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Robust stabilization of nonlinear systems via stable kernel representations with $L_2$-gain bounded uncertainty

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Abstract

The approach to robust stabilization of linear systems using normalized left coprime factorizations with $\mathcal{H}_\infty$ bounded uncertainty is generalized to nonlinear systems. A nonlinear perturbation model is derived, based on the concept of a stable kernel representation of nonlinear systems. The robust stabilization problem is then translated into a nonlinear disturbance feedforward $\mathcal{H}_\infty$ optimal control problem, whose solution depends on the solvability of a single Hamilton–Jacobi equation.

Keywords: Robust nonlinear control; Perturbation model; Kernel representation; Small-gain theorem; Nonlinear $\mathcal{H}_\infty$ control

1. Introduction

A wealth of literature is available on the problem of robustly stabilizing nonlinear uncertain systems. Here we propose a very particular approach, which directly generalizes the solution of the linear robust stabilization problem via normalized left coprime factorizations, as obtained in Glover and McFarlane [6] (see also [15]), to the nonlinear case. Essential ingredients in our approach are the stable kernel representation of nonlinear state space systems as introduced in [18, 17], the resulting nonlinear perturbation model, and the solution to a particular type of nonlinear $\mathcal{H}_\infty$ control problems. The theory is illustrated with a simple example admitting an explicit solution.

2. A nonlinear perturbation model

A very general perturbation model for linear systems is the numerator–denominator perturbation model, or coprime factor uncertainty model, as it is also known (see e.g. [23, 12, 24]). Let $G(s)$ be the transfer matrix of a linear system (i.e. $G(s)$ is a proper rational matrix). Left factorization of $G(s)$ over the stable proper rational
matrices yields \( G(s) = D^{-1}(s)N(s) \), with \( D(s) \), \( N(s) \) coprime stable proper rational matrices. The stable linear system

\[
e = [N(s) - D(s)] \begin{bmatrix} u \\ y \end{bmatrix}
\]

(with "inputs" \([u]\) and "outputs" \(e\)) will be called a stable kernel representation of \( G(s) \), since by setting \( e = 0 \) in (1) one recovers the input–output map \( y = G(s)u \). In the numerator–denominator perturbation model one considers the following class of perturbations:

\[
N(s) \mapsto N(s) + A_N(s),
\]
\[
D(s) \mapsto D(s) + A_D(s),
\]

with \( A_N(s) \), \( A_D(s) \) stable proper rational matrices. (In applications one would normally include some extra weighting filters; however, they can be incorporated in the system transfer matrix \( G(s) \), see [15, 13].) This results in the perturbed stable kernel representation

\[
e_p = [N(s) - D(s)] \begin{bmatrix} u \\ y \end{bmatrix} + [A_N(s) - A_D(s)] \begin{bmatrix} u \\ y \end{bmatrix}
\]

and the perturbed transfer matrix \( G_p(s) = [D(s) + A_D(s)]^{-1}[N(s) + A_N(s)] \). Usually it is convenient to normalize the kernel representation (1) by starting with left coprime factors \( D(s) \), \( N(s) \) satisfying

\[
N(s)N^T(-s) + D(s)D^T(-s) = I, \quad s \in \mathbb{C}.
\]

A detailed treatment of the robust stabilization problem based on this normalized coprime factor uncertainty model is given in [6, 15], see also [24] for the unnormalized case.

Now let us consider smooth nonlinear systems

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u, \quad u \in \mathbb{R}^m, \\
y &= h(x),
\end{align*}
\]

where \( x = (x_1, \ldots, x_n) \) are local coordinates for an \( n \)-dimensional state space manifold \( M \). Throughout we assume the existence of a distinguished equilibrium \( x_0 \), i.e. \( f(x_0) = 0 \). Without loss of generality we assume \( x_0 = 0 \), and furthermore \( h(0) = 0 \).

Before defining a stable kernel representation for \( \Sigma \) and the resulting perturbation model we need some preliminaries. Let \( \gamma > 0 \). \( \Sigma \) is said to have \( L_2 \)-gain \( \leq \gamma \) if there exists a nonnegative solution \( V : M \to \mathbb{R} \) (a storage function) to the dissipation inequality [25],

\[
\int_{t_0}^{t_1} \left( \gamma^2 \|u(t)\|^2 + \|y(t)\|^2 \right) dt, \quad V(0) = 0,
\]

for all \( t_1 \geq t_0 \) and all \( u \in L_2([t_0, t_1]) \) (with \( x(t_1) \) denoting the solution at time \( t_1 \) for initial condition \( x(t_0) \) at time \( t_0 \)). \( \Sigma \) is said to have \( L_2 \)-gain \( \leq \gamma \) if there exists some \( \gamma < \gamma \) such that \( \Sigma \) has \( L_2 \)-gain \( \leq \gamma \). Throughout we will assume that if there exists a solution \( V \geq 0 \) to (6) then there also exists a differentiable solution \( V \geq 0 \) to (6), and we will restrict ourselves to these differentiable solutions.

