Robust stabilization of nonlinear systems via stable kernel representations with $L_2$-gain bounded uncertainty

A.J. van der Schaft

Systems and Control Group, Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands

(Received 24 November 1993)

Abstract

The approach to robust stabilization of linear systems using normalized left coprime factorizations with $H_\infty$ bounded uncertainty is generalized to nonlinear systems. A nonlinear perturbation model is derived, based on the concept of a stable kernel representation of nonlinear systems. The robust stabilization problem is then translated into a nonlinear disturbance feedforward $H_\infty$ optimal control problem, whose solution depends on the solvability of a single Hamilton–Jacobi equation.

Keywords: Robust nonlinear control; Perturbation model; Kernel representation; Small-gain theorem; Nonlinear $H_\infty$ control

1. Introduction

A wealth of literature is available on the problem of robustly stabilizing nonlinear uncertain systems. Here we propose a very particular approach, which directly generalizes the solution of the linear robust stabilization problem via normalized left coprime factorizations, as obtained in Glover and McFarlane [6] (see also [15]), to the nonlinear case. Essential ingredients in our approach are the stable kernel representation of nonlinear state space systems as introduced in [18, 17], the resulting nonlinear perturbation model, and the solution to a particular type of nonlinear $H_\infty$ control problems. The theory is illustrated with a simple example admitting an explicit solution.

2. A nonlinear perturbation model

A very general perturbation model for linear systems is the numerator–denominator perturbation model, or coprime factor uncertainty model, as it is also known (see e.g. [23, 12, 24]). Let $G(s)$ be the transfer matrix of a linear system (i.e. $G(s)$ is a proper rational matrix). Left factorization of $G(s)$ over the stable proper rational
matrices yields \( G(s) = D^{-1}(s)N(s) \), with \( D(s), N(s) \) coprime stable proper rational matrices. The stable linear system

\[
e = \begin{bmatrix} \dot{N}(s) \colon - D(s) \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}
\]

(with “inputs” \( \dot{N}(s) \) and “outputs” \( e \)) will be called a stable kernel representation of \( G(s) \), since by setting \( e = 0 \) in (1) one recovers the input–output map \( y = G(s)u \). In the numerator–denominator perturbation model one considers the following class of perturbations:

\[
N(s) \leftrightarrow N(s) + \Delta_N(s),
\]

\[
D(s) \leftrightarrow D(s) + \Delta_D(s),
\]

with \( \Delta_N(s), \Delta_D(s) \) stable proper rational matrices. (In applications one would normally include some extra weighting filters; however, they can be incorporated in the system transfer matrix \( G(s) \), see [15, 13].) This results in the perturbed stable kernel representation

\[
e_p = \begin{bmatrix} \dot{N}(s) \colon - D(s) \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} + \begin{bmatrix} \Delta_N(s) \colon - \Delta_D(s) \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}
\]

and the perturbed transfer matrix \( G_p(s) = [D(s) + \Delta_D(s)]^{-1}[N(s) + \Delta_N(s)] \). Usually it is convenient to normalize the kernel representation (1) by starting with left coprime factors \( D(s), N(s) \) satisfying

\[
N(s)N^T(\cdot s) + D(s)D^T(\cdot s) = I, \quad s \in \mathbb{C}.
\]

A detailed treatment of the robust stabilization problem based on this normalized coprime factor uncertainty model is given in [6, 15], see also [24] for the unnormalized case.

Now let us consider smooth nonlinear systems

\[
\begin{align*}
\Sigma: & \quad \dot{x} = f(x) + g(x)u, \quad u \in \mathbb{R}^m, \\
y & = h(x), \quad y \in \mathbb{R}^n,
\end{align*}
\]

where \( x = (x_1, \ldots, x_n) \) are local coordinates for an \( n \)-dimensional state space manifold \( M \). Throughout we assume the existence of a distinguished equilibrium \( x_0 \), i.e. \( f(x_0) = 0 \). Without loss of generality we assume \( x_0 = 0 \), and furthermore \( h(0) = 0 \).

