Robust stabilization of nonlinear systems via stable kernel representations with $L_2$-gain bounded uncertainty

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Abstract

The approach to robust stabilization of linear systems using normalized left coprime factorizations with $H_\infty$ bounded uncertainty is generalized to nonlinear systems. A nonlinear perturbation model is derived, based on the concept of a stable kernel representation of nonlinear systems. The robust stabilization problem is then translated into a nonlinear disturbance feedforward $H_\infty$ optimal control problem, whose solution depends on the solvability of a single Hamilton–Jacobi equation.

Keywords: Robust nonlinear control; Perturbation model; Kernel representation; Small-gain theorem; Nonlinear $H_\infty$ control

1. Introduction

A wealth of literature is available on the problem of robustly stabilizing nonlinear uncertain systems. Here we propose a very particular approach, which directly generalizes the solution of the linear robust stabilization problem via normalized left coprime factorizations, as obtained in Glover and McFarlane [6] (see also [15]), to the nonlinear case. Essential ingredients in our approach are the stable kernel representation of nonlinear state space systems as introduced in [18, 17], the resulting nonlinear perturbation model, and the solution to a particular type of nonlinear $H_\infty$ control problems. The theory is illustrated with a simple example admitting an explicit solution.

2. A nonlinear perturbation model

A very general perturbation model for linear systems is the numerator–denominator perturbation model, or coprime factor uncertainty model, as it is also known (see e.g. [23, 12, 24]). Let $G(s)$ be the transfer matrix of a linear system (i.e. $G(s)$ is a proper rational matrix). Left factorization of $G(s)$ over the stable proper rational
matrices yields \( G(s) = D^{-1}(s)N(s) \), with \( D(s) \), \( N(s) \) coprime stable proper rational matrices. The stable linear system

\[
e = [N(s) : - D(s)] \begin{bmatrix} u \\ y \end{bmatrix}
\]

(with “inputs” \( u \) and “outputs” \( e \)) will be called a stable kernel representation of \( G(s) \), since by setting \( e = 0 \) in (1) one recovers the input–output map \( y = G(s)u \). In the numerator–denominator perturbation model one considers the following class of perturbations:

\[
N(s) \mapsto N(s) + A_N(s),
\]

\[
D(s) \mapsto D(s) + A_D(s),
\]

with \( A_N(s), A_D(s) \) stable proper rational matrices. (In applications one would normally include some extra weighting filters; however, they can be incorporated in the system transfer matrix \( G(s) \), see [15, 13].) This results in the perturbed stable kernel representation

\[
e_p = [N(s) : - D(s)] \begin{bmatrix} u \\ y \end{bmatrix} + [A_N(s) : - A_D(s)] \begin{bmatrix} u \\ y \end{bmatrix}
\]

and the perturbed transfer matrix \( G_p(s) = [D(s) + A_D(s)]^{-1}[N(s) + A_N(s)] \). Usually it is convenient to normalize the kernel representation (1) by starting with left coprime factors \( D(s), N(s) \) satisfying

\[
N(s)N^T(-s) + D(s)D^T(-s) = I, \quad s \in \mathbb{C}.
\]

A detailed treatment of the robust stabilization problem based on this normalized coprime factor uncertainty model is given in [6, 15], see also [24] for the unnormalized case.

Now let us consider smooth nonlinear systems

\[
\Sigma: \quad \dot{x} = f(x) + g(x)u, \quad u \in \mathbb{R}^m,
\]

\[
y = h(x), \quad y \in \mathbb{R}^p,
\]

where \( x = (x_1, \ldots, x_n) \) are local coordinates for an \( n \)-dimensional state space manifold \( M \). Throughout we assume the existence of a distinguished equilibrium \( x_0 \), i.e. \( f(x_0) = 0 \). Without loss of generality we assume \( x_0 = 0 \), and furthermore \( h(0) = 0 \).