Let \( \Sigma \) have \( L_2 \)-gain \( \leq \gamma \). From [7, 8, 21] we recall that if additionally \( \Sigma \) is zero-state observable (i.e. \( y(t) = 0, u(t) = 0, \forall t \geq 0 \), implying \( x(0) = 0 \)), then necessarily a solution \( V \geq 0 \) to (6) is positive definite \((V(x) > 0, x \neq 0)\), and \( 0 \) is a locally asymptotically stable equilibrium of (5) with Lyapunov function \( V \).

Next we consider the Hamilton–Jacobi–Bellman equation (corresponding to \( \Sigma \) with cost criterion \( \int_{-\infty}^{0} \left( \|u(t)\|^2 + \|y(t)\|^2 \right) dt \))

\[
W_x(x)f(x) + \frac{1}{2} W_x(x)g(x)g^T(x)W_x^T(x) - \frac{1}{2} h^T(x)h(x) = 0, \quad W(0) = 0,
\]

(7)
where

\[ W_x(x) = \left( \frac{\partial W}{\partial x_1}(x), \ldots, \frac{\partial W}{\partial x_n}(x) \right). \]

The following theorem is a generalized version of [18, 17].

**Theorem 2.1.** Consider the nonlinear system (5), and assume it is zero-state detectable \((y(t) = 0, u(t) = 0, \forall t \geq 0, \text{implying } x(t) \rightarrow 0, t \rightarrow \infty)\). Suppose there exists a smooth positive definite solution \(W\) to (7), and suppose there exists a smooth solution \(k(x)\) to

\[ W_x(x)k(x) = h^T(x). \quad (8) \]

Define the system \(\Sigma_s\) with inputs \([u, y]\) and outputs \(e\):

\[ \dot{x} = \left[ f(x) - k(x)h(x) \right] + \left[ g(x) : k(x) \right] u, \]

\[ \Sigma_s: \quad e = h(x) - y. \quad (9) \]

Then \(f(x) - k(x)h(x)\) is locally asymptotically stable (w.r.t. the equilibrium \(x = 0\)) with Lyapunov function \(W\), and globally asymptotically stable if \(W\) is proper (i.e. the sets \(\{x \in M | 0 \leq W(x) \leq c\} \) are compact for every \(c > 0\)). Furthermore, \(\Sigma_s\) has \(L_2\)-gain = 1. Setting \(e = 0\) in \(\Sigma_s\) yields \(\Sigma\), and \(\Sigma_s\) will be called a (nonlinear) stable kernel representation of \(\Sigma\).

**Proof** (sketch, see [18, 17] for details). From (7) and (8) we obtain

\[ W_x(x)\left[ f(x) - k(x)h(x) \right] + g(x)u + k(x)y \]

and (global) asymptotic stability follows from LaSalle's invariance principle. Similarly,

\[ W_x(x)\left[ f(x) - k(x)h(x) \right] + g(x)u + k(x)y \]

proving \(L_2\)-gain \(\leq 1\) by integration (see e.g. [21]).

**Remark 2.2.** If the linearized system \(\dot{x} = (\partial f/\partial u)(0)x + g(0)u, y = (\partial h/\partial x)(0)x\) is anti-stabilizable, and if the imaginary eigenvalues of \((\partial f/\partial x)(0)\) are \((\partial h/\partial x)(0)\)-observable, then at least locally around 0 there exists a smooth non-negative solution \(W \geq 0\) to (7) (see e.g. [11, 20]), which will be locally positive definite if the linearized system is observable.

**Remark 2.3.** Consider a star-shaped coordinate neighborhood of \(x = 0\). Since \(W_z(0) = 0\) and \(h(x) = 0\) we can write (see e.g. [16])

\[ W_x(x) = x^T M(x), \quad h(x) = C(x)x \quad (12) \]

for suitable matrices \(M(x), C(x)\), with entries depending smoothly on \(x\). Assume that \(M(x)\) is invertible for all \(x\) in the coordinate neighborhood of 0; then a solution \(k(x)\) to (8) is given as [18]

\[ k(x) = M^{-1}(x)C^T(x). \quad (13) \]
Consider the nonlinear system $\Sigma$ given by (5), and its perturbed model $\Sigma_p$ given by (17), (16). The robust stabilization problem is to find a controller

$$
C: \begin{align*}
\dot{\xi} &= l(\xi, y), \quad l(0, 0) = 0, \\
u &= m(\xi, y), \quad m(0, 0) = 0,
\end{align*}
$$

with $\xi \in \mathbb{R}^n$ the controller state, such that the $L_2$-gain of the closed-loop system (17), (18), from $w$ to $z = [y]^T$, is minimized, say equal to $\gamma^* \geq 0$. 

