(t)p(t)| \leq \kappa \|r\|_{\infty,[t_{h},t_{h})} \quad \forall t \in [(k_{h} + 1)T, t_{h}), \quad (49)$$

for some positive real $\kappa$, as it follows from Assumption 2 and the fact that $p(t) = e^{(\theta^2/2)p(t_{h})}$, where

$$p(t_{h}) = \xi_{p_{h}}^{-1}(t_{h}) \int_{t_{h}}^{t} e^{E(t-x)H_{T}T} \xi_{p_{h}}(x) \, dx,$$

$\xi_{p_{h}}(t_{h})$ stands for the observability Gramian of $(L, E)$ over the interval $[t_{h}, t_{h})$, and $t - t_{h} \geq (k_{h} + 1)T - t_{h} > 0$. Combining (48) and (49) we conclude that

$$|w(t)| \leq \theta^2 \mu^e e^{(q+\lambda)T} e^{-\lambda(t-h)} |w(t_{h})| + \hat{k} \|r\|_{\infty,[t_{h},t_{h})}, \quad \forall t \in [t_{h}, t_{h}), \quad (50)$$

where $\hat{k} := k_B (\theta^2/2)e^{2T} + \kappa$.

Consider finally the case where the $C_{p_{h}}$ switched-on in feedback with the plant at some instant $k_{h}T \in [t_{h}, t_{h})$ where $k_{h}$ is a positive integer. If $k_{h}T - t_{h} < 2T$, the analysis is the same as in the previous case, and the inequality in (48) holds true over $[t_{h}, t_{h})$. If instead $k_{h}T - t_{h} > 2T$, then, by Assumption 5, the closed-loop is in steady-state over the interval $[k_{h}T, (k_{h} + 1)T)$. Consequently, the inequality in (48) holds true over $[k_{h}T, (k_{h} + 1)T)$, and the inequality in (49) holds true over $[(k_{h} + 1)T, (k_{h} + 1)T)$. Therefore, the inequality in (50) holds true over $[k_{h}T, (k_{h} + 1)T)$. On the other hand, over the remaining subinterval $[(k_{h} + 1)T, t_{h})$, picking $\mu$ greater than one if necessary, we have

$$\|\Phi(t, (k_{h} + 1)T)\| \leq \theta e^T \mu^e e^{-\lambda((k_{h} + 1)T - t_{h})} = \theta e^T \mu^e e^{-\lambda((k_{h} + 1)T - t_{h})},$$

$\forall t \in [(k_{h} + 1)T, t_{h})$, as it follows from the fact that $C_{p_{h}}$ is switched-on in feedback with the plant at $k_{h}T$. This, combined with (46), yields

$$|w(t)| \leq \theta e^T \mu^e e^{-\lambda((k_{h} + 1)T - t_{h})} |w(k_{h}T)| + \kappa \theta (\theta^2 / 2)e^{2T} \|r\|_{\infty,[k_{h}T,(k_{h} + 1)T)} \leq \hat{k} \|r\|_{\infty,[t_{h},t_{h})} \quad \forall t \in [(k_{h} + 1)T, t_{h}), \quad (51)$$

where $\hat{k} := \theta e^T (\theta^2 + \lambda) + \kappa + \hat{k}$.

Proof of Theorem 1. Consider a generic interval $[t_{0}, t)$ and let $t_{i}$ denote the $h$-th discontinuity of $\rho$ in such an interval. Further, define with $\rho_{i} \in \mathcal{N}$ the value taken on by $\rho$ on $[t_{i-1}, t_{i})$. From Lemma 1 we have

$$|w(t)| \leq \beta e^{-\lambda(t-h)} |w(t_{i})| + \gamma \|r\|_{\infty,[t_{i},t_{i})} \quad \forall t \in [t_{i}, t_{i+1}), \quad (53)$$

where $\beta = \theta^2 \mu^e e^{(q+\lambda)T}$ and defining $\alpha_{s} := \lambda - \log \mu + 2 \log \theta + (n + 1) \lambda T / \tau_{D}$ and $\beta_{s} := \theta^2 \mu^e e^{(q+\lambda)T}$, the latter inequality can be rewritten as

$$\beta_{s} e^{-\lambda(t-h)} \leq \beta_{s} e^{-\alpha_{s}(t-h)}.$$ \hspace{1cm} (54)

Now, under Assumption 6, we have

$$\beta_{s} e^{-\alpha_{s}(t-h)} \leq \beta_{s} e^{-\alpha_{s}(t-h)}.$$ \hspace{1cm} (55)

Using (55), we finally get

$$\sum_{h=0}^{N_{e}(t_{0})} \beta_{s} e^{-\alpha_{s}(t-h)} \leq \sum_{h=0}^{N_{e}(t_{0})} \beta_{s} e^{-\alpha_{s}(t-h)}.$$ \hspace{1cm} (56)

which concludes the proof of (i).

As for (ii), let $t_{f}$ denote the final plant switching instant and let $\rho_{f} \in \mathcal{N}$ be the value taken on by $\rho$ from $t_{f}$ onwards. If $C_{p_{f}}$ (the controller tuned on $s_{p_{f}}$) is switched-on in feedback with the plant at or after $t_{f}$, the claim follows directly from Assumption 4 and the fact that, in agreement with the DTS, $C_{p_{f}}$ cannot be switched-off afterwards. If instead $C_{p_{f}}$ is never switched-on in feedback with the plant, then the switched system is in steady-state at and after $k_{f}T$, where $k_{f}$ denotes the least integer such that $k_{f}T > t_{f}$. From (18), we therefore conclude that $e(t) = 0$ for every $t \geq k_{f}T$. \hspace{1cm} □
References