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$p$-adic Whittaker groups


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p-ADIC WHITTAKER GROUPS

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An algebraic curve (non singular, irreducible and complete) over \( \mathbb{C} \) which is hyperelliptic can be uniformized by a Whittaker group (see [1], p. 247-249). We will treat the rigid analytic case for complete non-archimedean valued fields \( k \) with characteristic \( \neq 2 \). In order to avoid rationality problems the field \( k \) is supposed to be algebraically closed. A part of the results in this paper was independently proved by G. Van STEEN.

1. Combinations of discontinuous groups.

Let \( \Gamma \subset \text{PGL}(2, k) \) be a discontinuous group. We will assume that \( \infty \in \mathbb{P}^1(k) = \mathbb{P}^1 \) is an ordinary point for \( \Gamma \). A fundamental domain \( F \) for \( \Gamma \), containing \( \infty \), is a subset \( F \subset \mathbb{P}^1 \) satisfying:

(i) \( \mathbb{P}^1 - F \) is a finite union of open spheres \( B_1, \ldots, B_n \) in \( k \) such that the corresponding closed spheres \( B_1^+, \ldots, B_n^+ \) are disjoint,

(ii) The set \( \{ \gamma \in \Gamma ; \gamma F \cap F \neq \emptyset \} \) is finite,

(iii) if \( \gamma \neq 1 \) and \( \gamma F \cap F \neq \emptyset \) then \( \gamma F \cap F \subset \bigcup_{i=1}^{n} (B_i^+ - B_i) \),

(iv) \( \bigcup_{\gamma \in \Gamma} \gamma F = \Omega = \) the set of ordinary points of \( \Gamma \).

We will write \( F_j \) for \( \mathbb{P}^1 - \bigcup_{i=1}^{n} B_i^+ \).

One can show that a fundamental domain for \( \Gamma \) exists if \( \Gamma \) is finitely generated (see [2] and [3]).

**Proposition.** - Let \( \Gamma_1, \ldots, \Gamma_m \) be discontinuous groups with fundamental domains containing the point \( \infty \). Suppose that \( \mathbb{P}^1 - F_j \) for all \( i \neq j \). Then the group \( \Gamma \) generated by \( \Gamma_1, \ldots, \Gamma_m \) is discontinuous. Moreover \( \Gamma = \Gamma_1 * \ldots * \Gamma_m \) (the free product) and \( \bigcap F_1 \) is a fundamental domain for \( \Gamma \).

**Proof.** - Put \( F = \bigcap_{i=1}^{m} F_i \) and \( \mathcal{F} = \bigcap_{i=1}^{m} F_i^* \). Let \( W = \delta_0 \delta_{s-1} \ldots \delta_1 \) be a reduced word in \( \Gamma_1 * \ldots * \Gamma_n \), i.e. each \( \delta_i \in \bigcup \Gamma_j \{1\} \) and if \( \delta_i \in \Gamma_j \), then \( \delta_i \neq \Gamma_i \). Then \( W(F) \subset \mathbb{P}^1 - F \). Hence \( \Gamma \) is equal to \( \Gamma_1 * \ldots * \Gamma_n \). Further \( W(F) \cap F \neq \emptyset \) implies that \( W \in \bigcup \Gamma_j \). So we have shown that \( F \) satisfies the conditions (i), (ii) and (iii). Let \( \delta > 0 \), then there are finite sets \( W_1 \subset \Gamma_1, \ldots, W_m \subset \Gamma_m \) such that the complement of \( \bigcup_{w \in W_1} \gamma F_1 \) consists of

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finitely many spheres of radii < \delta.

Given \varepsilon > 0 then there is \delta > 0 and some \eta \gg 0 such that the complement of
\bigcup_{W \in W} W, where W consists of all reduced words in \mathcal{W}_1, \ldots, \mathcal{W}_n of length
\leq \eta, is a finite union of spheres of radii < \varepsilon.

This shows that the set of limit points of \Gamma is equal to the compact set
\mathbb{P}^1 - \bigcup_{w \in W} w.

