On Realization of Nonlinear Systems Described by Higher-Order Differential Equations

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Abstract. We consider systems of smooth nonlinear differential and algebraic equations in which some of the variables are distinguished as "external variables." The realization problem is to replace the higher-order implicit differential equations by first-order explicit differential equations and the algebraic equations by mappings to the external variables. This involves the introduction of "state variables." We show that under general conditions there exist realizations containing a set of auxiliary variables, called "driving variables." We give sufficient conditions for the existence of realizations involving only state variables and external variables, which can then be split into input and output variables. It is shown that in general there are structural obstructions for the existence of such realizations. We give a constructive procedure to obtain realizations with or without driving variables. The realization procedure is also applied to systems defined by interconnections of state space systems. Finally, a theory of equivalence transformations of systems of higher-order differential equations is developed.

1. Introduction

Realization theory for nonlinear systems is by now a well-established subject. We refer to [9] and [21], the survey [10], and the references cited therein. The central problem is one of constructing, for a given input-output map

\[ y(t) = F(u(\tau); 0 \leq \tau \leq t), \quad t \geq 0, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p \]  

(sometimes given as a Volterra series [1], [7] or in generating power series form [2], [7]), a (minimal) state space manifold \( M \), a groundstate \( x_0 \in M \), and an
input-state-output system
\[\dot{x} = f(x, u), \quad x(0) = x_0 \in M,\]
\[y = h(x, u), \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p,\]
living on \(M\), which reproduces the same input-output map (1.1).

As recently argued by Willems [22-24] this is, however, not the only realization problem one might wish to consider. Moreover, it is often not the most natural one. As a matter of fact, for (finite-dimensional) linear systems it is well known [13], [25] that instead of starting from a linear input-output map one can also consider linear higher-order differential equations in the inputs and outputs
\[D\left(\frac{d}{dt}\right)y(t) = N\left(\frac{d}{dt}\right)u(t), \quad y \in \mathbb{R}^p, \quad u \in \mathbb{R}^m,\]
where \(D(s)\) and \(N(s)\) are polynomial matrices of compatible dimensions. In order to justify the nomenclature of inputs \(u\) and outputs \(y\) one has to assume that \(D(s)\) is square with \(\det D(s) \neq 0\) and that \(D^{-1}(s)N(s)\) is a proper rational matrix. The realization problem is now to look for a linear input-state-output system
\[\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}\]
\[x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p,\]
such that the totality of input functions \(u(t), \ t \in \mathbb{R}\), and resulting output functions \(y(t), \ t \in \mathbb{R}\), for different states \(x(0)\) at time 0 coincides with the set of function pairs \((u(t), y(t)), \ t \in \mathbb{R}\), satisfying (1.3) (where equality in (1.3) has to be understood in a distributional sense [23], [24], [15]). Notice that in this case we no longer specify the groundstate \(x_0\) as in (1.2). This also enables us to treat autonomous systems \(\dot{x} = Ax, \ y = Cx\). Of course, in the linear case there exists a natural groundstate \(x_0 = 0\), for which (1.4) reproduces the linear input–output map given by the inverse Laplace transform of \(D^{-1}(s)N(s)\). As a matter of fact, apart from pole-zero cancellations, not much computational difference exists between both realization approaches in the linear case.

A further generalization of (1.3) was proposed by Willems [22], by arguing that in many instances it is not necessary, or even desirable, to distinguish between inputs and outputs \textit{a priori}. Instead, one may wish to start with a vector \(w \in \mathbb{R}^q\) of external variables, and instead of (1.3) one considers higher-order differential equations in \(w\):
\[R\left(\frac{d}{dt}\right)w(t) = 0,\]
where \(R(s)\) is an arbitrary \(l \times q\) polynomial matrix. It is now a \textit{modeling question} which part of the \(w\)-vector can be correctly called inputs and which complementary part outputs.

For the nonlinear case all this suggests considering input–output systems given by smooth nonlinear higher-order differential equations
\[D_i(y, \dot{y}, \ldots, y^{(k)}) = N_i(u, \dot{u}, \ldots, u^{(k)}), \quad i = 1, \ldots, p, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p,\]
or, if we combine inputs and outputs into one vector of external variables \( w \in \mathbb{R}^q \), looking at systems described as

\[
R_i(w, \dot{w}, \ldots, w^{(k)}) = 0, \quad i = 1, \ldots, l, \quad w \in \mathbb{R}^q,
\]

where the \( R_i \) are smooth functions.

For both descriptions one may try to find a state space realization (without fixed groundstate as in (1.2))

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x, u)
\end{align*}
\]

\( x \in M, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p. \)

(1.8)

In the first description (1.6) the inputs \( u \) and outputs \( y \) are already specified, while in the second case (1.7) \( w \) has to be split into an input and an output part, i.e., \( Tw = \text{col}(y, u) \) for some permutation matrix \( T \).

In the linear case, as shown in [22] and [24], the input–output description (1.3), the external description (1.5), and the input–state–output description (1.4) are all equivalent, in the sense that one may freely pass from (1.3) to (1.4) and vice versa, and, given (1.5), one can always find a permutation matrix \( T \) such that \( R(s)T^{-1} = [D(s) \quad -N(s)] \) with \( D(s) \) and \( N(s) \) as in (1.5).

For the nonlinear case the situation is much more delicate, as already discussed in [14], [15], and [22], see also [4]. First, it is not obvious in what sense the equalities (1.6) and (1.7) have to be understood. A nonlinear analogue of distributional equality seems hard to get by, while it is in many cases too restrictive to assume that the output and input functions are all \( C^k \). In this paper we will adopt the pragmatic approach of first assuming \( C^\infty \)-smoothness of \( u, y, \) and \( w \), then constructing a state space realization (1.8), and, finally, relaxing the smoothness assumptions on \( u \), especially, as long as (1.8) continues to make sense.

There are also structural problems. Most importantly, as shown in [4], given an input–output system (1.6), an input–state–output system (1.8) which realizes (1.6) may not always exist. A fortiori an external system (1.7) may not always be realizable by an input–state–output system. Secondly, due to the nonlinear structure of equations (1.7) it is clear that we cannot always rewrite (1.7) in the form (1.6). Furthermore, it is not immediate which additional assumptions have to be added to (1.6) in order that the variables \( u \) and \( y \) are correctly called inputs and outputs. Thirdly, it is not clear whether an input–state–output system always defines an input–output system (1.6) or external system (1.7).

The main goal of this paper is to give sufficient conditions in order that an external system (1.7) can be realized as an input–state–output system (1.8). Furthermore, we shall give a constructive procedure to obtain such an input–state–output system if these conditions are satisfied. This procedure is closely related to a realization procedure for the linear case recently proposed by Schumacher [19], who was in turn inspired by the work of Willems [22–24]. If these conditions are not all met, then we shall show that in many cases it is still possible to realize the external system as a so-called driven state space system

\[
\begin{align*}
\dot{x} &= f(x, v) \\
w &= g(x, v)
\end{align*}
\]

\( x \in M, \quad v \in \mathbb{R}^r, \quad w \in \mathbb{R}^q. \)

(1.9)
where the $v$ are arbitrary time-functions, called driving variables. As a matter of fact, we shall show that the main structural obstruction to obtaining an input-output realization is in the transition from a driven state space system (1.9) to an input–state–output system (1.8). This obstruction is geometrical in nature and has to do with certain integrability conditions, which generalize the conditions obtained in [4].

Usually one is not interested only in obtaining arbitrary driven state space or input–state–output systems realizing an external system, but the realization has to be *minimal* in some sense. Since autonomous (i.e., without inputs) systems are also included in our theory it is clear that we cannot require “controllability” of the realization. Instead, it will be shown, under some technical assumptions, that an arbitrary realization can always be reduced to a realization with *minimal* dimension of its state space. This minimal realization is, roughly speaking, *observable*.

There is one appealing further generalization of the description of an external system as given in (1.7). Instead of describing the external behavior in terms of higher-order differential equations solely in the external variables $w$, it is in many cases (e.g., electrical networks) more natural to describe it with higher-order differential equations involving also some auxiliary, or *internal*, variables $\xi$:

$$P_i(w, \dot{w}, \ldots, w^{(k)}, \xi, \dot{\xi}, \ldots, \xi^{(k)}) = 0, \quad i = 1, \ldots, l, \quad w \in \mathbb{R}^q, \quad \xi \in \mathbb{R}^s. \quad (1.10)$$

The external behavior described by (1.10) is defined as the set of (smooth) time-functions $w(t)$ satisfying (1.10) for some (smooth) time-function $\xi(t)$. Notice also that input–state–output and driven state space systems are in this form. (In the first case the internal variables consist of the state variables, and in the second case of the state and the driving variables.) In the linear case this point of view was introduced in [22] and [23], and stressed in [19]; as a matter of fact, it is close to the approach proposed earlier by Rosenbrock [13]. Although equations (1.10) should serve as the “first definition” for systems described by higher-order differential equations, we postpone the treatment of this more general case to Section 5 in order to keep the notation and formulation of our results as simple as possible. We show in Section 5 that the realization results obtained for systems described by the more simple form (1.7) can be immediately generalized to this general case (1.10).