Before defining a stable kernel representation for \( \Sigma \) and the resulting perturbation model we need some preliminaries. Let \( \gamma > 0 \). \( \Sigma \) is said to have \( L_2 \)-gain \( \leq \gamma \) if there exists a nonnegative solution \( V: M \rightarrow \mathbb{R} \) (a storage function) to the dissipation inequality [25],

\[
V(x(t_1)) - V(x(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 \|u(t)\|^2 - \|y(t)\|^2) \, dt, \quad V(0) = 0,
\]

for all \( t_1 \geq t_0 \) and all \( u \in L_2([t_0, t_1]) \) (with \( x(t_1) \) denoting the solution at time \( t_1 \) for initial condition \( x(t_0) \) at time \( t_0 \)). \( \Sigma \) is said to have \( L_2 \)-gain \( < \gamma \) if there exists some \( \gamma \) \( \leq \gamma \) such that \( \Sigma \) has \( L_2 \)-gain \( < \gamma \). Throughout we will assume that if there exists a solution \( V \geq 0 \) to (6) then there also exists a differentiable solution \( V \geq 0 \) to (6), and we will restrict ourselves to these differentiable solutions.

Let \( \Sigma \) have \( L_2 \)-gain \( \leq \gamma \). From [7, 8, 21] we recall that if additionally \( \Sigma \) is zero-state observable (i.e. \( y(t) = 0, u(t) = 0, \forall t \geq 0 \), implying \( x(0) = 0 \), then necessarily a solution \( V \geq 0 \) to (6) is positive definite (\( V(x) > 0, x \neq 0 \)), and 0 is a locally asymptotically stable equilibrium of (5) with Lyapunov function \( V \).

Next we consider the Hamilton–Jacobi–Bellman equation (corresponding to \( \Sigma \) with cost criterion \( \int_{-\infty}^{0} (\|u(t)\|^2 + \|y(t)\|^2) \, dt \))

\[
W_x(x)f(x) + \frac{1}{2} W_x(x)g(x)g^T(x)W_x^T(x) - \frac{1}{2} h^T(x)h(x) = 0, \quad W(0) = 0,
\]
The following theorem is a generalized version of [18, 17].

**Theorem 2.1.** Consider the nonlinear system (5), and assume it is zero-state detectable (y(t) = 0, u(t) = 0, \( \forall t \geq 0 \)) implying \( x(t) \to 0, t \to \infty \). Suppose there exists a smooth positive definite solution \( W \) to (7), and suppose there exists a smooth solution \( k(x) \) to

\[
W_x(x)k(x) = h^T(x). \tag{8}
\]

Define the system \( \Sigma \) with inputs \( \begin{bmatrix} u \\ y \end{bmatrix} \) and outputs \( e \):

\[
\dot{x} = \left[ f(x) - k(x)h(x) \right] + \left[ g(x) : k(x) \right] \begin{bmatrix} u \\ y \end{bmatrix},
\]

\[
\Sigma: \quad e = h(x) - y. \tag{9}
\]

Then \( f(x) - k(x)h(x) \) is locally asymptotically stable (w.r.t. the equilibrium \( x = 0 \)) with Lyapunov function \( W \), and globally asymptotically stable if \( W \) is proper (i.e. the sets \( \{ x \in M \mid 0 \leq W(x) \leq c \} \) are compact for every \( c \geq 0 \)). Furthermore, \( \Sigma \) has \( L_2 \)-gain = 1. Setting \( e = 0 \) in \( \Sigma \) yields \( \Sigma \), and \( \Sigma \) will be called a (nonlinear) stable kernel representation of \( \Sigma \).

**Proof (sketch, see [18, 17] for details).** From (7) and (8) we obtain

\[
W_x(x)\left[ f(x) - k(x)h(x) \right] = -\frac{1}{2}W_x(x)g(x)g^T(x)W_x(x) - \frac{1}{2}h^T(x)h(x) \leq 0 \tag{10}
\]

and (global) asymptotic stability follows from LaSalle's invariance principle. Similarly,

\[
W_x(x)\left[ f(x) - k(x)h(x) \right] + g(x)u + k(x)y
\]

\[
= -\frac{1}{2}\|u - g^T(x)W_x^T(x)\|^2 - \frac{1}{2}\|e\|^2 + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|y\|^2, \tag{11}
\]

proving \( L_2 \)-gain \( \leq 1 \) by integration (see e.g. [21]).