Before defining a stable kernel representation for \( \Sigma \) and the resulting perturbation model we need some preliminaries. Let \( \gamma > 0 \). \( \Sigma \) is said to have \( L_2 \)-gain \( \leq \gamma \) if there exists a nonnegative solution \( V: M \to \mathbb{R} \) (a storage function) to the dissipation inequality [25],

\[
V(x(t_1)) - V(x(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 \|u(t)\|^2 - \|y(t)\|^2) \, dt, \quad V(0) = 0,
\]

for all \( t_1 \geq t_0 \) and all \( u \in L_2[I_0, t_1] \) (with \( x(t_0) \) denoting the solution at time \( t_1 \) for initial condition \( x(t_0) \) at time \( t_0 \)). \( \Sigma \) is said to have \( L_2 \)-gain \( < \gamma \) if there exists some \( \bar{\gamma} < \gamma \) such that \( \Sigma \) has \( L_2 \)-gain \( \leq \bar{\gamma} \). Throughout we will assume that if there exists a solution \( V \geq 0 \) to (6) then there also exists a differentiable solution \( V \geq 0 \) to (6), and we will restrict ourselves to these differentiable solutions.

Let \( \Sigma \) have \( L_2 \)-gain \( \leq \gamma \). From [7, 8, 21] we recall that if additionally \( \Sigma \) is zero-state observable (i.e. \( y(t) = 0, u(t) = 0, \forall t \geq 0 \), implying \( x(0) = 0 \)), then necessarily a solution \( V \geq 0 \) to (6) is positive definite \((V(x) > 0, x \neq 0)\), and \( 0 \) is a locally asymptotically stable equilibrium of (5) with Lyapunov function \( V \).

Next we consider the Hamilton–Jacobi–Bellman equation (corresponding to \( \Sigma \) with cost criterion \( \int_{t_0}^{t_1} (\|u(t)\|^2 + \|y(t)\|^2) \, dt \))

\[
W_x(x)f(x) + \frac{1}{2} W_x(x)g(x)g^T(x)W_x^T(x) - \frac{1}{2} h^T(x)h(x) = 0, \quad W(0) = 0,
\]

(7)
Theorem 2.1. Consider the nonlinear system (5), and assume it is zero-state detectable \((y(t) = 0, u(t) = 0, \forall t \geq 0, \text{implying } x(t) \rightarrow 0, t \rightarrow \infty)\). Suppose there exists a smooth positive definite solution \(W\) to (7), and suppose there exists a smooth solution \(k(x)\) to
\[
W(x)k(x) = h(x). \tag{8}
\]

Define the system \(\Sigma\) with inputs \(\begin{bmatrix} u \\ y \end{bmatrix}\) and outputs \(e\):
\[
\dot{x} = \begin{bmatrix} f(x) - k(x)h(x) \\ g(x) \end{bmatrix} + \begin{bmatrix} k(x) \\ 0 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}, \quad \Sigma:\ e = h(x) - y. \tag{9}
\]

Then \(f(x) - k(x)h(x)\) is locally asymptotically stable (w.r.t. the equilibrium \(x = 0\)) with Lyapunov function \(W\), and globally asymptotically stable if \(W\) is proper (i.e. the sets \(\{x \in \mathbb{R} | 0 \leq W(x) \leq c\}\) are compact for every \(c \geq 0\)). Furthermore, \(\Sigma\) has \(L_2\)-gain \(= 1\). Setting \(e = 0\) in \(\Sigma\) yields \(\Sigma\), and \(\Sigma\) will be called a (nonlinear) stable kernel representation of \(\Sigma\).

Proof (sketch, see [18, 17] for details). From (7) and (8) we obtain
\[
W(x)\left[f(x) - k(x)h(x)\right] = -\frac{1}{2}W(x)g(x)g^T(x)W^T(x) - \frac{1}{2}h^T(x)h(x) \leq 0 \tag{10}
\]
and (global) asymptotic stability follows from LaSalle's invariance principle. Similarly,
\[
W(x)\left[f(x) - k(x)h(x)\right] + g(x)u + k(x)y
= -\frac{1}{2}\|u - g(x)W^T(x)y\|^2 - \frac{1}{2}\|e\|^2 + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|y\|^2, \tag{11}
\]
proving \(L_2\)-gain \(\leq 1\) by integration (see e.g. [21]).

Remark 2.2. If the linearized system \(\dot{x} = (\partial f/\partial u)(0)x + g(0)u, y = (\partial h/\partial x)(0)x\) is anti-stabilizable, and if the imaginary eigenvalues of \((\partial f/\partial x)(0)\) are \((\partial h/\partial x)(0)\)-observable, then at least locally around \(0\) there exists a smooth non-negative solution \(W \geq 0\) to (7) (see e.g. [11, 20]), which will be locally positive definite if the linearized system is observable.