2. Example. If each \Gamma_i \cong \mathbb{Z}, so \Gamma_i is generated by an hyperbolic element, then
\Gamma is a free group on \eta generators. We will call such a \Gamma a Schottky group of
rank \eta. It can be shown that any group \Gamma, which satisfies :

(i) \Gamma discontinuous;

(ii) \Gamma is finitely generated;

(iii) \Gamma has no elements of finite order (\neq 1),

is a Schottky group of rank \eta. Moreover \sqrt{\Gamma} turns out to be an algebraic curve
over \mathbb{k} with genus \eta.

3. Definition of the \textit{p}-adic Whittaker groups. (characteristic \mathbb{k} \neq 2.)

Let \sigma be an element of order two in \text{PGL}(2, \mathbb{k}). Then \sigma has two fixed points
a and b. Moreover \sigma is determined by \{a, b\}. Let B be an open sphere in
\mathbb{P}^1 maximal, w. r. t. the condition \sigma(B) \cap B = \emptyset and let c be a point of B.

There exists a \tau \in \text{PGL}(2, \mathbb{k}) with \sigma(a) = 1, \sigma(b) = -1, \sigma(c) = 0. Then
\tau = \sigma \sigma^{-1} has the form \tau(z) = 1/z; \tau has 1, -1 as fixed points and
\sigma(B) = \{z \in \mathbb{P}^1; |z| < 1\}. It follows that \mathbb{P}^1 - B is a fundamental domain for
the group \{1, \sigma\}.

Let (g + 1) elements \sigma_0, \ldots, \sigma_g of order two in \text{PGL}(2, \mathbb{k}) be given.
Suppose that their fixed points \{a_0, b_0\}, \{a_1, b_1\}, \ldots, \{a_g, b_g\} are all
finite and are such that the smallest closed spheres \mathbb{B}^+_0, \ldots, \mathbb{B}^+_g in \mathbb{k}
containing \{a_0, b_0\}, \{a_1, b_1\}, \ldots, \{a_g, b_g\}, are disjoint.

Choose points \mathcal{C}_i \in \mathbb{B}^+_i such that the open sphere \mathcal{B}_i with center \mathcal{C}_i and
radius = radius of \mathbb{B}^+_i does not contain \mathcal{A}_i and \mathcal{B}_i.

According to Prop. 1 the group \Gamma = \langle \sigma_0, \sigma_1, \ldots, \sigma_g \rangle generated by
\{\sigma_0, \ldots, \sigma_g\} is discontinuous, has \mathbb{F} = \mathbb{P}^1 - \bigcup_{i=0}^g \mathcal{B}_i as fundamental domain
and is equal to
\langle \sigma_0, \sigma_1 \rangle \ast \ldots \ast \langle \sigma_g \rangle \cong \mathbb{Z}/2 \ast \ldots \ast \mathbb{Z}/2.

Let \varphi: \Gamma \to \mathbb{Z}/2 be the group homomorphism given by \varphi(\sigma_i) = 1 for all i.
The kernel W of \varphi is called a \textit{Whittaker} group. The group W is generated by
\{\sigma, \sigma_0, \sigma_2, \sigma_0, \ldots, \sigma_g, \sigma_0\}. An easy exercise shows that W is a free group on
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s 2 ... ,
s g *0~ ° So W is a Schottky group of rank g .
The groups W and Γ have the same set E of limit points. Let \( \Omega = P^1 - E \).
Then \( \Omega/W \) and \( \Omega/\Gamma \) have a canonical structure of an algebraic curve over \( k \). The
natural map \( \Omega/W \to \Omega/\Gamma \) is a morphism of algebraic curves of degree 2 .

4. PROPOSITION. \( \Omega/\Gamma \cong P^1 \).

Proof. - Consider
\[
\theta(a, b, z) = \prod_{\gamma \in \Gamma} \frac{x - \gamma(a)}{x - \gamma(b)},
\]
where \( a, b \in \Omega \) and \( a \notin \Gamma b \) and \( \omega \notin \Gamma a \cup \Gamma b \).

This function converges uniformly on the affinoid subsets of \( \Omega \) since
\[
\lim |\gamma(a) - \gamma(b)| = 0.
\]
So \( F(z) = \theta(a, b, z) \) is a meromorphic function on \( \Omega \). For any \( \delta \in \Gamma \) we
have \( F(\delta z) = c(\delta) F(z) \) where \( c(\delta) \in k^* \). Clearly \( c : \Gamma \to k^* \) is a group homo-
morphism and hence \( c(\delta) = \pm 1 \).