3. The robust stabilization problem
By the small-gain theorem (see e.g. [3]) this will mean that the overall closed-loop system (16)–(18) will be $L_2$-stable for all perturbations $\Delta$ with $L_2$-gain strictly less than $1/\gamma^*$.

In state space terms, if there exist proper positive definite solutions $V_{xc}$, $V_{xd}$ to the dissipation inequalities

$$V_{xc}(x(t_1), \xi(t_1)) - V_{xc}(x(t_0), \xi(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 \|w(t)\|^2 - \|z(t)\|^2) \, dt, \quad \gamma \geq \gamma^*;$$

$$V_{xd}(\phi(t_1)) - V_{xd}(\phi(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 \|z(t)\|^2 - \|w(t)\|^2) \, dt, \quad \gamma_\Delta < 1/\gamma,$$

then, assuming zero-state detectability [7], the overall closed-loop system (16)–(18) will be globally asymptotically stable with Lyapunov function $\gamma V_{xc} + \gamma V_{xd}$, as can be readily checked from (19).

The problem of minimizing the $L_2$-gain from $w$ to $z = [\xi]$ for (17) is a standard $H_\infty$ optimal nonlinear control problem [19, 21, 10, 1, 22, 9]. Usually one first considers the suboptimal $H_\infty$ problem of finding for given $\gamma > 0$ a controller $C$ (if existing!) which makes the $L_2$-gain from $w$ to $z = [\xi]$ less than or equal to $\gamma$. For the solution to the suboptimal $H_\infty$ control problem we follow the approach of [2, 22]. For the state feedback suboptimal $H_\infty$ control problem we consider the pseudo-Hamiltonian

$$K(x, p, u, w) = p^T[f(x) + g(x)u + k(x)w]$$

$$- \frac{1}{2} \gamma^2 \|w\|^2 + \frac{1}{2} \|u\|^2 + \frac{1}{2} \|h(x) + w\|^2.$$

Solving $\partial K/\partial u = 0$, $\partial K/\partial d = 0$ leads to the saddle point $u^* = -g^T(x)p$, $w^* = (\gamma^2 - 1)^{-1}[h(x) + k^T(x)p]$.

Substitution of $u^*, w^*$ into $K$ yields the Hamiltonian $H(x, p) = K(x, p, u^*, w^*)$ and the Hamilton-Jacobi-Isaacs equation $H(x, P^T_{x}(x)) = 0$ given as

$$P_{x}(x)[f(x) + (\gamma^2 - 1)^{-1}k(x)h(x)] + \frac{1}{2}P_{x}(x)[(\gamma^2 - 1)^{-1}k^T(x) - g(x)g^T(x)]P^T_{x}(x) = 0,$$

$$P(0) = 0. \quad (21)$$

If there exists a solution $P \geq 0$ to (21) then the suboptimal state feedback $H_\infty$ control problem (for $\gamma$) is solvable by the state feedback

$$u = -g^T(x)P^T_{x}(x). \quad (22)$$

Following the certainty equivalence principle of [2] the solution to the output feedback suboptimal $H_\infty$ control problem is, under appropriate conditions, given as

$$u = -g^T(\hat{x})P^T_{x}(\hat{x}), \quad (23)$$

with $\hat{x}(t)$ denoting the worst-case estimate of $x(t)$ given the measurements $y(t)$, $-\infty < \tau \leq t$, see [2, 22]. (Currently there is intense research activity about the precise conditions for the validity of the worst-case certainty equivalence principle, but we will not elaborate on this.) In general (see e.g. [22]), this will yield an infinite dimensional controller. In the present case, however, the situation is much simpler. Indeed, the suboptimal $H_\infty$ control problem for (17) with $z = [\xi]$ is an example of the so-called disturbance feedforward problem, discussed for the linear case in [5], and for the nonlinear case in [14]. In fact, by asymptotic stability
of $\dot{x} = f(x) - k(x)h(x)$ it follows that for a given control function $u(\tau)$, $-\infty < \tau \leq t$, the measurement record $y(\tau)$, $-\infty < \tau \leq t$, uniquely specifies the disturbance $\psi(\tau) = y(\tau) - h(\dot{x}(\tau))$ and the state trajectory $\hat{x}(\cdot)$. Indeed the state trajectory $\hat{x}(\cdot)$ is generated by the differential equations

$$\dot{\hat{x}} = f(\hat{x}(\tau)) + g(\hat{x}(\tau))u(\tau) + k(\hat{x}(\tau))[y(\tau) - h(\dot{x}(\tau))], \quad x(-\infty) = 0. \quad (24)$$