Remark 2.3. Consider a star-shaped coordinate neighborhood of \(x = 0\). Since \(W_x(0) = 0\) and \(h(x) = 0\) we can write (see e.g. [16])
\[
W(x) = x^T M(x), \quad h(x) = C(x)x \tag{12}
\]
for suitable matrices \(M(x), C(x)\), with entries depending smoothly on \(x\). Assume that \(M(x)\) is invertible for all \(x\) in the coordinate neighborhood of \(0\); then a solution \(k(x)\) to (8) is given as [18]
\[
k(x) = M^{-1}(x)C^T(x). \tag{13}
\]
(See [9] for similar considerations in a different context.) Note furthermore that \( M(0) = (\partial^2 W/\partial x^2)(0) \) (the Hessian matrix of \( W \) at 0). Thus, under the conditions of Remark 1, \( M(0) \) will be positive definite, implying that \( M(x) \) will be invertible for \( x \) near 0.

**Remark 2.4** ([18]). If \(-V\) is a negative definite solution to (7), then

\[
\Sigma_{ei} : \dot{u} = f(p) - g(p)g^T(p) V_p^T(p) - k(p)e, \quad p \in M, \quad y = h(p) - e
\]

is a right inverse system to \( \Sigma_e \), i.e., if \( x(0) = p(0) \), then the input–output map of \( \Sigma_e \circ \Sigma_{ei} \) is the identity mapping. Furthermore, \( f(p) - g(p)g^T(p) V_p^T(p) \) is locally asymptotically stable (w.r.t. \( p = 0 \)) with Lyapunov function \( V \) (and globally asymptotically stable if \( V \) is proper). Hence \( \Sigma_e \) has a stable right inverse, generalizing the linear notion of coprimeness.

**Remark 2.5.** For a linear system \( \Sigma \), the stable kernel representation \( \Sigma_e \) reduces to the left normalized coprime factorization (1), (4).

Analogously to the linear case (cf. (3)), we will now consider perturbed nonlinear stable kernel representations

\[
\dot{x} = \left[ f(x) - k(x)h(x) \right] + \left[ g(x) \right] u, \quad e_p = e + w,
\]

where \( w \) is the output of an arbitrary nonlinear state space system with input \( \left[ u \right] \),

\[
\begin{align*}
\phi &= \alpha(p, u, y), \quad \alpha(0, 0, 0) = 0, \\
w &= \beta(p, u, y), \quad \beta(0, 0, 0) = 0,
\end{align*}
\]

having finite \( L_2 \)-gain. (More generally we could consider families of nonlinear input–output maps from \( r \) to \( w \), parametrized by the set of initial conditions.) Setting \( e_p = 0 \) in (15) yields the perturbed system

\[
\Sigma_p : \begin{align*}
\dot{x} &= f(x) + g(x)u + k(x)w, \\
y &= h(x) + w,
\end{align*}
\]

with \( w \) the output of (16).

### 3. The robust stabilization problem

Consider the nonlinear system \( \Sigma \) given by (5), and its perturbed model \( \Sigma_p \) given by (17), (16). The robust stabilization problem is to find a controller

\[
\begin{align*}
C : \dot{\xi} &= l(\xi, y), \quad l(0, 0) = 0, \\
u &= m(\xi, y), \quad m(0, 0) = 0,
\end{align*}
\]

with \( \xi \in \mathbb{R}^* \) the controller state, such that the \( L_2 \)-gain of the closed-loop system (17), (18), from \( w \) to \( z = \left[ y \right] \), is minimized, say equal to \( \gamma^* \geq 0 \).
By the small-gain theorem (see e.g. [3]) this will mean that the overall closed-loop system (16)–(18) will be $L_2$-stable for all perturbations $\Delta$ with $L_2$-gain strictly less than $1/\gamma^*$. In state space terms, if there exist proper positive definite solutions $V_{xc}, V_d$ to the dissipation inequalities

$$V_{xc}(x(t_l), \xi(t_l)) - V_{xc}(x(t_0), \xi(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_l} (\gamma^2 \|w(t)\|^2 - \|z(t)\|^2) \, dt,$$

$$V_d(\phi(t_l)) - V_d(\phi(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_l} (\gamma^2 \|z(t)\|^2 - \|w(t)\|^2) \, dt,$$

then, assuming zero-state detectability [7], the overall closed-loop system (16)–(18) will be globally asymptotically stable with Lyapunov function $\gamma V_{xc} + \gamma V_d$, as can be readily checked from (19).