For given \( a \) one can take \( b \) close to \( a \) such that \( |F(\omega) - F(s_0 \omega)| < \frac{1}{2} \). For
this choice of \( a \) and \( b \), we find that \( F \) is invariant under \( \Gamma \). So \( F \) defines
a morphism \( \bar{F} : \Omega/\Gamma \to P^1 \). This morphism has only one pole. Hence \( \bar{F} \) is an isomorphism.

Second proof (G. Van STEEN). - If \( \Gamma \) is finitely generated then \( \Omega/\Gamma \) is an al-
gebraic curve of genus = rank of \( \Gamma = \Gamma/[\Gamma, \Gamma] \).

In our case the rank is clearly zero.

5. THEOREM. \( \Omega/W \) is an hyperelliptic curve of genus \( g \). The affine equation of
\( \Omega/W \) is
\[
y^2 = \prod_{i=0}^{2g} (x - F(s_i))(x - F(s_i^*)).
\]
Proof. - It follows from 3 and 4 that \( \Omega/W \) is indeed hyperelliptic of genus \( g \).
Therefore \( \Omega/W \) must have \( 2g + 2 \) ramification points over \( \Omega/\Gamma \). A point \( \omega \in \Omega/W \),
image of \( \omega \in \Omega \), is a ramification if, and only if, \( s_0 \in W \). The points
\( a_0, b_0, \ldots, a_g, b_g \) satisfy this condition, and their images in \( \Omega/\Gamma \) are dif-
ferent. So the equation follows.

6. COROLLARY. - Let \( s_0, \ldots, s_g \in \text{PGl}(2, k) \) be elements of order 2 such that
the group \( \Gamma \) generated by them satisfies : \( \Gamma \) is discontinuous and
\[
\Gamma = \langle s_0 \rangle = \langle s_1 \rangle = \cdots = \langle s_g \rangle.
\]
Then there are elements \( s_0^*, \ldots, s_g^* \) of order 2 in \( \text{PGl}(2, k) \) with the
\( 2g + 2 \) fixed points in the position required in 3, and such that \( \Gamma = \langle s_0^*, \ldots, s_g^* \rangle \).

Proof. - In 3, 4 and 5, the position of the \( 2g + 2 \) fixed points of
\( \{s_0, \ldots, s_g\} \) is only used to prove that \( \Gamma \) is discontinuous and equal to
\[ \langle s_0 \rangle \ast \cdots \ast \langle s_g \rangle \text{, so we can also form } W = \langle s_0 s_1, s_0 s_2, \ldots, s_0 s_g \rangle \in \Gamma \text{ and conclude that } \Omega/W \xrightarrow{f} \Omega/\Gamma = P^1 \text{ has degree 2 and has } 2g + 2 \text{ ramification points, called } A_1, \ldots, A_{2g+2}. \]

Let \( \sigma : \Omega/W \rightarrow \Omega/W \) be the automorphism of order 2 defined by \( f \). Then \( A_1, \ldots, A_{2g+2} \) are the fixed points of \( \sigma \).

Write \( t_1 = s_0 s_1, \ldots, t_g = s_0 s_g \). Every element in \( \Gamma \) of order 2 must have the form \( a s_i a^{-1} \) (\( a \in \Gamma; i = 0, \ldots, g \)) (see \([5]\)). Further \( a \in \Gamma \) has the form \( w s_0 \) or \( w \), with \( w \in W \). Since \( s_0 s_i s_0 = t_i s_i t_i^{-1} \), we find that every element in \( \Gamma \) of order 2 has the form \( w s_i w^{-1} \), with \( w \in W \) and \( i \in \{0, \ldots, g\} \). It is easily verified that this presentation is unique.

Further \( \Omega \xrightarrow{\pi} \Omega/W \) is a universal covering (see \([4]\)). Hence for any \( e, f \in \Omega \) with \( \sigma(\pi(e)) = \pi(f) \), there exists a unique lifting \( s : \sigma \rightarrow \Omega \) of \( \sigma \) with \( s(e) = f \). Moreover \( s \in \Gamma \).

Take now \( e \in \pi^{-1}(A_j) \) and a lifting \( s \) of \( \sigma \) with \( s(e) = e \). Then \( e^2 = 1 \).

