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Harmonic analysis on $p$-torsional groups


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HARMONIC ANALYSIS ON p-TORSIONAL GROUPS
(after A. M. M. Gommers)

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The following is a presentation of results obtained by A. M. M. GOMMERS. Full
details will appear in his forthcoming thesis prepared under the guidance of A. C.
M. van Rooij.

1. The groups G that we consider are torsional, i.e. G satisfies the equivalent
conditions:

(a) G is a commutative topological group, and G has a zero-dimensional open
compact subgroup H such that G/H is a torsion group.

(b) G is a commutative, locally compact, zero-dimensional group such that every
finite subset of G lies in a compact subgroup.

Let p be a prime number; then G is called p-torsional (resp. p-free) when
for any open compact subgroup H of G the group G/H is a p-torsion group
(resp. has no p-torsion).

The field k is supposed to be a non-archimedean valued complete field with
residue field \( \mathbb{F} \) of characteristic p.

(1.1) LEMMA. - G has a unique decomposition as a topological product
\( G = G_1 \times G_2 \),
where \( G_1 \) is p-torsional and \( G_2 \) is p-free.

Proof. - For a compact zero-dimensional group G this decomposition is well
known. In the general case, each open compact subgroup H of G has an unique de-
composition \( H_1 \times H_2 \). Then \( G_i = \bigcup \{ H_i \; ; \; H \text{ open compact subgroup of } G \} \) (i=1,2)
provides the unique decomposition of G.

(1.2) Remarks. - On the part \( G_2 \) of G there exists a (k-valued) Haar measure
\( \mu \). Let \( C_\infty(G_2) \) denote the Banach space of the continuous functions \( G_2 \to k \)
which are "zero at \( \infty \)" provided with the supremum norm. On \( C_\infty(G_2) \) we have a
convolution

\[
(f \ast g)(a) = \int f(b) g(a - b) \, d\mu(b)
\]

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and $L(G_2)$ denotes $C_\omega(G_2)$ with the algebra structure given by the convolution.

Let us suppose, for convenience, that $k$ is algebraically closed. Then the dual of $G_2$ is $\hat{G}_2 = \text{the continuous homomorphisms } G_2 \to k^*$, provided with the compact open topology. The Fourier theory ([2], [3]) states:

$$F : L(G_2) \to C_\omega(\hat{G}_2)$$

is an isometric isomorphism of Banach algebra's where the Fourier transform $F$ is defined by:

$$F(f)(\chi) = \int f(b) \chi(-b) \, d\mu(b) \quad \text{with } f \in L(G_2) \quad \text{and } \chi \in \hat{G}_2.$$  

On the part $G_1$ of $G$ there is (in general) no Haar-measure. So $L(G_1)$ is meaningless. One studies instead $M(G_1)$. In general, $M(G)$ is the Banach space of tight measures on $G = \text{inj lim} \{C_\omega(H)'; \ H \text{ compact in } G\}$. In particular, if $G$ is compact then $M(G) = C_\omega(G)' = \text{the topological dual of the Banach space } C_\omega(G)$. On $M(G)$ the convolution is defined by

$$(\mu \ast \nu)(f) = \int f(a+b) \, d\mu(a) \, d\nu(b).$$

If $G$ is p-free then $M(G) \cong BUC(\hat{G}) = \text{the bounded uniformly continuous functions on } \hat{G}$. This isomorphism is given by:

$$\mu \mapsto \hat{\mu} \quad \text{and} \quad \hat{\mu}(\chi) = \int \chi(a) \, d\mu(a), \quad \text{where } \mu \in M(G) ; \ \chi \in \hat{G}.$$  

In general, the algebra $M(G)$ is (morally speaking) determined by $M(G_1)$ and $M(G_2)$. Since the part $M(G_2)$ is well known as an algebra, the remaining part $M(G_1)$ will have most of our attention.

We can formulate the connection between $M(G)$, $M(G_1)$, $M(G_2)$ as follows:

(1.3) PROPOSITION. - If $G_2$ is compact then $M(G) \cong M(G_1) \circ M(G_2)$ (as Banach algebra's).