Finally, we like to mention the possible connections with some recent work by Fliess [3], where higher-order nonlinear differential equations (of a somewhat restricted nature) in the inputs and outputs are considered in problems of (left and right) invertibility and input–output decoupling of nonlinear systems.

### 2. Preliminaries

Before proceeding with the realization procedure let us first fix notations and terminology. A (smooth) *external system* is given by a set of higher-order differential equations

$$R_i(w, \dot{w}, \ldots, w^{(k)}) = 0, \quad i = 1, \ldots, l. \quad (2.1)$$
where \( w \in \mathbb{R}^q \) denotes the vector of external variables and the equations \( R_i \) are \( C^\infty \). (We mention that without too much difficulty this can be extended to the case when \( w \) belongs to an arbitrary manifold, see [15].) The (smooth) external behavior \( \Sigma_e(R) \) of the external system (2.1) is defined as

\[
\Sigma_e(R) = \{ w: \mathbb{R} \to \mathbb{R}^q | w \text{ is } C^\infty \text{ and } R_i(w(t), \dot{w}(t), \ldots, w^{(k)}(t)) = 0, i = 1, \ldots, l, \forall t \in \mathbb{R} \}. \tag{2.2}
\]

Notice that we restrict the solution set of (2.1) to the \( C^\infty \) time-functions. For simplicity we shall, throughout, make:

**Assumption 1.** Every time-function \( w(t) \) defined on an interval \([0, T] , T > 0\), and satisfying (2.1) can be extended to a time-function \( w: \mathbb{R} \to \mathbb{R}^q \) satisfying (2.1).

In the case when \( w = \text{col}(y, u) \), with \( y \in \mathbb{R}^p \) and \( u \in \mathbb{R}^m \), and the external system (2.1) is given by equations

\[
D_i(y, \dot{y}, \ldots, y^{(k)}) = N_i(u, \dot{u}, \ldots, u^{(k)}), \quad i = 1, \ldots, p, \tag{2.3}
\]

the external system is called an input-output system. Its (smooth) external behavior is defined as in (2.2) and is denoted by \( \Sigma_e(D, N) \).

We now come to the state space systems. Let us call a system

\[
\begin{align*}
\dot{x} &= f(x, v) \quad x \in M, \quad v \in \mathbb{R}^r, \quad w \in \mathbb{R}^q, \\
w &= g(x, v)
\end{align*} \tag{2.4}
\]

where \( M \) is a smooth manifold, \( f \) and \( g \) are smooth, and \( v \) are the driving variables (arbitrary smooth functions of time), a driven state space system. Its external behavior \( \Sigma_d(f, g) \) is defined as

\[
\Sigma_d(f, g) = \{ w: \mathbb{R} \to \mathbb{R}^q | \exists C^\infty \text{ function } v: \mathbb{R} \to \mathbb{R}^r \text{ and a state } x_0 \in M \text{ such that } w(t) = g(x(t), v(t)), \quad t \in \mathbb{R}, \text{ where } x(t) \text{ is the solution of } \dot{x}(t) = f(x(t), v(t)), \quad x(0) = x_0 \}. \tag{2.5}
\]

**Remark.** In a more general (and more natural) definition of a driven state space system the space \( \mathbb{R}^r \) of the driving variables may also depend on the state \( x \). This is formalized by defining a (vector) bundle \( B \) over \( M \), the fibers of which contain the driving variables \( v \), and by defining a bundle morphism \( F: B \to TM \), which is in local coordinates \((x, v)\) for \( B \) and \((x, \dot{x})\) for \( TM \) of the form \( F(x, v) = (x, \dot{x} = f(x, v)) \). For a discussion of these issues see [22], [14], and [15].

A system

\[
\begin{align*}
\dot{x} &= f(x, u) \quad x \in M, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p, \\
y &= h(x, u)
\end{align*} \tag{2.6}
\]
with $M$ a smooth manifold and $f$ and $h$ smooth mappings, is called an **input-state-output system**. Its (smooth) external behavior $\Sigma_{i/s/o}(f, h)$ is given as

$$
\Sigma_{i/s/o}(f, h) = \{(y(\cdot), u(\cdot)) : \mathbb{R} \to \mathbb{R}^p \times \mathbb{R}^m | u(\cdot) \text{ is smooth and } \exists x_0 \in M \text{ such that } y(t) = h(x(t), u(t)), t \in \mathbb{R}, \text{ where } x(t) \text{ is the solution of } \dot{x}(t) = f(x(t), u(t)), x(0) = x_0 \}.
$$

(2.7)

In analogy with Assumption 1 we shall, throughout, make the following completeness assumption on the state space systems:

**Assumption 2.** The solution of $\dot{x} = f(x, v)$, $x(0) = x_0$ is defined for any $t \in \mathbb{R}$ and any smooth $v(\cdot)$ and $x_0 \in M$.

It is clear that every input-state-output system (2.6) can also be written as a driven state space system. Simply, define $g(x, v)$ as

$$
g(x, v) = \text{col}(h(x, v), v)
$$

(2.8)

and identify the driving variables $v$ with the inputs $u$. The other way around does not work in general. For example, if $g$ in (2.4) does not depend on $v$ then clearly the driven state space system cannot be written as an input-state-output system. See [14] for a discussion of these issues.

Given an external system (2.1) with external behavior $\Sigma_e(R)$ the following realization problems can be posed:

1. Does a driven state space system (2.4) such that $\Sigma_d(f, g) = \Sigma_e(R)$ exist? If so, then (2.4) is called a **driven realization** of (2.1).
2. Does an input-state-output system (2.6) and a permutation matrix $T$ on $\mathbb{R}^q$ such that $\Sigma_{i/s/o}(f, h) = \Sigma_e(R)$, where $Tw = \text{col}(y, u)$, exist? If so, then (2.6) is called an **input-output realization** of (2.1).

These realization problems form the contents of the next two sections.

### 3. The Linear Case

Before going to a realization procedure for nonlinear external systems, we shall first briefly describe a realization procedure for **linear** external systems. This procedure is very close to the one recently proposed by Schumacher [19]. In Section 4 we will try to mimic this procedure for nonlinear systems. A linear external system is given by equations

$$
R \left( \frac{d}{dt} \right) w(t) = 0, \quad w \in \mathbb{R}^q,
$$

(3.1)

where $R(d/dt)$ is the linear differential operator given by the $l \times q$ polynomial matrix

$$
R(s) = R_0 + R_1 s + \cdots + R_k s^k
$$

(3.2)

for some $k \in \mathbb{N}$ and constant matrices $R_0, \ldots, R_k$. 

A linear driven state space system is denoted as
\[
\begin{align*}
\dot{x} &= Ax + Bu, \\
w &= Fx + Gu,
\end{align*}
\] (3.3)
and its smooth external behavior as \( \Sigma_d(A, B, F, G) \). A linear input-state-output system is denoted as
\[
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &= Cx + Du,
\end{align*}
\] (3.4)
and its external behavior as \( \Sigma_{i/s/o}(A, B, C, D) \).

The realization procedure consists of three steps. First we shall realize (3.1) by a driven state space system. In order to do so define a \((k+1) \times q\)-dimensional state vector \( \text{col}(w, \dot{w}, \ldots, \dot{w}^{(k)}) \), and driving variables \( v \in \mathbb{R}^q \). Then
defines a driven state space system. Since we are looking for all time-functions \( w(t) \) satisfying (3.1), we can equivalently look for all state trajectories of (3.5a) which are contained in the kernel of the linear output map (3.5b). Using geometric control theory [26] this can be rephrased as: Find the maximal controlled invariant subspace of (3.5a) contained in the kernel of (3.5b).

This space exists and can be calculated [26]. Call it \( X_1 \subset \mathbb{R}^{(k+1) \times q} \). After feedback, system (2.5a) can then be restricted to this subspace \( X_1 \), yielding a system of the form
\[
\dot{x} = A_{1}x_1 + B_1v_1, \quad x_1 \in \mathbb{R}^k
\] (3.6a)
with \( v_1 \) the new driving variables. Since \( X_1 \) is a subspace of \( \mathbb{R}^{(k+1) \times q} \) we can define the projection of \( X_1 \) onto the first \( q \) components of \( \mathbb{R}^{k \times q} \). Call this \( F_1 \).

By construction this driven state space system (3.6) is a driven realization of (3.1).