Hence \( s = ws_i w^{-1} \) for some \( i \in \{0, \ldots, g\} \) and \( w \in W \). The \( i \) does not depend on the choice of \( e \in \pi^{-1}(A_j) \). Hence we have constructed a map

\[ \tau : \{A_1, \ldots, A_{2g+2}\} \rightarrow \{0, 1, \ldots, g\}. \]

Further any \( ws_i w^{-1} \) has at most two fixed points in \( \pi^{-1}(\{A_1, \ldots, A_{2g+2}\}) \).

It follows that \( \tau^{-1}(i) \) consists of at most two points. Hence \( \tau \) is surjective and every \( s_i \) has both fixed points in \( \pi^{-1}(\{A_1, \ldots, A_{2g+2}\}) \). The generators for \( \Gamma \) can be changed into \( s_0, t_2 s_1 t_2^{-1}, s_2, \ldots, s_g \). With a sequence of changes of this type one finds generators \( s_0, \ldots, s_g \) for \( \Gamma \) with their \((2g+2)\) fixed points in the required position.

7. **Theorem.** Suppose that \( X \) is a hyperelliptic curve of genus \( g \) over \( k \) which is totally split. Then there exists a Whittaker group \( W \), unique up to conjugation in \( \text{PGl}(2, k) \), with \( X \cong \Omega/W \).

**Proof.** We will use freely the results of \([3]\) and \([4]\). We know that \( \Omega \xrightarrow{\pi} \Omega/W \cong X \)

exists where \( W \) is a Schottky group of rank \( g \), unique up to conjugation. We have to show that \( W \) is in fact a Whittaker group.

Let \( \sigma \) be the automorphism of \( X \) with order two such that \( \tau : X \rightarrow X/\sigma \cong P^1 \).

Then \( \sigma \) has \( A_1, \ldots, A_{2g+2} \in X \) as fixed points. Let \( \Gamma \) denote the set of all lifts \( s : \Omega \rightarrow \Omega \) of \( \sigma : X \rightarrow X \) and of \( \text{id} : X \rightarrow X \). Then \( \Gamma \) is a group and \( W \) has index 2 in \( \Gamma \). The set

\[ X = \pi^{-1}(\{A_1, \ldots, A_{2g+2}\}) \subseteq P^1 \]

is a compact set with limit points \( \mathcal{L} = P^1 - \Omega = \) the limit points of \( W = \) the limit points of \( \Gamma \). Let \( \overline{\Omega} \) denote the reduction of \( \Omega \) with respect to \( K \). Then
\( \overline{\mathcal{V}}/\Gamma \) is a reduction of \( \mathcal{P}^1 \) and it is in fact the reduction of \( \mathcal{P}^1 \) with respect to the finite set \( \{ \tau(A_1), \ldots, \tau(A_{2g+2}) \} \).

Let \( \overline{X} \) denote the reduction induced by \( \overline{\mathcal{U}} \), i.e. \( \overline{\mathcal{U}} \) is given with respect to a pure covering \( \mathcal{U} \), and \( \overline{X} \) is the reduction with respect to \( \overline{\pi(\mathcal{U})} \).

One easily sees that \( \overline{X} = \overline{\mathcal{U}}/W \) and consists of projective lines over the residue field \( \mathbb{F} \) of \( \mathcal{K} \). The intersection graph \( G(\overline{X}) \) is defined by:

- vertices = the components of \( \overline{X} \)
- edges = the intersection points.

The map \( \sigma \) induces an automorphism of \( \overline{X} \) and \( G(\overline{X}) \), again denoted by \( \sigma \).

Further \( \overline{X} \xrightarrow{\sigma} \overline{X}/\sigma \cong \overline{\mathcal{V}}/\Gamma \) and \( G(\overline{X})/\sigma \cong G(\overline{\mathcal{V}}/\Gamma) \) is a connected finite tree.

Through the image \( \overline{A}_1 \) of \( A_1 \) on \( \overline{X} \) goes only one component of \( \overline{X} \) since \( \overline{\tau(A_1)} \) lies on only one component of \( \overline{\mathcal{V}}/\Gamma \). Call this vertex of \( G(\overline{X}) \) the vertex \( g_1 \).

Then \( \sigma(g_1) = g_1 \) and the homeomorphism \( \sigma \) of \( G(\overline{X}) \) induces an automorphism \( \hat{\sigma} \) of \( \pi_1(G(\overline{X}) , g_1) \) = the fundamental group of \( G(\overline{X}) \).