Proof. - The operation $\circ$ is a variant of the tensor product of Banach spaces. We define $\circ$ only for pairs $(E, F')$, where $F'$ is the dual of some Banach space $F$.

Definition. - $E \circ F' = \text{proj lim} \{E \otimes F_0' ; \ F_0' \text{ finite dimensional subspace of } F\}$. In our case, $M(G_2)$ is naturally given as the dual of $C_\omega(G_2)$. One easily verifies the formula when $G_2$ is finite (then $\circ$ and $\otimes$ agree). From this the general case follows.

(1.4) Remarks.

1° If $G_2$ is not compact then $M(G) \cong \text{inj lim} \ M(G_1) \circ M(H_2)$, where $H_2$ runs in the set of all open compact subgroups of $G_2$. The isomorphism is again an isomorphism of Banach algebras.

2° If $G_2$ is compact, and $k$ is algebraically closed, then $M(G_2) = B(\hat{G}_2) =$
the bounded functions on \( \hat{G}_2 \). Proposition (1.3) yields
\[
M(G) = \prod_{x \in G_2} M(G_1) \times x \quad \text{and every} \quad M(G_1) \times x \cong M(G_1).
\]

3° In many cases, one can show that there is a \((1 - 1)\)-correspondance between the homomorphisms \( \varphi : M(G) \rightarrow k \) and the pairs of homomorphisms
\[
\varphi_i : M(G_i) \rightarrow k \quad (i = 1, 2).
\]
This holds for instance if \( k \) is not locally compact.

2. In this section, we assume that \( G \) is a \( p \)-torsional group.

Let \( T \) denote the discrete \( p \)-torsion group \( \mathbb{Q}_p / \mathbb{Z}_p \). If the field \( k \) has characteristic 0 and is algebraically closed then we can identify \( T \) with the subgroup of \( k^* \) consisting of the elements of order \( p^n \) \((n \geq 0)\).

For a \( p \)-torsional group \( G \) we define a dual \( G^* \) as the continuous homomorphisms \( G \rightarrow T \), provided with the compact open topology.

\( G^* \) is again \( p \)-torsional; \( G \cong G^{**} \); \( G \) is compact if, and only if, \( G^* \) is discrete.

There are two extreme cases for \( p \)-torsional groups:

- **Type (1):** \( G \) has no elements \((\neq 0)\) of finite order.
- **Type (2):** The elements of finite order are dense in \( G \).

For compact \( G \) one has: \( G \) is of type (1) if, and only if, \( G^* \) is a \( p \)-divisible group; \( G \) is of type (2) if, and only if, \( G^{**} \) has no \( p \)-divisible subgroups \( \neq 0 \). Further, if \( G \) is compact then \( G = G_1 \times G_2 \) where \( G_1 \) is of type (1). This follows from \( G^* = H_1 \times H_2 \) where \( H_1 \) is a maximal \( p \)-divisible subgroup of \( G^* \) and so \( G = H_1^* \times H_2^* \).

The compact groups \( G \) of type (1) are easily determined: \( G^* \) is \( p \)-divisible and (as is well known) it follows that \( G^* = T(I) \) for some index set \( I \). Then \( G \cong Z_p^I \) since \( T^* \cong Z_p \).

The compact groups \( G \) of type (2) (or their duals \( G^* \)) are very complicated in general. One can however prove the following:

(2.1) **Proposition.** - Let \( G \) be compact, then there exists an exact sequence of topological groups
\[
0 \rightarrow Z^I_p \rightarrow G \rightarrow \prod_{j \in J} Z/p^n_j \rightarrow 0.
\]

If \( \sup(x_j) < \infty \) then the sequence splits topologically.