The second step of the realization procedure is to construct an input–output realization. Roughly speaking we have to decide which components of the \( w \)-vector can serve as inputs, respectively outputs. This is done by maximally reducing the “number of integrations” from the driving variables \( v_1 \) to the outputs \( w \). Specifically, since the driving variables \( v_1 \) in (3.6a) are arbitrary smooth
functions, it follows that the components of the state $x_1$ contained in $\text{Im} B_1$ are smooth functions. Therefore we can define these state components as the new driving variables for a system with state space $X_t/\text{Im} B_1$. Furthermore, the image of these state components under the mapping $F_1$ qualify as input components of $w$. This procedure can be repeated. More formally we construct the nondecreasing sequence of subspaces

$$S_1 = \text{Im} B_1,$$

$$S_2 = S_1 + A_1(S_1 \cap \text{Ker } F_1)$$

$$S_{i+1} = S_i + A_1(S_i \cap \text{Ker } F_1), \quad i = 2, \ldots.$$

Due to the finite-dimensionality we get, for a certain $i$, $S_{i+1} = S_i$. Denote $S^* := S_i = S_{i+1}$, then it follows that $S^*$ is \textit{conditioned invariant}, i.e.,

$$A_1(S^* \cap \text{Ker } F_1) \subset S^*.$$  \hspace{1cm} (3.8)

Furthermore, it can be proved that $S^*$ is the \textit{minimal} conditioned invariant subspace containing $\text{Im} B_1$. Now let us choose a basis of $X_1$ adapted to $S^*$ and $F_1$ in such a way that

$$S^* = \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}, \quad \text{Ker } F_1 = \begin{pmatrix} 0 \\ * \\ 0 \\ 0 \end{pmatrix}, \quad x_1 = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}. \hspace{1cm} (3.9)$$

By (3.8) and $\text{Im} B_1 \subset S^*$, the matrices $A_1$, $B_1$, and $F_1$ in this basis have the form

$$A_1 = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{11} & A_{22} & A_{23} & A_{24} \\ A_{31} & 0 & A_{33} & A_{34} \\ A_{41} & 0 & A_{43} & A_{44} \end{pmatrix}, \quad B_1 = \begin{pmatrix} B_{11} \\ B_{21} \\ 0 \\ 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} F_{11} & 0 & 0 & F_{14} \end{pmatrix}.$$  \hspace{1cm} (3.10)

Hence, on $X/S^*$ we obtain the equations

$$\dot{z}_3 = A_{33}z_3 + Z_{34}z_4 + A_{31}z_1,$$

$$\dot{z}_4 = A_{43}z_3 + A_{44}z_4 + A_{41}z_1.$$  \hspace{1cm} (3.11)

Since $\text{Ker } F_{11} = 0$ there exists a permutation matrix $T$ such that

$$TF_{11} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix},$$

with $G_2$ a regular matrix. Denote

$$T(F_{11} : F_{14}) = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{pmatrix} \quad \text{and} \quad Tw = \begin{pmatrix} y \\ u \end{pmatrix}.$$

Then $u = G_2z_3 + G_4z_4$. It follows from the algorithm (3.7) that $z_1$ is an arbitrary smooth function and hence $u$ is an arbitrary smooth function, and so qualifies...
as an input. Now substitute $z_l = G_2^{-1}u - G_2^{-1}G_4z_4$ into (3.11) to obtain equations of the form (with $x_2 = \text{col}(z_3, z_4)$)

$$\dot{x}_2 = A_2x_2 + B_2u, \quad x_2 \in X_2 := X_1/S^*.$$  

(3.12a)

Moreover, since $y = G_3z_4 + G_1z_1$ we obtain output equations

$$y = C_2x_2 + D_2u.$$  

(3.12b)

By construction the external behavior of the input-state-output system (3.12) equals the external behavior of the driven state space system (3.6) and hence the external behavior of the external system (3.1). So (3.12) is an input-output realization.

The third and final step of the realization procedure is to reduce (3.12) to a minimal input-output realization. This is simply done by computing the maximal unobservability subspace

$$0 = \bigcap_{i=1}^{k} \text{Ker } C_2A_2^{i-1}$$  

(3.13)

and by factoring out $X_2$ by the subspace 0. Then we obtain the observable input-state-output system

$$\begin{bmatrix} \dot{x}_3 = A_3x_3 + B_3u \\ y = C_3x_3 + D_3u \end{bmatrix} \quad x_3 \in X_3 = X_2/0.$$  

(3.14)

**Remark.** As already indicated, in general (3.14) will not be controllable. However, we have the following nice condition directly in terms of the defining polynomial matrix $R(s)$: (3.14) is controllable if and only if $\dim \text{Ker } R(s)$ does not depend on $s \in \mathbb{C}$ (for a proof see [15]).

Summarizing the construction of a minimal input-state-output system realizing the external system (3.1) consists of three steps (consult [19] for more details):

**Step 1.** Compute the maximal controlled invariant subspace of (3.5a) contained in $\text{Ker } (R_0 \mid \cdots \mid R_k)$. Restrict the system to this subspace to obtain the driven realization (3.6).

**Step 2.** Compute the minimal conditioned invariant subspace of (3.6) containing $\text{Im } B_1$. Factor out by this subspace to obtain the input-output realization (3.12).

**Step 3.** Compute the maximal unobservability subspace of (3.12). Factor out by this subspace to obtain the minimal input-output realization (3.14).

4. The Nonlinear Case

In this section we try to mimic for nonlinear systems the realization procedure described in section 3 for linear systems.
First we note that, as in the linear case, we can associate to any nonlinear external system
\[ R_i(w, \dot{w}, \ldots, w^{(k)}) = 0, \quad i = 1, \ldots, l, \quad w \in \mathbb{R}^q, \]
(4.1)
a driven state space system with linear dynamics
\[ \frac{d}{dt} \begin{pmatrix} w \\ \dot{w} \\ \vdots \\ \dot{w}^{(k)} \end{pmatrix} = \begin{pmatrix} 0 & I_q & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_q \\ 0 & \cdots & 0 & I_q \end{pmatrix} \begin{pmatrix} w \\ \dot{w} \\ \vdots \\ \dot{w}^{(k)} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \dot{\vartheta} \end{pmatrix}, \]
(4.2a)
where \( \vartheta \in \mathbb{R}^l \) are the driving variables, and with smooth nonlinear output functions
\[ z_i = R_i(w, \dot{w}, \ldots, w^{(k)}), \quad i = 1, \ldots, l \]
(4.2b)
In step 1 of the linear realization procedure we computed the maximal controlled invariant subspace \( X_1 \) contained in the subspace \( z_1 = \cdots = z_l = 0 \). Similarly, in the nonlinear case we compute the "maximal controlled invariant submanifold contained in the set \( z_1 = \cdots = z_l = 0 \)." The notion of controlled invariant submanifold was introduced in [17], [18]:

**Definition 4.1.**
Let
\[ \dot{x} = g_0(x) + \sum_{j=1}^{m} u_j g_j(x), \quad x \in M, \]
(4.3)
be a nonlinear state space system. A submanifold \( N \subset M \) is called **controlled invariant** for (4.3) if there exists a smooth feedback
\[ u_j = \alpha_j(x), \quad j = 1, \ldots, m, \]
(4.4)
such that the feedback transformed vector field
\[ \dot{x} = g_0(x) + \sum_{j=1}^{m} \alpha_j(x) g_j(x), \quad x \in M, \]
(4.5)
is **tangent** to \( N \) in any point of \( N \) (and so the solutions of (4.5) starting in \( N \) remain in \( N \)).

In the linear case this is just the definition of a controlled invariant subspace. On the other hand, let \( V \subset \mathbb{R}^n \) be a controlled invariant subspace of a linear system. Then \( V \) also defines a controlled invariant distribution, namely the (constant) distribution \( V \subset T_x \mathbb{R}^n = \mathbb{R}^n \) for any \( x \in \mathbb{R}^n \). Hence the generalization of linear controlled invariance to the nonlinear case can be performed in at least two different ways: controlled invariant **submanifolds** as in Definition 3.1, or the controlled invariant **distributions** as introduced in [5] and [8]. In our case we need the first generalization. A major problem of this first generalization is that the existence of a maximal controlled invariant submanifold contained in a given subset is not guaranteed (contrary to the case of controlled invariant distributions, see [5] and [8]). For instance, consider the system
\[ \dot{x} = u, \quad y_1 = x, \quad y_2 = x - 1. \]
(4.6)
Clearly, the maximal controlled invariant submanifold contained in \( y_1 = y_2 = 0 \) does not exist. Hence, we have to impose extra conditions on the system.