The problem of minimizing the $L_2$-gain from $w$ to $z = [\gamma]$ for (17) is a standard $\mathcal{H}_\infty$ optimal nonlinear control problem [19, 21, 10, 1, 22, 9]. Usually one first considers the suboptimal $\mathcal{H}_\infty$ problem of finding for given $\gamma > 0$ a controller $C$ (if existing!) which makes the $L_2$-gain from $w$ to $z = [\gamma]$ less than or equal to $\gamma$. For the solution to the suboptimal $\mathcal{H}_\infty$ control problem we follow the approach of [2, 22]. For the state feedback suboptimal $\mathcal{H}_\infty$ control problem we consider the pseudo-Hamiltonian

$$K(x, p, u, w) = p^T [f(x) + g(x)u + k(x)w] - \frac{1}{2} \gamma^2 \|w\|^2 + \frac{1}{2} \|u\|^2 + \frac{1}{2} \|h(x) + w\|^2.$$

Solving $\partial K/\partial u = 0, \partial K/\partial d = 0$ leads to the saddle point $u^* = - g^T(x)p, w^* = (\gamma^2 - 1)^{-1}[h(x) + k^T(x)p]$. Substitution of $u^*, w^*$ into $K$ yields the Hamiltonian $H(x, p) = K(x, p, u^*, w^*)$ and the Hamilton-Jacobi-Isaacs equation $H(x, P^T(x)) = 0$ given as

$$P_s(x)[f(x) + (\gamma^2 - 1)^{-1}k(x)h(x)] + \frac{1}{2} \gamma^2 (\gamma^2 - 1)^{-1}h^T(x)h(x)$$

$$+ \frac{1}{2} P_s(x)[(\gamma^2 - 1)^{-1}k^T(x)k(x) - g(x)g^T(x)]P_s(x) = 0, \quad P(0) = 0. \quad (21)$$

If there exists a solution $P \geq 0$ to (21) then the suboptimal state feedback $\mathcal{H}_\infty$ control problem (for $\gamma$) is solvable by the state feedback

$$u = - g^T(x)P_s(x). \quad (22)$$

Following the certainty equivalence principle of [2] the solution to the output feedback suboptimal $\mathcal{H}_\infty$ control problem is, under appropriate conditions, given as

$$u = - g^T(\hat{x})P^T_s(\hat{x}). \quad (23)$$

with $\hat{x}(t)$ denoting the worst-case estimate of $x(t)$ given the measurements $y(\tau), - \infty < \tau \leq t$, see [2, 22]. (Currently there is intense research activity about the precise conditions for the validity of the worst-case certainty equivalence principle, but we will not elaborate on this.) In general (see e.g. [22]), this will yield an infinite dimensional controller. In the present case, however, the situation is much simpler. Indeed, the suboptimal $\mathcal{H}_\infty$ control problem for (17) with $z = [\gamma]$ is an example of the so-called disturbance feedforward problem, discussed for the linear case in [5], and for the nonlinear case in [14]. In fact, by asymptotic stability
of \( \dot{x} = f(x) - k(x)h(x) \) it follows that for a given control function \( u(\tau) \). \(-\infty < \tau \leq t \), the measurement record \( y(\tau) \), \(-\infty < \tau \leq t \), uniquely specifies the disturbance \( \psi(\tau) = y(\tau) - h(\dot{x}(\tau)) \) and the state trajectory \( \dot{x}(\tau) \).

Indeed the state trajectory \( \dot{x}(\cdot) \) is generated by the differential equations

\[
\dot{x}(\tau) = f(\dot{x}(\tau)) + g(\dot{x}(\tau))u(\tau) + k(\dot{x}(\tau))[y(\tau) - h(\dot{x}(\tau))], \quad x(-\infty) = 0. \tag{24}
\]

Hence a controller solving the suboptimal \( \mathcal{H}_\infty \) control problem for (17) is given as (substitute (23) into (24))

\[
\begin{align*}
\dot{x} &= f(\dot{x}) - g(\dot{x})g^T(\dot{x})P^T_\infty(\dot{x}) + k(\dot{x})[y - h(\dot{x})], \\
u &= -g^T(\dot{x})P^T_\infty(\dot{x}). \tag{25}
\end{align*}
\]

Based on the linear case, the same controller for the general nonlinear disturbance feedforward problem has been recently proposed in [14]. In this paper also a direct proof is provided showing that (25) solves the suboptimal \( \mathcal{H}_\infty \) control problem at least locally, i.e. for initial states in a neighborhood of the origin and for disturbances \( w(\cdot) \) which keep the state trajectories within this neighborhood. Summarizing, we have the following theorem.