Next, we have the following:
(2.2) PROPOSITION. - The following properties of the p-torsional group \( G \) are equivalent:

(a) \( G \) has no elements \( \neq 0 \) of finite order (i.e., \( G \) of type (i)),

(b) \( \mathbb{Z}_p^I \subset G \subset \mathbb{Z}_p^I \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \), where \( \mathbb{Z}_p^I \), with the product topology, is an open compact subgroup of \( G \),

(c) the norm on \( M(G) \) is multiplicative,

(d) for any \( \mu \in M(G), \, \mu \neq 0 \), one has:

\[ \mu \text{ is invertible in } M(G) \iff \|\mu\| = |\mu(G)| \cdot \]

Proof. - Since \( M(G) = \text{proj lim}(M(H); \, H \text{ open compact subgroup of } G) \), it suffices to consider compact groups \( G \). In this case, (b) can be replaced by (b'):

\[ G = \mathbb{Z}_p^I. \]

Another argument shows that the general case will follow from the case where \( G \) is topologically finitely generated. Such a group has the form

\[ G = \prod_{i=1}^n \mathbb{Z}_p / p^{m_i} \mathbb{Z}_p \quad \text{with } 0 < m_i < \infty. \]

In (2.3) and (2.4), \( M(G) \) is explicitly given and one can verify (2.2).

(2.3) PROPOSITION. - Let \( G = \mathbb{Z}_p^n \), then \( M(G) = \mathbb{C}[X_1, \ldots, X_n] \) is the Banach algebra of all power series \( \sum_{\alpha} a_{\alpha} X_1^{\alpha_1} \cdots X_n^{\alpha_n} \) with \( \sup|a_{\alpha}| < \infty \).

Proof. - \( C(\mathbb{Z}_p^n) \) has the orthonormal base

\[ (X_1, \ldots, X_n), \]

considered as a function : \( \mathbb{Z}_p^n \to \mathbb{C} \). The isomorphism of (2.3) is given by the map

\[ \mu \mapsto \sum_{\alpha} \mu(X_{\alpha}) X_{\alpha_1} \cdots X_{\alpha_n}. \]

(2.4) COROLLARY. - Let \( G = \prod_{i=1}^n \mathbb{Z}_p / p^{m_i} \mathbb{Z}_p \), then \( M(G) = \mathbb{C}[X_1, \ldots, X_n] / I \), where \( I \) is the ideal generated by \( (X_i + 1)^{p^{-m_i}} - 1 \) (all \( i \) with \( m_i \neq \infty \)).

(2.5) Remark. - If \( G \) is compact then there exists a surjective map \( \mathbb{Z}_p^I \to G \). Hence \( M(G) \) is a quotient of \( M(\mathbb{Z}_p^I) = \mathbb{C}[X_i | i \in I] \). If \( I \) is infinite then it is not clear what the kernel \( M(\mathbb{Z}_p^I) \to M(G) \) should be.

3. We suppose in this section that \( G \) is a compact p-torsional group.

(3.1) PROPOSITION. - Suppose that \( k \) has characteristic \( p \). Then

(a) \( M(G) \) has no idempotents \( \neq 0, 1 \),

(b) any character \( \chi : G \to k^* \) with open kernel is trivial,

(c) if \( \mu \in M(G) \) satisfies \( \|\mu\| = |\mu(G)| \neq 0 \), then \( \mu \) is invertible and \( \|\mu^{-1}\| = |\mu(G)|^{-1} \).
Proof. Let \( \mu = \mu^2 \in M(G) \), let \( H \) be an open compact subgroup of \( G \), and let \( \psi \in M(G/H) \) be the image of \( \mu \). Then \( \psi^2 = \psi \) and \( \psi = 0 \) or \( 1 \) since \( M(G/H) \) is a local ring. It follows easily that \( \mu = 0 \) or \( 1 \). Statement (b) follows since \( k \) contains no \( p \)-th roots of unity. Statement (c) is easily seen for finite groups and follows from that special case.

Suppose now that \( k \) has characteristic zero (hence \( k \supseteq \mathbb{Q}_p \)). If \( k \) is algebraically closed then we can identify \( G^* \) with the characters \( x : G \to k^* \) with open kernel. Further \( M(G) \) can contain idempotents \( \neq 0, 1 \). Namely, let \( H \) be a finite subgroup of \( G \), and let \( \chi : H \to k^* \) be a character then \( \mu_\chi = \frac{1}{p^n} \sum_{h \in H} \chi(-h) \delta_h \in M(H) \subseteq M(G) \), where \( p^n \) is the order of \( H \), is clearly an idempotent. For any finite set \( E \subseteq H^* \) one can form

\[
\mu_E = \sum_{\chi \in E} \mu_\chi.
\]

In this way we have described all idempotents, with support in \( H \). Now A. M. M. GOMMERS conjectures that there are no other idempotent elements in \( M(G) \). We can state this as follows:

(3.2) CONJECTURE. Every idempotent in \( M(G) \) has finite support.