**Theorem 4.2** [17], [18]. Let

\[
\dot{x} = g_0(x) + \sum_{j=1}^{m} u_j g_j(x), \quad x \in M, \\
y_i = H_i(x), \quad i = 1, \ldots, p, 
\]

be a nonlinear system. For each output \( H_i \) define the characteristic number \( \rho_i \) as the smallest nonnegative integer for which (see [7], [8])

\[
\exists j \in \{1, \ldots, m\} \quad \text{such that} \quad L_{g_j} L_{g_0}^{\rho_i} H_i(x) \neq 0 \quad \text{for some} \quad x \in M. 
\]

Suppose that for \( i = 1, \ldots, p, \rho_i \) exists. Define the \( p \times m \) matrix \( A(x) \) with \((i,j)\)th element

\[
A_{ij}(x) = L_{g_j} L_{g_0}^{\rho_i} H_i(x). 
\]

Suppose that rank \( A(x) = p \) for any \( x \in M \). Then the maximal controlled invariant submanifold contained in \( y_1 = \cdots = y_p = 0 \) exists and is given as

\[
N^* = \{ x | L_{g_0}^k H_i(x) = 0, k = 0, 1, \ldots, \rho_i, i = 1, \ldots, p \}, 
\]

where the functions \( L_{g_0}^k H_i, k = 0, 1, \ldots, \rho_i, i = 1, \ldots, p, \) are all independent. Furthermore, the needed feedback (4.4) is given as a solution of

\[
A(x) \alpha(x) + b(x) = \gamma(x), 
\]

where \( b(x) \) is the vector with \( i \)th component \( L_{g_0}^{\rho_i+1} H_i(x) \), and \( \gamma(x) \) is a vector of functions which are zero on \( N^* \). Finally, let \( \beta(x) \) be an \( m \times (p - m) \) matrix with rank \( p - m \) satisfying

\[
A(x) \beta(x) = 0. 
\]

Then the (degenerate) feedback

\[
u_j = \alpha_j(x) + \sum_{k=1}^{p-m} \beta_{jk}(x) v_k, 
\]

with \( \alpha_j \) a solution of (4.11), yields a control system

\[
\dot{x} = g_0(x) + \sum_{j=1}^{m} \alpha_j(x) g_j(x) + \sum_{k=1}^{p-m} v_k \left( \sum_{j=1}^{k} \beta_{jk}(x) g_j(x) \right) 
\]

with \( p - m \) inputs \( v_k \), which can be restricted to a control system on \( N^* \).

**Proof.** The above procedure was introduced in [7] and [8] for calculating the maximal controlled invariant distribution contained in the distribution \( dH_1 = \cdots = dH_p = 0 \). We shall show that it also yields the maximal controlled invariant submanifold \( N^* \) given in (4.10). From Lemma 3.10 of [7] it follows that the functions \( L_{g_0}^k H_i, k = 0, 1, \ldots, \rho_i, i = 1, \ldots, p, \) are independent, so that \( N^* \) defined
by (4.10) is a submanifold. \( N^* \) is controlled invariant since we have (with \( d/dt \) differentiation along the system (4.7))

\[
\frac{d}{dt} L_{g_0}^k H_i = L_{g_0}^{k+1} H_i + \sum_{j=1}^{m} u_j L_{g_j} L_{g_0}^k H_i.
\]

(4.15)

It now follows from the definition of \( \rho_i \) that

\[
\frac{d}{dt} L_{g_0}^k H_i = L_{g_0}^{k+1} H_i, \quad k = 0, 1, \ldots, \rho_i - 1,
\]

\[
\frac{d}{dt} L_{g_0}^0 H_i = L_{g_0}^{0+1} H_i + \sum_{j=1}^{m} u_j L_{g_j} L_{g_0}^0 H_i
\]

(4.16)

\[
= b_i + \sum_{j=1}^{m} u_j A_{g_j}.
\]

Therefore, if \( u_j = \alpha_l(x) \) is a solution of (4.11) then the vector field \( g_0 + \sum_{j=1}^{m} \alpha_j g_j \) is tangent to \( N^* \).

Furthermore, by (4.16) all functions \( L_{g_0}^k H_i, \quad k = 0, 1, \ldots, \rho_i, \quad i = 1, \ldots, p \), have to be zero on any controlled invariant submanifold contained in \( H_i(x) = \cdots = H_p(x) = 0 \). Hence \( N^* \) is the maximal controlled invariant submanifold contained in \( H_i(x) = \cdots = H_p(x) = 0 \). It also follows from (4.16) that, with \( \beta(x) \) defined by (4.12), the derivatives of the functions \( L_{g_0}^k H_i, \quad k = 0, 1, \ldots, \rho_i, \quad i = 1, \ldots, p \), along the feedback transformed system (4.14) equal

\[
\frac{d}{dt} L_{g_0}^k H_i = L_{g_0}^{k+1} H_i, \quad k = 0, 1, \ldots, \rho_i - 1,
\]

(4.17)

\[
\frac{d}{dt} L_{g_0}^0 H_i = \gamma_i, \quad i = 1, \ldots, p,
\]

and hence (4.14) can be restricted to a control system on \( N^* \).

\[ \square \]

**Remark 1.** Notice that \( N^* \) can be empty.

**Remark 2.** In order to define a feedback transformed system (4.14) which can be restricted to \( N^* \) the assumption that the matrix \( A(x) \) is surjective for any \( x \in M \) can be relaxed to the assumption that \( A(x) \) is surjective for any \( x \in N^* \) (see [17]).

Let us now apply Theorem 4.2 to the driven state space system (4.2), with

\[
g_0(w, \dot{w}, \ldots, w^{(k)}) = \text{col}(\dot{w}, \ldots, w^{(k)}, 0),
\]

(4.18)

\[
g_j(w, \dot{w}, \ldots, w^{(k)}) = \text{col}(0, \ldots, 0, e_j), \quad j = 1, \ldots, q,
\]

and output mappings \( R_i(w, \dot{w}, \ldots, w^{(k)}), \quad i = 1, \ldots, l \). In order to compute \( \rho_i \) notice that

\[
L_{g_j} L_{g_0}^0 R_i = L_{g_j} R_i = \frac{\partial R_i}{\partial w^{(k)}_j}.
\]

(4.19)

Suppose that for a certain \( i, \partial R_i/\partial w^{(k)}_j \equiv 0 \) for all \( j = 1, \ldots, q \). Then \( \rho_i \geq 1 \) and in order to compute \( \rho_i \) we have to proceed with the calculation of

\[
L_{g_0} R_i = \sum_{s=1}^{q} \left( \frac{\partial R_i}{\partial \dot{w}_s} \dot{w}_s + \frac{\partial R_i}{\partial \ddot{w}_s} \ddot{w}_s + \cdots + \frac{\partial R_i}{\partial w^{(k-1)}_s} w^{(k-1)}_s \right).
\]

(4.20)
Since \( \partial R_i/\partial w_j^{(k)} = 0 \), \( j = 1, \ldots, q \), we obtain
\[
L_{g_i}L_{b_0}R_i = \frac{\partial R_i}{\partial w_j^{(k-1)}}, \quad j = 1, \ldots, q. \tag{4.21}
\]

Suppose that for this \( i \) also \( \partial R_i/\partial w_j^{(k-1)} = 0 \) for all \( j = 1, \ldots, q \). Then \( \rho_i \geq 2 \) and we compute from (4.20) that
\[
L^2_{g_0}R_i = \frac{\partial R_i}{\partial w_j^{(k-2)}}, \quad j = 1, \ldots, q. \tag{4.22}
\]

Since \( \partial R_i/\partial w_j^{(k)} = 0 \), \( j = 1, \ldots, q \), we obtain
\[
L_{g_i}L^2_{g_0}R_i = \frac{\partial R_i}{\partial w_j^{(k-2)}}, \quad j = 1, \ldots, q. \tag{4.23}
\]

We continue in this way to obtain:

**Theorem 4.3.** Consider the driven state space system (4.2), and let the vector fields \( g_0 \) and \( g_j \) be defined as in (4.18). The characteristic number \( \rho_i \) is the smallest nonnegative integer for which
\[
\exists j \in \{1, \ldots, q\} \text{ such that } \frac{\partial R_i}{\partial w_j^{(k-\rho_j)}}(x) \neq 0 \text{ for some } x = (w, \ldots, w^{(k)}) \tag{4.24}
\]
and the matrix \( A(x) \) has \((i,j)\)th element
\[
A_{ij}(x) = \frac{\partial R_i}{\partial w_j^{(k-\rho_j)}}(x), \quad x = (w, \ldots, w^{(k)}). \tag{4.25}
\]

**Proof.** First we prove by induction that if for any \( r < k \)
\[
\frac{\partial R_i}{\partial w_j^{(k)}} = \frac{\partial R_i}{\partial w_j^{(k-1)}} = \ldots = \frac{\partial R_i}{\partial w_j^{(k-r-1)}} = 0 \text{ for all } \quad j = 1, \ldots, q, \tag{4.26}
\]
then \( L^r_{g_0}R_i \) is of the form
\[
L^r_{g_0}R_i = \frac{\partial R_i}{\partial w_j^{(k-r)}} \left( \frac{\partial R_i}{\partial w_j^{(r-1)}} + \frac{\partial R_i}{\partial w_j^{(r-2)}} + \ldots + \frac{\partial R_i}{\partial w_j^{(r-r)}} \right) \tag{4.27}
\]
for certain constants \(c_1, c_2, \ldots, c_{r-1}\). Indeed, for \(r=0,1\) we have proved this in (4.20) and (4.22). Let now \(L_{g_0}^r R_i\) be of the form (4.27) and let

\[
\frac{\partial R_i}{\partial w_j^{(k-r)}} = 0 \quad \text{for all} \quad j = 1, \ldots, q.
\]  

(4.28)

Then all the terms in (4.27) involving \(\frac{\partial R_i}{\partial w_j^{(k-r)}}\) vanish, and an easy calculation shows that \(L_{g_0}^{r+1} R_i = L_{g_0}(L_{g_0}^r R_i)\) is again of the form (4.27) with \(r\) replaced by \(r+1\).