**Remark 2.2.** If the linearized system \( \dot{x} = (\partial f/\partial u)(0)x + g(0)u \), \( y = (\partial h/\partial x)(0)x \) is anti-stabilizable, and if the imaginary eigenvalues of \((\partial f/\partial x)(0)\) are \((\partial h/\partial x)(0)\)-observable, then at least locally around 0 there exists a smooth non-negative solution \( W \geq 0 \) to (7) (see e.g. [11, 20]), which will be locally positive definite if the linearized system is observable.

**Remark 2.3.** Consider a star-shaped coordinate neighborhood of \( x = 0 \). Since \( W_x(0) = 0 \) and \( h(x) = 0 \) we can write (see e.g. [16])

\[
W_x(x) = x^T M(x), \quad h(x) = C(x)x \tag{12}
\]

for suitable matrices \( M(x), C(x) \), with entries depending smoothly on \( x \). Assume that \( M(x) \) is invertible for all \( x \) in the coordinate neighborhood of 0; then a solution \( k(x) \) to (8) is given as [18]

\[
k(x) = M^{-1}(x)C^T(x). \tag{13}
\]
(See [9] for similar considerations in a different context.) Note furthermore that \( M(0) = (\partial^2 W/\partial x^2)(0) \) (the Hessian matrix of \( W \) at 0). Thus, under the conditions of Remark 1, \( M(0) \) will be positive definite, implying that \( M(x) \) will be invertible for \( x \) near 0.

**Remark 2.4** ([18]). If \( -V \) is a negative definite solution to (7), then

\[
\Sigma_{ri} : \quad \dot{p} = f(p) - g(p)g^T(p) V_p^T(p) - k(p)e, \quad p \in M,
\]

is a right inverse system to \( \Sigma_s \), i.e., if \( x(0) = p(0) \), then the input–output map of \( \Sigma_s \circ \Sigma_{ri} \) is the identity mapping. Furthermore, \( f(p) - g(p)g^T(p) V_p^T(p) \) is locally asymptotically stable (w.r.t. \( p = 0 \)) with Lyapunov function \( V \) (and globally asymptotically stable if \( V \) is proper). Hence \( \Sigma_s \) has a stable right inverse, generalizing the linear notion of coprimness.

**Remark 2.5.** For a linear system \( \Sigma \), the stable kernel representation \( \Sigma_s \) reduces to the left normalized coprime factorization (1), (4).

Analogously to the linear case (cf. (3)), we will now consider *perturbed* nonlinear stable kernel representations

\[
\dot{x} = [f(x) - k(x)h(x)] + [g(x); k(x)] \begin{bmatrix} u \\ y \end{bmatrix},
\]

\[
e_p = e + w,
\]

where \( w \) is the output of an arbitrary nonlinear state space system with input \( \begin{bmatrix} u \\ y \end{bmatrix} \),

\[
\begin{align*}
\phi &= \varphi(p, u, y), \quad \varphi(0, 0, 0) = 0, \\
w &= \beta(p, u, y), \quad \beta(0, 0, 0) = 0,
\end{align*}
\]

having finite \( L_2 \)-gain. (More generally we could consider families of nonlinear input–output maps from \( [\gamma] \) to \( w \), parametrized by the set of initial conditions.) Setting \( e_p = 0 \) in (15) yields the *perturbed* system

\[
\Sigma_p : \quad \dot{x} = f(x) + g(x)u + k(x)w, \\
y = h(x) + w,
\]

with \( w \) the output of (16).

### 3. The robust stabilization problem

Consider the nonlinear system \( \Sigma \) given by (5), and its perturbed model \( \Sigma_p \) given by (17), (16). The *robust stabilization problem* is to find a controller

\[
\begin{align*}
\dot{\zeta} &= l(\zeta, y), \quad l(0, 0) = 0, \\
u &= m(\zeta, y), \quad m(0, 0) = 0,
\end{align*}
\]

with \( \zeta \in \mathbb{R}^n \) the controller state, such that the \( L_2 \)-gain of the closed-loop system (17), (18), from \( w \) to \( z = [\gamma] \), is minimized, say equal to \( \gamma^* \geq 0 \).
By the small-gain theorem (see e.g. [3]) this will mean that the overall closed-loop system (16)–(18) will be $L_2$-stable for all perturbations $\Delta$ with $L_2$-gain strictly less than $1/\gamma^\ast$.