Hence a controller solving the suboptimal $H_\infty$ control problem for (17) is given as (substitute (23) into (24))

$$\dot{x} = f(\hat{x}) - g(\hat{x})g^T(\hat{x})P_\Sigma(\hat{x}) + k(\hat{x})[y - h(\dot{x})],$$

$$u = -g^T(\hat{x})P_\Sigma(\hat{x}). \quad (25)$$

Based on the linear case, the same controller for the general nonlinear disturbance feedforward problem has been recently proposed in [14]. In this paper also a direct proof is provided showing that (25) solves the suboptimal $H_\infty$ control problem at least locally, i.e. for initial states in a neighborhood of the origin and for disturbances $w(\cdot)$ which keep the state trajectories within this neighborhood. Summarizing, we have the following theorem.

**Theorem 3.1.** Suppose (cf. Theorem 2.1) that $\Sigma$ given by (9) is a stable kernel representation of $\Sigma$ such that $f(x) - k(x)h(x)$ is globally asymptotically stable. Suppose there exists a solution $P > 0$ to (21) for given $\gamma > 0$, and assume the certainty equivalence principle for the suboptimal $H_\infty$ control problem for (17) holds. Then the controller (25) stabilizes the closed-loop system (16), (17), (25) for every perturbation system $\Delta$ as in (16), having $L_2$-gain $< 1/\gamma$.

**Remark 3.2.** From the linear theory (cf. [6, 15]) and the local existence of solutions to (21) based on existence of solutions to the corresponding Riccati equation (cf. [20, 21]), it follows that the minimal $\gamma^*$, such that locally around 0 there exist solutions $P > 0$ to (21) for $\gamma > \gamma^*$, is given by

$$\gamma^* = [1 + \sigma_{\max}(XZ)]^{1/2}, \quad (26)$$

with $X$ the Hessian matrix $(\partial^2 V/\partial x^2)(0)$ and $Z$ the inverse Hessian matrix $[(\partial^2 W/\partial x^2)(0)]^{-1}$ of the solutions $W > 0$ and $-V \leq 0$ to (7).

**Remark 3.3.** A related approach to nonlinear robust stabilization will be found in [4].

**Example 3.4.** Let $\Sigma$ be a lossless system, i.e. there exists $H: M \to \mathbb{R}$, $H(0) = 0$, $H(x) > 0$, $x \neq 0$, called the internal energy, such that $(d/dt)H = u^T \nu$ or, equivalently,

$$H_x(x)f(x) = 0, \quad H_x(x)g(x) = h^T(x). \quad (27)$$

Clearly, positive and negative definite solutions to (7) are given as $H$, and $-H$, respectively. Furthermore, $k(x)$ solving (8) is given as $g(x)$, and thus the perturbed system $\Sigma_\nu$ is given as

$$\dot{x} = f(x) + g(x)[u + w], \quad y = g^T(x)H^T_x(x) + w. \quad (28)$$

The Hamilton–Jacobi–Isaacs equation (21) takes the form

$$P_x(x)[f(x) + (\gamma^2 - 1)^{-1}g(x)g^T(x)H^T_x(x)] + \frac{1}{2}[\gamma^2 - 1 - 1]P_x(x)g(x)g^T(x)P_x(x)$$

$$+ \frac{1}{2}P_x(x)H^T_x(x)H_x(x) = 0, \quad P(0) = 0, \quad (29)$$

having the positive definite solution $P(x) = (\gamma^2/(\gamma^2 - 2))H(x)$ for $\gamma > \sqrt{2}$. It follows that the controller

$$\dot{x} = f(\hat{x}) - \frac{\gamma^2}{\gamma^2 - 2}g(\hat{x})g^T(\hat{x})H^T_x(\hat{x}) + g(\hat{x})[y - g^T(\hat{x})H^T_x(\hat{x})],$$

$$u = -\frac{\gamma^2}{\gamma^2 - 2}g^T(\hat{x})H^T_x(\hat{x}). \quad (30)$$
robustly stabilizes $\Sigma$ for every perturbation $A$ with $L_2$-gain $< 1/\gamma$. By Remark 3.2, $\gamma^*$ is given by (26). Since $W = H$ and $V = -H$ we conclude that $\gamma^* = \sqrt{2}$, in accordance with the lower bound $\gamma > \sqrt{2}$ as derived above. From a physical point of view, if in (28) $u$'s denote external forces and $y$'s are the corresponding (disturbed) generalized velocities, then (30) corresponds to adding damping with regard to the estimated generalized velocities with a damping factor $\gamma^2 (\gamma^2 - 2)^{-1}$, tending to $\infty$ for $\gamma \downarrow \gamma^* = \sqrt{2}$.

References