It immediately follows from (4.27) that if (4.26) is satisfied then

\[
L_{g_0}^r L_{g_0}^r R_i = \frac{\partial R_i}{\partial w_j^{(k-r)}}, \quad j = 1, \ldots, q.
\]  

(4.29)

The theorem is now an immediate consequence of the definition of the integers \(\rho_i\) and the matrix \(A(x)\). \(\Box\)

In order to apply Theorem 4.2 we have to assume that for any \(i = 1, \ldots, l\), \(\rho_i\) exists and that rank \(A(x) = l\) for any \(x = (w, \dot{w}, \ldots, w^{(k)})\). By Theorem 4.3 we see that \(\rho_i\) exists if and only if the function \(R_i\) is not equal to a constant function, and if \(\rho_i\) exists then \(\rho_i \leq k\). Therefore we make the standing assumption:

**Assumption 3.** For any \(i = 1, \ldots, l\), the functions \(R_i\) are not constant.

Of course this assumption is harmless since we may always leave out the equation \(R_i(w, \dot{w}, \ldots, w^{(k)}) = 0\) in (4.1) if \(R_i\) is identically zero and if \(R_i\) is constant there is no solution to (4.1) (see also Section 6). The next standing assumption is more serious.

**Assumption 4.** The rank of the matrix

\[
\left( \frac{\partial R_i}{\partial w_j^{(k-r)}} \right)_{(w, \dot{w}, \ldots, w^{(k)})} \quad i = 1, \ldots, l, \quad j = 1, \ldots, q,
\]  

(4.30)

equals \(l\) for any \((w, \dot{w}, \ldots, w^{(k)})\).

In Section 6 we analyze the generality of Assumption 4, starting from the observation that in the linear case (3.1) Assumption 4 amounts to assuming the polynomial matrix \(R(s)\) to be row proper; an assumption which can always be made if we allow for certain operations on the defining equations \(R_i\) which leave the external behavior invariant (see Proposition 6.1).

As a consequence of Theorems 4.2 and 4.3 and Assumptions 3 and 4 we conclude that the maximal controlled invariant submanifold of the driven state space system (4.2a) contained in the output mappings (4.2b) exists and is given by

\[
N^* = \{(w, \dot{w}, \ldots, w^{(k)}) \in \mathbb{R}^{(k+1)q} | L_{g_0}^r R_i(w, \dot{w}, \ldots, w^{(k)}) = 0, \quad r = 0, 1, \ldots, \rho_i, \quad i = 1, \ldots, l\},
\]  

(4.31)
where $\rho_i$ is defined as in (4.24) and the equations $L_{\mathbf{R}_0}^c R_i$ are of the form (4.27). Furthermore, by Theorem 4.2 there exists a feedback

$$\dot{v} = \alpha(x) + \beta(x) v, \quad x = (w, \dot{w}, \ldots, w^{(k)}),$$

(4.32)

with $v \in \mathbb{R}^m$, $m := q - l$, $\rank \beta(x) = q - l$, such that the feedback transformed driven state space system (4.2) can be restricted to a system on $N^*$:

$$\dot{x} = g_0^1(x^1) + \sum_{j=1}^m v_j g_j^1(x^1), \quad x^1 \in M^1 := N^*.$$  

(4.33a)

Since $M^1 = N^*$ is a submanifold of $\mathbb{R}^{(k+1)q}$, the projection of $N^*$ onto the first $q$ components of $\mathbb{R}^{(k+1)q}$ is a smooth mapping. Hence we obtain smooth output functions

$$w_j = G_j^1(x^1), \quad j = 1, \ldots, q.$$  

(4.33b)

By construction, as in the linear case, the driven state space system (4.33) is a driven realization of the external system (4.1).

**Remark.** Actually, at this point it becomes clear that we should use the more general definition of a driven state space system alluded to in Section 2. As a matter of fact, the kernel of the matrix $A(w, \dot{w}, \ldots, w^{(k)})$ given by (4.30) defines at every point $x^1 \in M^1$ an $m$-dimensional subspace of the tangent space of $M^1$ in $x^1$, and hence defines an $m$-dimensional distribution on $M^1$. The input vector fields $g_1^1, \ldots, g_m^1$ in (4.33a) have to be such that they span this distribution; and there may be structural obstructions to finding such globally defined vector fields. However, as can be readily checked, the results of this section, rephrased to this more general setting, remain the same.

Now we are heading for an input-output realization of (4.1). Recall that in step 2 of the linear realization procedure we had to compute the minimal conditioned invariant subspace containing the directions corresponding to the driving variables. In the nonlinear case, conditioned invariance is defined as follows (see [8], [12]):

**Definition 4.4.** Let

$$\dot{x} = g_0(x) + \sum_{j=1}^m u_j g_j(x), \quad x \in M,$$

$$y_i = H_i(x), \quad i = 1, \ldots, p,$$

(4.34)

be a nonlinear system. An involutive distribution $S$ on $M$ is conditioned invariant if

$$\left[ g_j, S \cap \left( \bigcap_{i=1}^p \ker dH_i \right) \right] \subseteq S, \quad j = 0, 1, \ldots, m.$$  

(4.35)
Of special importance for us are conditioned invariant distributions containing the input vector fields \( g_j, j = 1, \ldots, m \), i.e.,

\[
\left[ g_0, S \cap \left( \bigcap_{i=1}^p \ker dH_i \right) \right] \subseteq S, \tag{4.36a}
\]

\[
g_j \in S, \quad j = 1, \ldots, m. \tag{4.36b}
\]

If \( S \) is a regular distribution (i.e., involutive and of constant dimension), then, equivalently, we can switch over to the codistribution level. Let \( P \) be a regular (i.e., involutive and of constant dimension) codistribution, then we define its annihilating (regular) distribution as

\[
\ker P(x) = \{ X(x) \in T_xM \mid \alpha(X(x)) = 0, \text{ for any one-form } \alpha \text{ in } P \}. \tag{4.37}
\]

Now let \( P \) be a regular codistribution such that \( \ker P = S \). Then it is proved by Corollary 4.3 of [8] that (4.35) is equivalent to

\[
L_{g_j} P \subseteq P + \text{span}\{dH_1, \ldots, dH_p\}, \quad j = 0, 1, \ldots, m, \tag{4.38}
\]

and (4.36) is therefore equivalent to

\[
L_{g_j} P \subseteq P + \text{span}\{dH_1, \ldots, dH_p\}, \quad j = 0, 1, \ldots, m. \tag{4.39a}
\]

\[
g_j \in \ker P, \quad j = 1, \ldots, m. \tag{4.39b}
\]

The following theorem is a global version of Proposition 3 in [16].

**Theorem 4.5.** Let (4.34) be a nonlinear system. Let \( P \) be a regular codistribution satisfying (4.39). Suppose that the regular distribution \( \ker P \) can be globally factored out, i.e., there exists a manifold \( \tilde{M} \) and a surjective submersion \( \pi: M \to \tilde{M} \) such that \( \ker \pi_* = \ker P \). Furthermore, assume that \( \tilde{M} \) is simply connected. Then (4.34) projects under \( \pi \) to a nonlinear system on \( \tilde{M} \), driven by the outputs \( y = (y_1, \ldots, y_p) \):

\[
\dot{x} = \tilde{g}_0(\tilde{x}, y), \quad \tilde{x} \in \tilde{M}, \tag{4.40}
\]

i.e.,

\[
\pi_* g_0(\pi(x)) = \tilde{g}_0(\pi(x), H_1(x), \ldots, H_p(x)), \tag{4.41a}
\]

\[
\pi_* g_j = 0, \quad j = 1, \ldots, m. \tag{4.41b}
\]

**Proof.** Equation (4.41b) immediately follows from (4.39b). In a coordinate neighborhood \( U \) for \( M \) we can write \( \pi = (\pi_1, \ldots, \pi_k) \). It is clear that on \( \pi^{-1}(U) \) \( P = \text{span}\{d\pi_1, \ldots, d\pi_k\} \), with \( k = \dim P \). Then (4.39a) yields

\[
dL_{g_j} \pi_i = L_{g_j} d\pi_i \in \text{span}\{d\pi_1, \ldots, \pi_k\} + \text{span}\{dH_1, \ldots, dH_p\},
\]

\[
i = 1, \ldots, k. \tag{4.42}
\]