**Theorem 3.1.** Suppose (cf. Theorem 2.1) that \( \Sigma \) given by (9) is a stable kernel representation of \( \Sigma \) such that \( f(x) - k(x)h(x) \) is globally asymptotically stable. Suppose there exists a solution \( P \geq 0 \) to (21) for given \( \gamma > 0 \), and assume the certainty equivalence principle for the suboptimal \( \mathcal{H}_\infty \) control problem for (17) holds. Then the controller (25) stabilizes the closed-loop system (16), (17), (25) for every perturbation system \( \Delta \) as in (16), having \( L_2 \)-gain < \( 1/\gamma \).

**Remark 3.2.** From the linear theory (cf. [6, 15]) and the local existence of solutions to (21) based on existence of solutions to the corresponding Riccati equation (cf. [20, 21]), it follows that the minimal \( \gamma^* \), such that locally around 0 there exist solutions \( P \geq 0 \) to (21) for \( \gamma > \gamma^* \), is given by

\[
\gamma^* = [1 + \sigma_{\text{max}}(XX^T)]^{1/2}, \tag{26}
\]

with \( X \) the Hessian matrix \( (\partial^2 V/\partial x^2)(0) \) and \( Z \) the inverse Hessian matrix \( [(\partial^2 W/\partial x^2)(0)]^{-1} \) of the solutions \( W > 0 \) and \( -V \leq 0 \) to (7).

**Remark 3.3.** A related approach to nonlinear robust stabilization will be found in [4].

**Example 3.4.** Let \( \Sigma \) be a lossless system, i.e. there exists \( H: M \to \mathbb{R}, H(0) = 0, H(x) > 0, x \neq 0 \), called the internal energy, such that \( (d/dt)H = u^T \gamma \) or, equivalently,

\[
H_x(x)f(x) = 0, \quad H_x(x)g(x) = h^T(x). \tag{27}
\]

Clearly, positive and negative definite solutions to (7) are given as \( H \), and \(-H\), respectively. Furthermore, \( k(x) \) solving (8) is given as \( g(x) \), and thus the perturbed system \( \Sigma_\rho \) is given as

\[
\dot{x} = f(x) + g(x)[u + w], \quad y = g^T(x)H^T_x(x) + w. \tag{28}
\]

The Hamilton–Jacobi–Isaacs equation (21) takes the form

\[
P_x(x)[f(x) + (\gamma^2 - 1)^{-1}g(x)g^T(x)H^T_x(x)] + \frac{1}{2}[(\gamma^2 - 1)^{-1} - 1] \cdot P_x(x)g(x)g^T(x)P^T_\infty(x)
+ \frac{1}{2}\gamma^2(\gamma^2 - 1)^{-1}H_x(x)g(x)g^T(x)H^T_x(x)(0) = 0, \quad P(0) = 0,
\tag{29}
\]

having the positive definite solution \( P(x) = (\gamma^2/(\gamma^2 - 2))H(x) \) for \( \gamma > \sqrt{2} \). It follows that the controller

\[
\begin{align*}
\dot{x} &= f(\dot{x}) - \gamma^2 \sqrt{2} g(\dot{x})g^T(\dot{x})H^T_x(\dot{x}) + g(\dot{x})[y - g^T(\dot{x})H^T_x(\dot{x})], \\
u &= -\gamma^2 \sqrt{2} g^T(\dot{x})H^T_x(\dot{x}). \tag{30}
\end{align*}
\]
robustly stabilizes $\Sigma$ for every perturbation $A$ with $L_2$-gain $< 1/\gamma$. By Remark 3.2, $\gamma^*$ is given by (26). Since $W = H$ and $V = -H$ we conclude that $\gamma^* = \sqrt{2}$, in accordance with the lower bound $\gamma \geq \sqrt{2}$ as derived above. From a physical point of view, if in (28) $u$'s denote external forces and $y$'s are the corresponding (disturbed) generalized velocities, then (30) corresponds to adding damping with regard to the estimated generalized velocities with a damping factor $\gamma^2 (\gamma^2 - 2)^{-1}$, tending to $\infty$ for $\gamma \downarrow \gamma^* = \sqrt{2}$.

References