We know further that \( \pi_1(G(\overline{X}) , g_1) \) is in a natural way isomorphic to \( W \). Suppose that we can find a base for the fundamental group, \( t_1, \ldots, t_g \), such that \( \hat{\sigma}(t_1) = t_1^{-1} \) for all \( i \). Then we can lift this situation to \( \Omega \) as follows:

Choose an element \( e \in \pi_1(A_1) \); let \( s_0 \) be the lift of \( \sigma \) satisfying \( s_0(e) = e \); let \( h_0 \) be the component of \( \overline{\mathcal{U}} \) on which \( e \) lies; let the curve in \( G(\overline{\mathcal{U}}) \) with begin point \( h_0 \) and lying above \( A_1 \) have endpoint \( h_1 \in G(\overline{\mathcal{U}}) \); let \( T_1 \in W \) be defined by \( T_1(e) \) lies on \( h_1 \).

Then \( W = \langle T_1, \ldots, T_g \rangle \) and \( s_0 T_1 s_0 = T_1^{-1} \) for all \( i \). Put
\[
\begin{align*}
s_1 &= s_0 T_1, \\
\cdots & \\
s_g &= s_0 T_g.
\end{align*}
\]

Then \( \Gamma = \langle s_0, s_1, \ldots, s_g \rangle \) and easy inspection yields
\[
\Gamma = \langle s_0 \rangle \ast \langle s_1 \rangle \ast \cdots \ast \langle s_g \rangle.
\]

According to Corollary 6, we have shown that \( W \) is a Whittaker group.

Finally we have to show the following lemma:

8. LEMMA. - Let \( G \) be a finite connected graph with Betti number \( g \). Let \( \sigma \) be an homeomorphism of \( G \) such that:

(i) \( \sigma \) has order 2;
(ii) \( G/\sigma \) is a tree;
(iii) \( \sigma \) fixes a vertex \( p \in G \).

Then the fundamental group \( \pi_1(G , p) \) has generators \( t_1, \ldots, t_g \) such that the induced automorphism \( \hat{\sigma} \) of \( \pi_1(G , p) \) has the form \( \hat{\sigma}(t_i) = t_i^{-1} \) for all \( i \).

Proof. - Induction on the number of vertices of \( G \).
(1) \( p \) is the only vertex of \( G \). Then \( G \) is a wedge of \( g \) circles. As generators for \( \pi_1 \) we take the \( g \) circles together with an orientation. Call them \( t_1, \ldots, t_g \). Since \( \sigma \) is an homeomorphism we must have

\[
\hat{\sigma}(t_i) \in \{t_1, \ldots, t_g, t_i^{-1} \}
\]

for all \( i \). Since \( G/\sigma \) has a trivial fundamental group, one finds that \( \hat{\sigma}(t_i) = t_i^{-1} \) for all \( i \).

(2) Induction step. Choose an edge \( \lambda \) of \( G \) with endpoints \( p \) and \( q \neq p \).

If \( \sigma(\lambda) = \lambda \) then we make a new graph \( G^* \) by identifying \( p \) and \( q \) and deleting the edge \( \lambda \).

If \( \sigma(\lambda) \neq \lambda \), but \( \sigma(\lambda) \) has also endpoints \( p \) and \( q \), then we make \( G^* \) by identifying \( p \) and \( q \) and also identifying \( \lambda \) on \( \sigma(\lambda) \).

If \( \sigma(\lambda) \) has endpoints \( p, r \) with \( r \neq q \), then we make \( G^* \) by identifying \( q \) and \( r \) with \( p \) and deleting \( \lambda \) and \( \sigma(\lambda) \).

In all cases, \( G^* \) is homotopic to \( G \); \( \sigma \) acts again on \( G^* \) and induces the same automorphism of the fundamental group.


1° An easy calculation gives that the number of moduli for Whittaker groups of rank \( g \) is \( 2g - 1 \). This is the same as the number of moduli for hyperelliptic curves of genus \( g \).

2° Is it possible to give an explicit calculation of the numbers \( F(a_1), F(b_1) \) in theorem 5?

3° Hyperelliptic curves and Whittaker groups in characteristic 2 will be treated by G. Van STEEN.

REFERENCES