Define the map \( F: M \to \tilde{M} \times \mathbb{R}^p \) by \( F(x) = (\pi(x), H_1(x), \ldots, H_p(x)) \). Then by (4.42) there exist one-forms \( \alpha_i \) on \( U \times \mathbb{R}^p \) such that

\[
dL_{g_j} \pi_i = F^* \alpha_i, \quad i = 1, \ldots, k. \tag{4.43}
\]

Since \( F^* d\alpha_i = dF^* \alpha_i = d(dL_{g_0} \pi_i) = 0 \) there exist functions \( \tilde{g}_0^i, i = 1, \ldots, k, \) on \( U \times \mathbb{R}^p \) such that \( \alpha_i = d\tilde{g}_0^i \). Since \( \pi \) is a submersion the functions \( \pi_1, \ldots, \pi_k \) are
a partial coordinate system on \( M \). Recalling that \( L_{\omega_i} \pi_i = d\pi_i(\omega_0) \) is just the component of the vector field \( \omega_0 \) corresponding to the \( i \)th coordinate function \( \pi_i \), it follows that on \( U \times \mathbb{R}^p \)
\[
\pi_* \omega_0(\pi(x)) = \tilde{\omega}_0(\pi(x), H_1(x), \ldots, H_p(x)),
\]
with \( \tilde{\omega}_0(x, y) = \text{col}(\omega_0^1(x, y), \ldots, \omega_0^p(x, y)) \). Since \( \tilde{M} \) is simply connected the definition of \( \tilde{\omega}_0 \) extends to a global map \( \tilde{\omega}_0 : \tilde{M} \times \mathbb{R}^p \rightarrow T\tilde{M} \) satisfying (4.41a).

Roughly speaking, the above theorem shows that the dynamics of the part of the state corresponding to a conditioned invariant codistribution \( P \) for which \( g_j \in \ker P \), is only driven by the outputs \( H_1(x), \ldots, H_p(x) \). We need one easy corollary.

**Corollary 4.6.** Let (4.34) be a nonlinear system and let \( P \) be a codistribution as in Theorem 4.5. Suppose that the functions \( H_1, \ldots, H_p \) (possibly after a permutation) are such that for a certain \( s \):
\[
P + \text{span}\{dH_1, \ldots, dH_p\} = P + \text{span}\{dH_{s+1}, \ldots, dH_p\}.
\]
Correspondingly, denote the first \( s \) components of \( y \) by \( y^1 \), and the last \( p-s \) components by \( y^2 \). Then the system projects under \( \pi \) to a system of the form
\[
\begin{align*}
\dot{x} &= \tilde{\omega}_0(x, y^2), \\
y^1 &= h(x, y^2).
\end{align*}
\]

**Proof.** Equation (4.46a) immediately follows from the proof of Theorem 4.5. By (4.45) we have
\[
\text{span}\{dH_1, \ldots, dH_p\} \subset P + \text{span}\{dH_{s+1}, \ldots, dH_p\}
\]
from (4.46b) follows. 

Let us return to Definition 4.4 and assume that \( S_1 \) and \( S_2 \) are two involutive conditioned invariant distributions, i.e., satisfying (4.35). It immediately follows that the distribution \( S_1 \cap S_2 \) also satisfies (4.35) and so is conditioned invariant. Hence, by an application of Zorn's lemma there always exists a *minimal* conditioned invariant distribution containing a given distribution, denoted as \( S^* \). This \( S^* \) can be *computed* using the \( S^* \)-algorithm, as follows. Consider the nonlinear system (4.34). Define the distribution \( \Delta_0 \) as \( \Delta_0(x) = \text{span}\{g_1(x), \ldots, g_m(x)\} \). Then define the increasing sequence of distributions (see [8])
\[
\begin{align*}
S_1 &= \Delta_0, \\
S_2 &= \bar{S}_1 + \left[ g_0, \bar{S}_1 \cap \left( \bigcap_{i=1}^p \ker dH_i \right) \right], \\
\vdots \\
S_k &= \bar{S}_{k-1} + \left[ g_0, \bar{S}_{k-1} \cap \left( \bigcap_{i=1}^p \ker dH_i \right) \right],
\end{align*}
\]
where \( \bar{S} \) denotes the involutive closure of a distribution \( S \). If we *assume* that all the distributions \( S_k \) have constant dimension then by the finite-dimensionality
of $M$ the algorithm terminates in a finite number of steps, i.e., there exists a $k < \dim M$ such that

$$S_{k+1} = S_k \quad \text{and} \quad S_{k+1} = S_k, \quad \forall l \geq 1.$$  

(4.49)

Denote $S_{k+1} = S_k = S^*$ then it follows that $S^*$ is a regular distribution and is the minimal conditioned invariant distribution containing $\Delta_0$. (Of course, by replacing $\Delta_0$ with an arbitrary distribution $D$ we can compute the minimal conditioned invariant distribution containing $D$ in the same way.) Define the regular codistribution $P^*$ by $\ker P^* = S^*$, then it immediately follows that $P^*$ is the maximal condition invariant codistribution such that $\Delta_0 \subseteq \ker P^*$. Assume now that $P^*$ also satisfies the additional technical assumptions of Theorem 4.5, then it follows that under $\pi: M \rightarrow \bar{M}$, with $\ker \pi_* = \ker P^*$, the nonlinear system (4.34) projects to

$$\dot{x} = \tilde{g}_0(\bar{x}, y), \quad \bar{x} \in \bar{M},$$  

(4.50)

and that $\bar{x}$ is the maximal part of the state of (4.34) which is driven only by the outputs.

Let us now try, as in the linear case, to apply all this to the driven state space system (4.33) which is a driven realization of the external system (4.1). In order to do this we shall, throughout, make the following additional assumptions:

**Assumption 5.** In the $S^*$-algorithm for (4.33) the distributions $S_k$ and $\bar{S}_k \cap (\bigcap_{i=1}^p \ker dH_i)$ all have constant dimension. As a consequence $S^*$ is a regular distribution.

**Assumption 6.** Let $P^*$ be the regular codistribution on $M^1$ such that $\ker P^* = S^*$. Assume $P^*$ satisfies the assumptions of Theorem 4.5, i.e., there exists a surjective submersion $\pi: M^1 \rightarrow M^2$ such that $\ker \pi_* = \ker P^* S^*$, with $M^2$ a simply connected manifold.

**Assumption 7.** By Assumption 5 $P^* + \text{span}\{dG_1^1, \ldots, dG_q^1\}$ has constant dimension $k$. Furthermore, assume that the functions $G_1^1, \ldots, G^1_q$ are independent and can be permuted in such a way that

$$p^* + \text{span}\{dG_1^1, \ldots, dG_q^1\} = P^* + \text{span}\{dG_{s+1}^1, \ldots, dG_q^1\},$$  

(4.51)

where $(q-s) + \dim P^* = k$.

Under these assumptions it follows from Theorem 4.5 and Corollary 4.6 that we obtain from the driven state space system (4.33), i.e.,

$$\begin{align*}
\dot{x}^1 &= g_0^1(x^1) + \sum_{j=1}^m v_j g_j^1(x^1), \quad x^1 \in M^1, \\
\dot{w}_j &= G_j^1(x^1), \quad j = 1, \ldots, q,
\end{align*}$$  

(4.52)

an input-state-output system living on $M^2$ of the form

$$\begin{align*}
\dot{x}^2 &= f(x^2, u), \quad u \in \mathbb{R}^{q-s}, \\
y &= h(x^2, u), \quad y \in \mathbb{R}^q, \\
w &= \text{col}(y, u).
\end{align*}$$  

(4.53)
As in the linear case we would like to conclude that (4.53) is an input–output realization of the external system (4.1). However, this is not generally true. The problem is that the time-functions \( u(t) \), i.e., the last \( q-s \) components of \( w(t) \), are in general not arbitrary smooth functions, as is needed for (4.53) to be an input–output realization. This can already be seen in the following simple example of a driven state space system:

\[
\begin{align*}
\dot{x}_1 &= v_1, \\
\dot{x}_2 &= v_2, \\
\dot{x}_3 &= x_1 v_2, \\
w_i &= x_i, & i = 1, 2, 3.
\end{align*}
\]

Here \( \Delta_0 = \text{span}\{\partial/\partial x_1, \partial/\partial x_2 + x_1(\partial/\partial x_3)\} \). Since \[ \partial/\partial x_1, \partial/\partial x_2 + x_1(\partial/\partial x_3) \] we have \( \Delta_0 = T\mathbb{R}^3 = S^* \). Factoring out by \( S^* \) we obtain an input-state-output system without state, where all the components of \( w \) are declared to be inputs, i.e.,

\[
w_i = u_i, & i = 1, 2, 3. \quad (4.55)
\]

However, it is clear from (4.54) that the mapping from \( \mathbb{R} \) to \( \mathbb{R}^3 \) given by \( t \rightarrow \text{col}(w_1(t), w_2(t), w_3(t)) \) is not an arbitrary smooth map. (As a matter of fact, the components \( w_i \) satisfy the differential relation \( \dot{w}_3 = \dot{w}_1 \dot{w}_2 + w_1 w_2 \).) Therefore the external behavior of the input–state–output system (4.5)) is larger than the external behavior of the driven state space system (4.54). Hence, in general, we only have:

**Proposition 4.7.** The external behavior of the input–state–output system (4.53) contains the external behavior of the driven state space system (4.52) and hence contains the external behavior of the external system (4.1).