In state space terms, if there exist proper positive definite solutions $V_{xc}, V_d$ to the dissipation inequalities

$$V_{xc}(x(t_1), \zeta(t_1)) - V_{xc}(x(t_0), \zeta(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 \|w(t)\|^2 - \|z(t)\|^2) \, dt, \quad \gamma \geq \gamma^\ast,$$

then, assuming zero-state detectability [7], the overall closed-loop system (16)–(18) will be globally asymptotically stable with Lyapunov function $\gamma_d V_{xc} + \gamma V_d$, as can be readily checked from (19).

The problem of minimizing the $L_2$-gain from $w$ to $z = [\xi]$ for (17) is a standard $H_\infty$ optimal nonlinear control problem [19, 21, 10, 1, 22, 9]. Usually one first considers the suboptimal $H_\infty$ problem of finding for given $\gamma > 0$ a controller $C$ (if existing!) which makes the $L_2$-gain from $w$ to $z = [\xi]$ less than or equal to $\gamma$. For the solution to the suboptimal $H_\infty$ control problem we follow the approach of [2, 22]. For the state feedback suboptimal $H_\infty$ control problem we consider the pseudo-Hamiltonian

$$K(x, p, u, w) = p^T[f(x) + g(x)u + k(x)w]$$

$$- \frac{1}{2} \gamma^2 \|w\|^2 + \frac{1}{2} \|u\|^2 + \frac{1}{2} \|h(x) + w\|^2. \quad (20)$$

Solving $\partial K/\partial u = 0, \partial K/\partial d = 0$ leads to the saddle point $u^\ast = - g^T(x)p, w^\ast = (\gamma^2 - 1)^{-1} [h(x) + k^T(x)p]$. Substitution of $u^\ast, w^\ast$ into $K$ yields the Hamiltonian $H(x, p) = K(x, p, u^\ast, w^\ast)$ and the Hamilton-Jacobi-Isaacs equation $H(x, P^T_1(x)) = 0$ given as

$$P_d(x) [f(x) + (\gamma^2 - 1)^{-1} k(x)h(x)] + \frac{1}{2} \gamma^2 (\gamma^2 - 1)^{-1} h^T(x)h(x)$$

$$+ \frac{1}{2} P_d(x) [(\gamma^2 - 1)^{-1} k(x)k^T(x) - g(x)g^T(x)] P^T_d(x) = 0, \quad P(0) = 0. \quad (21)$$

If there exists a solution $P \geq 0$ to (21) then the suboptimal state feedback $H_\infty$ control problem (for $\gamma$) is solvable by the state feedback

$$u = - g^T(x)P^T_d(x). \quad (22)$$

Following the certainty equivalence principle of [2] the solution to the output feedback suboptimal $H_\infty$ control problem is, under appropriate conditions, given as

$$u = - g^T(\hat{x})P^T_\xi(\hat{x}), \quad (23)$$

with $\hat{x}(t)$ denoting the worst-case estimate of $x(t)$ given the measurements $y(\tau), -\infty < \tau \leq t$, see [2, 22]. (Currently there is intense research activity about the precise conditions for the validity of the worst-case certainty equivalence principle, but we will not elaborate on this.) In general (see e.g. [22]), this will yield an infinite dimensional controller. In the present case, however, the situation is much simpler. Indeed, the suboptimal $H_\infty$ control problem for (17) with $z = [\xi]$ is an example of the so-called disturbance feedforward problem, discussed for the linear case in [5], and for the nonlinear case in [14]. In fact, by asymptotic stability
of \( \dot{x} = f(x) - k(x)h(x) \) it follows that for a given control function \( u(t) \), \(- \infty < t \leq t \), the measurement record \( y(t) \), \(- \infty < t \leq t \), uniquely specifies the disturbance \( \psi(t) = y(t) - h(\dot{x}(t)) \) and the state trajectory \( \dot{x}(t) \). Indeed the state trajectory \( \dot{x}(\cdot) \) is generated by the differential equations

\[
\dot{x}(\cdot) = f(x) - k(x)[y - h(x)], \quad x(- \infty) = 0. \tag{24}
\]

Hence a controller solving the suboptimal \( H_\infty \) control problem for (17) is given as (substitute (23) into (24))

\[
\dot{x} = f(\dot{x}) - g(\dot{x})g^T(x)P(x) + k(x)[y - h(x)], \quad u = -g^T(x)P(x). \tag{25}
\]

Based on the linear case, the same controller for the general nonlinear disturbance feedforward problem has been recently proposed in [14]. In this paper also a direct proof is provided showing that (25) solves the suboptimal \( H_\infty \) control problem at least locally, i.e. for initial states in a neighborhood of the origin and for disturbances \( w(\cdot) \) which keep the state trajectories within this neighborhood. Summarizing, we have the following theorem.