One has to work with \( G^* \) the characters of \( G \) to find a proof. The elements in \( G^* \) are linearly independent functions on \( G \), but they are by no means orthogonal. This is the main difficulty in the verification of (3.2).

A. M. M. GOMMERS gives a proof of a special case:

(3.3) PROPOSITION. For \( G = (\mathbb{Z}/p)^I \) every element \( \mu \in M(G) \) with \( \mu = \mu^2 \) has finite support.

We give some comment on the conjecture. Let \( G \) be a group of order \( p^n \). Let \( E \subseteq G^* \) be given, then

\[
\mu_E = \sum_{\chi \in E} \mu_\chi = \sum_{\chi \in G} \left( \frac{1}{p^n} \sum_{\chi \in E} \chi(-g) \right) \delta_g
\]

is an idempotent.

It has the property \( \mu_E(\chi) = 1 \) or \( 0 \) according to \( \chi \in E \) or \( \chi \notin E \). One sees that in general \( \|\mu_E\| = p^n \). If \( \mu_E \) has support in a subgroup \( H \) of \( G \) with order \( p^k \), then \( \|\mu_E\| \leq p^k \). This yields the following.

(3.4) CONJECTURE. Let \( G \) be a group of order \( p^n \), let \( \mu \in M(G) \) be an idempotent with norm \( \leq p^k \). If \( n \) is "large with respect to \( k \)" then \( \mu \) has support in a proper subgroup of \( G \).

We note that (3.4) implies (3.2). A first step towards (3.4) is estimating the
absolute value of sums of $p$-th roots of unity. This is done in:

(3.5) **Lemma.** - Let $\omega \in k$ be a primitive $p$-th root of unity and let $\lambda, n_1 \in \mathbb{Z}$; $\lambda \geq 1$.

Then equivalent are:

(a) $|\sum_{i=0}^{p-1} n_i \omega^i| \leq \frac{1}{p}$,

(b) For all $0 \leq i, j < p - 1$ with $i \equiv j (p-1)$ one has $n_i = n_j (p')$.

**Proof.** - (b) $\implies$ (a) follows easily from the minimal equation $(x^d - 1/x^{d-1} - 1)$ satisfied by $\omega$.

Further, we note that it suffices to show (a) $\implies$ (b) for $\lambda = 1$; $\lambda > 1$ follows easily by induction.

We consider $\mathbb{Z}[\omega] = \mathbb{Z}[\xi]$ where $\omega = 1 + \xi$. This is a subring of $k$. Since $|\xi|^{p-1}(p-1) = \frac{1}{p}$, it follows that the elements in $\mathbb{Z}[\xi]$ with absolute value $\leq \frac{1}{p}$ form the ideal $I = p\mathbb{Z}[\xi]$. Dividing by this ideal we find:

$$\mathbb{Z}[\omega]/I = \frac{\mathbb{F}_p[T]}{(1 + T^{p-1} + T^{2p-1} + \cdots + T^{(p-1)p-1})}$$

where $T$ has image $\omega$. Hence $|\sum_{i=0}^{p-1} n_i \omega^i| \leq \frac{1}{p}$ implies $\sum n_i t^i = 0$ (where $n_i$ is the image of $n_i$ in $\mathbb{F}_p$).

This means

$$\sum n_i T^i = (a_0 + a_1 T + \cdots + a_{p-1} T^{p-1})(1 + T^{p-1} + T^{2p-1} + \cdots + T^{(p-1)p-1})$$

for certain $a_0, a_1, \ldots, a_{p-1} \in \mathbb{F}_p$. This is equivalent with statement (b) for $\lambda = 1$.

**REFERENCES**