Now we wish to give a set of sufficient conditions in order that the external behavior of (4.53) equals the external behavior of (4.52), and so (4.52) is an input–output realization. Recall that in the linear case no extra conditions are needed. Notice also that the problems in the foregoing example (4.54) are due to the noninvolutivity of the distribution \( \Delta_0 \). This suggests the following set of sufficient conditions.

**Theorem 4.8.** Consider the driven state space system (4.52). Consider the \( S^* \)-algorithm for (4.52). Suppose that the distributions \( S_k \), \( k = 0, 1, 2, \ldots \), in this algorithm are all involutive. Then the input–state–output system (4.53) on \( M^2 \) (where \( \pi: M^1 \rightarrow M^2 \) is a surjective submersion such that \( \ker \pi_* = S^* \)) has the same external behavior as (4.52).

**Proof.** The proof is very similar to the proof of the feedback linearization theorem due to Jakubczyk and Respondek [11] and Hunt and Su [6]. For ease of notation let us denote

\[
\begin{align*}
A &:= g_0, \\
B_j &:= g_j, & j = 1, \ldots, m, \\
C_j &:= G_j, & j = 1, \ldots, q.
\end{align*}
\]
Define $N_1$ as the smallest integer $\geq 1$ such that
\[
\alpha_1 := \dim S_{N_1} - \dim \left( S_{N_1} \cap \bigcap_{j=1}^{q} \ker dC_j \right) > 0.
\]
Then let $N_2$ be the smallest integer $> N_1$ such that
\[
\alpha_2 := \dim S_{N_2} - \dim \left( S_{N_2} \cap \bigcap_{j=1}^{q} \ker dC_j \right) > \alpha_1.
\]
Inductively let, for $i > 1$, $N_i$ be the smallest integer $> N_{i-1}$ such that
\[
\alpha_i := \dim S_{N_i} - \dim \left( S_{N_i} \cap \bigcap_{j=1}^{q} \ker dC_j \right) > \alpha_{i-1}.
\]
The distributions $S_i$ in the increasing sequence
\[
S_1 \subset \cdots \subset S_{N_1} \subset \cdots \subset S_{N_2} \subset \cdots \subset S_{N_k} = S^\ast
\]
are by assumption all involutive and of constant dimension. Hence, by a general-
ized Frobenius' theorem [11, Lemma 1] there exist local coordinates $x = (x_1, \ldots, x_n)$ for $M^1$ such that the integral manifold of $S_i$ are of the form
\[
x_j = C_j, \quad j = \mu_i + 1, \ldots, n, \quad C_j \text{ constant} \quad (\mu_i = \dim S_i).
\]
Denote $x = (\bar{x}^1, \ldots, \bar{x}^N_i, \bar{x})$, where $\bar{x}^1$ consists of the first $P_1 = \mu_1$ coordinates of $x$, $\bar{x}^2$ consists of the next $P_2 = \mu_2 - \mu_1$ coordinates, $\ldots$, $\bar{x}^N_i$ consists of the last $P_{N_i} = \mu_{N_i} - \mu_{N_i-1}$ coordinates of $(\bar{x}^1, \ldots, \bar{x}^N_i)$. Also, the distributions $S_{N_1} \cap (\bigcap_{j=1}^{q} \ker dC_j)$ for $i = 1, \ldots, k$ are involutive and of constant dimension. Hence the coordinates $\bar{x}^{N_i}$, $i = 1, \ldots, k$, can be chosen as $\bar{x}^{N_i} = (C^{N_i}, \bar{x}^{N_i})$ in such a way that the integral manifolds of $S_{N_i} \cap (\bigcap_{j=1}^{q} \ker dC_j)$ are of the form
\[
x_j = C_j, \quad j = \mu_i + 1, \ldots, n, \quad C_j \text{ constants},
\]
\[
\bar{x}^{N_i} = C_i, \quad j = 1, \ldots, r.
\]
From the definition of $N_1, \ldots, N_k$ it then follows that for $N_i \leq i < N_s$ ($1 \leq r < s \leq k$) the integral manifolds of $S_i \cap (\bigcap_{j=1}^{q} \ker dC_j)$ are of the form
\[
x_j = C_j, \quad j = \mu_i + 1, \ldots, n,
\]
\[
\bar{x}^{N_i} = C_i, \quad j = 1, \ldots, r.
\]
Write $A = (A^1, \ldots, A^k, \bar{A})$ corresponding to the splitting $x = (\bar{x}^1, \ldots, \bar{x}^N_i, \bar{x})$. To simplify notation define for $j \neq N_i$, $i = 1, \ldots, k$,
\[
x^j := \bar{x}^j.
\]
As in the feedback linearization theorem [11, Theorem 1], the condition
\[
S_{i+1} = S_i + \left[ A, S_i \cap \left( \bigcap_{j=1}^{q} \ker dC_j \right) \right], \quad i = 1, \ldots, N_k,
\]
implies that:
\begin{itemize}
  \item[(a)] $f^j$ does not depend on $x^1, \ldots, x^{j-1}, j = 3, 4, \ldots$
  \item[(b)] $\text{rank}(\partial f^j/\partial x^{j-1}) = p_j, j = 2, 3, \ldots$
\end{itemize}
The difference with the feedback linearization case lies in the following:

(c) \( f^j \) may depend on \( \tilde{x}^{N_1}, \ldots, \tilde{x}^{N_k}, \ j = 1, \ldots, N_k. \)

Notice furthermore that:

(d) \( \text{span}\{\partial/\partial \tilde{x}^{N_1}, \ldots, \partial/\partial \tilde{x}^{N_k}\} \cap (\bigcap_{j=1}^{q} \ker dC_j) = 0. \)

Now let us finish the proof of the theorem.

By definition \( S^1 = \text{span}\{B_1, \ldots, B_m\}. \) Hence the time derivative \( \dot{x}^1 \) can be made into an arbitrary time-function by proper choice of the (arbitrary) driving variables. Hence the functions \( x^1 \) can be made arbitrary. Since by (b) \( \text{rank}(\partial f^2/\partial x^1) = p_2 \) it follows that the time-derivative \( \dot{x}^2 \) and hence \( \ddot{x} \) can be made arbitrary. Continuing, since by (b) \( \text{rank}(\partial f^3/\partial x^2) = p_3 \) and by (a) \( f^3 \) does not depend on \( x^1 \), we obtain that \( \dddot{x} \) and therefore \( \ddddot{x} \) can be made arbitrary time-functions. By induction using (a) and (b) it follows that \( \dddot{x}^{N_1} \) can be made arbitrary. Since \( \tilde{x}^{N_1} = (x^{N_1}, \tilde{x}^{N_1}) \) the time-functions \( \tilde{x}^{N_1} \) can be made arbitrary. We continue the process with the arbitrary time-functions \( x^{N_1}. \) Using (a) and (b) for \( j = N_1 + 1 \) it follows that the time-functions \( x^{N_1+1} \) can be made arbitrary. Again by induction \( x^{N_2} \) are arbitrary time-functions. Write \( \tilde{x}^{N_2} = (x^{N_2}, \tilde{x}^{N_2}) \) and continue again with \( x^{N_2} \) till \( \tilde{x}^{N_1} \) is reached. By Assumption 7 and (d) it follows that the functions \( C_1, \ldots, C_q \) can be permuted in such a way that (with \( \ker P^* = S^* \))

\[
P^* + \text{span}\{dC_1, \ldots, dC_q\} = P^* + \text{span}\{dC_{s+1}, \ldots, dC_q\},
\]

where \( (q - s) + \dim P^* = \dim(P^* + \text{span}\{dC_1, \ldots, dC_q\}) \), and the Jacobian matrix

\[
(\partial C_i/\partial \tilde{x}^{N_j}), \quad i = s + 1, \ldots, q, \quad j = 1, \ldots, k,
\]

is invertible. Since the time-functions \( \tilde{x}^{N_1}, \ldots, \tilde{x}^{N_k} \) can all be made arbitrary, it follows that the last \( (q - s) \) components of \( w \in \mathbb{R}^q \) are arbitrary time-functions. The theorem now follows from Corollary 4.6. \( \square \)