**Theorem 3.1.** Suppose (cf. Theorem 2.1) that \( \Sigma \), given by (9) is a stable kernel representation of \( \Sigma \) such that \( f(x) - k(x)h(x) \) is globally asymptotically stable. Suppose there exists a solution \( P \geq 0 \) to (21) for given \( \gamma > 0 \), and assume the certainty equivalence principle for the suboptimal \( H_\infty \) control problem for (17) holds. Then the controller (25) stabilizes the closed-loop system (16), (17), (25) for every perturbation system \( A \) as in (16), having \( L_2 \)-gain \( < 1/\gamma \).

**Remark 3.2.** From the linear theory (cf. [6, 15]) and the local existence of solutions to (21) based on existence of solutions to the corresponding Riccati equation (cf. [20, 21]), it follows that the minimal \( \gamma^* \), such that locally around 0 there exist solutions \( P \geq 0 \) to (21) for \( \gamma > \gamma^* \), is given by

\[
\gamma^* = [1 + \sigma_{\max}(XZ)]^{1/2}, \tag{26}
\]

with \( X \) the Hessian matrix \( (\partial^2 V/\partial x^2)(0) \) and \( Z \) the inverse Hessian matrix \( [(\partial^2 W/\partial x^2)(0)]^{-1} \) of the solutions \( W \geq 0 \) and \( -V \leq 0 \) to (7).

**Remark 3.3.** A related approach to nonlinear robust stabilization will be found in [4].

**Example 3.4.** Let \( \Sigma \) be a lossless system, i.e. there exists \( H : M \rightarrow \mathbb{R} \), \( H(0) = 0 \), \( H(x) > 0 \), \( x \neq 0 \), called the internal energy, such that \( (d/dt)H = u^Ty \) or, equivalently,

\[
H_x(x)f(x) = 0, \quad H_x(x)g(x) = h^T(x). \tag{27}
\]

Clearly, positive and negative definite solutions to (7) are given as \( H \), and \( -H \), respectively. Furthermore, \( k(x) \) solving (8) is given as \( g(x) \), and thus the perturbed system \( \Sigma_p \) is given as

\[
\dot{x} = f(x) + g(x)[u + w], \quad y = g^T(x)H_x(x) + w. \tag{28}
\]

The Hamilton–Jacobi–Isaacs equation (21) takes the form

\[
P_x(x)[f(x) + (\gamma^2 - 1)^{-1}g(x)g^T(x)H_x^2(x)] + \frac{1}{2}(\gamma^2 - 1)^{-1} - 1 \cdot P_x(x)g(x)g^T(x)P_x(x) \tag{29}
\]

having the positive definite solution \( P(x) = (\gamma^2/(\gamma^2 - 2))H(x) \) for \( \gamma > \sqrt{2} \). It follows that the controller

\[
\dot{x} = f(x) - \frac{\gamma^2}{\gamma^2 - 2}g(x)g^T(x)H_x^2(x) + g(x)[y - g^T(x)H_x(x)], \tag{30}
\]

\[
u = -\frac{\gamma^2}{\gamma^2 - 2}g^T(x)H_x^2(x)
\]
robustly stabilizes $\Sigma$ for every perturbation $A$ with $L_2$-gain $< \frac{1}{\gamma}$. By Remark 3.2, $\gamma^*$ is given by (26). Since $W = H$ and $V = -H$ we conclude that $\gamma^* = \sqrt{2}$, in accordance with the lower bound $\gamma > \sqrt{2}$ as derived above. From a physical point of view, if in (28) $u$'s denote external forces and $y$'s are the corresponding (disturbed) generalized velocities, then (30) corresponds to adding damping with regard to the estimated generalized velocities with a damping factor $\gamma^2(\gamma^2 - 2)^{-1}$, tending to $\infty$ for $\gamma \downarrow \gamma^* = \sqrt{2}$.

References