Now let us assume that we have obtained an input–output realization (4.53)
of the external system (4.1). As in the third step of the linear realization procedure we now wish to obtain a minimal input–output realization. This can be done in the following way. Consider the extended system [14] of (4.53):

\[
\begin{align*}
\dot{x} &= f(x^2, u) \\
\dot{u} &= v
\end{align*}
\]

where \( v \) are the new inputs (i.e., we have added \( (q - s) \) integrators to the system and \( u \) has become part of the state). Take as the output mapping the mapping from the extended state \( (x^2, u) \) to the whole vector of external variables

\[
w = \text{col}(h(x^2, u), u) =: C(x^2, u).
\]
Define for simplicity the following vector fields on $M^2 \times \mathbb{R}^{q-s}$:

$$A(x^2, u) = f(x^2, u) \frac{\partial}{\partial x},$$

(4.57)

$$B_j(x^2, u) = \frac{\partial}{\partial u_j}, \quad j = 1, \ldots, q-s.$$  

Furthermore, define the distribution $B$ and codistribution $B^\perp$ as

$$B(x^2, u) = \text{span} \left\{ \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{q-s}} \right\},$$

$$\ker B^\perp = B.$$  

(4.58)

Now set up the controlled invariant codistribution algorithm (introduced in [8] and [7]) for (4.56), i.e., define the increasing sequence of codistributions

$$\Omega_0 = dC,$$

(4.59)

$$\Omega_k = \Omega_{k-1} + L_A(B^\perp \cap \Omega_{k-1}) + \sum_{j=1}^{q-s} L_{B_j}(B^\perp \cap \Omega_{k-1}), \quad k = 1, 2, \ldots.$$  

with $C = (C_1, \ldots, C_q)$ and $dC$ the codistribution $\text{span}\{dC_i, i = 1, \ldots, q\}$. We shall make the following assumption throughout.

**Assumption 8.** The dimensions of the codistributions $\Omega_k$ and $B^\perp \cap \Omega_k$, $k = 0, 1, \ldots$, are constant.

By the finite-dimensionality of $M^2 \times \mathbb{R}^{q-s}$ it follows that algorithm (4.59) terminates in a finite number of steps, i.e., there exists a $k < \dim(M^2 \times \mathbb{R}^{q-s})$ such that

$$\Omega_{k+1} = \Omega_k \quad \text{and} \quad \Omega_{k+l} = \Omega_k, \quad l = 2, 3, \ldots.$$  

(4.60)

Denote $\Omega^* = \Omega_k$. It follows that $\Omega^*$ is the **minimal locally controlled invariant** codistribution containing the codistribution $dC$ (see [8], [7]). Recall that a codistribution $\Omega$ on $M^2 \times \mathbb{R}^{q-s}$ is locally controlled invariant for (4.56) if

$$L_A(B^\perp \cap \Omega) \subseteq \Omega,$$

(4.61a)

$$L_{B_j}(B^\perp \cap \Omega) \subseteq \Omega, \quad j = 1, \ldots, q-s.$$  

(4.61b)

It follows from (4.61b) (see [14]) that the distribution $\ker \Omega^*$ on $M^2 \times \mathbb{R}^{q-s}$ projects (under the natural projection from $M^2 \times \mathbb{R}^{q-s}$ to $M^2$) to a distribution $\Delta^*$ on $M^2$. Furthermore, since $\Omega$ is regular and $B^\perp \cap \Omega^*$ has constant dimension, $\Delta^*$ is a regular distribution. Hence $M^2$ can be (at least) **locally** factored out by $\Delta^*$ to obtain a smaller state space manifold $M^3$. In order to do this globally, we need:

**Assumption 9.** There exists a manifold $M^3$ and a surjective submersion $\pi: M^2 \to M^3$ such that $\ker \pi_* = \Delta^*$. 

Under these assumptions the original input-state-output system (4.53) living on $M^2$ projects (under $\pi$) to an input-state-output system living on $M^3$,
\[
\begin{align*}
\dot{x}^3 &= \bar{f}(x^3, u), \quad u \in \mathbb{R}^{q-r} \\
y &= \bar{h}(x^3, u), \quad y \in \mathbb{R}^r
\end{align*}
\]
with the same external behavior. Furthermore, (4.62) is minimal in the sense defined in [14] and [15]. In particular, if $\bar{h}$ in (4.62) does not depend on $u$ then (4.62) is locally weakly observable [15].

In case we do not have an input-output realization (4.53) of the external system (4.1), but only the driven state space realization (4.52) = (4.33) we can apply the same reduction procedure as above to the driven state space system. That is, we have to compute the minimal locally controlled invariant distribution $\Omega^*$ on $M^1$ for (4.52), i.e., the minimal codistribution such that
\[
\begin{align*}
L^1_{\kappa_0}(\Omega^* \cap g^\perp) &\subseteq \Omega^*, \\
L^1_{g_j}(\Omega^* \cap g^\perp) &\subseteq \Omega^*, \quad j = 1, \ldots, m, \\
\text{span}\{dG^1_1, \ldots, dG^1_m\} &\subseteq \Omega^*,
\end{align*}
\]
where $g^\perp$ is the codistribution defined by
\[
\ker g^\perp = \text{span}\{g^1_1, \ldots, g^1_m\}. \quad (4.64)
\]
As a matter of fact, even if (4.52) admits an input-output realization (4.53), it is still advisable first to reduce (4.52) in the above way, and then to convert this reduced-order driven state space system to an input-state-output system.

Remark. In the linear case it can be proved (see [19]) that in this case the resulting input-state-output system is automatically minimal.

We remark that in the procedure for obtaining an input-output realization (4.53) from a driven state space realization (4.52), so far we have only considered permutations in the $w$-vector such that $w = \text{col}(y, u)$. It is clear that if we allow for more general transformations on $\mathbb{R}^q$, the space of external variables, then we have more freedom in the selection of inputs and outputs. Assumptions 7, especially, can obviously be relaxed. In the linear case it is easily proven that with general (nonsingular) transformations on $\mathbb{R}^q$ we can always obtain an input-output realization without the feedthrough term $D$ [22-24].

Finally, let us illustrate the realization procedure given in this section by the following example, considered in a related context by Freedman and Willems [4]. Let $Y = U = \mathbb{R}^m$, and let $W = Y \times U$ with coordinates $w = (y, u)$. Consider the external system (in input-output form (1.6))
\[
R_i(y, u, \dot{y}, \dot{u}) := \dot{y}_i - a_i(y, u, \dot{u}) = 0, \quad i = 1, \ldots, m, \quad (4.65)
\]
for certain smooth functions $a_i$. 
Denote $a(y, u, \dot{u}) = \text{col}(a_1(y, u, \dot{u}), \ldots, a_m(y, u, \dot{u}))$. In the first step of the realization procedure we consider the driven state space system

$$
\begin{align*}
\frac{d}{dt} & \begin{pmatrix} y \\ u \\ \dot{y} \\ \dot{u} \end{pmatrix} = \\
& \begin{pmatrix} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ u \\ \dot{y} \\ \dot{u} \end{pmatrix} + \\
& \begin{pmatrix} 0 \\ 0 \\ I_m \\ 0 \end{pmatrix} \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix},
\end{align*}
$$

\begin{equation}
(4.66)
\end{equation}

$$
z = \dot{y} - a(y, u, \dot{u}).
$$

The characteristic numbers $\rho_i, i = 1, \ldots, m$, are all zero and the $A$-matrix is given as

$$
A(y, u, \dot{y}, \dot{u}) = \left[ I_m \quad -\frac{\partial a}{\partial \dot{u}}(y, u, \dot{u}) \right].
$$

\begin{equation}
(4.67)
\end{equation}

Clearly, rank $A = m$ and so Assumption 4 is satisfied. By Theorems 4.2 and 4.3 the maximal controlled invariant submanifold contained in $z = 0$ is given as

$$
N^* = \{(y, u, \dot{y}, \dot{u}) \mid \dot{y} - a(y, u, \dot{u}) = 0\}
$$

and the driven state space system on $M^1 := N^*$

$$
\begin{align*}
\frac{d}{dt} & \begin{pmatrix} y \\ u \\ \dot{u} \end{pmatrix} = \\
& \begin{pmatrix} a(y, u, \dot{y}) \\ \dot{u} \\ 0 \end{pmatrix} + \\
& \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{align*}
$$

\begin{equation}
(4.68)
\end{equation}

$w = (y, u)$

(with $v = \bar{v}_2$) is a driven state space realization of (4.65).

In the second step of the realization procedure we have to consider the $S^*$-algorithm corresponding to (4.68). First, $S_1 = \text{span}\{\partial/\partial u_1, \ldots, \partial/\partial u_m\}$ is clearly involutive and of constant dimension. $S_2$ is spanned by $S_1$ and all Lie brackets

$$
\left[ \sum_{i=1}^{m} \left( a_i(y, u, \dot{u}) \frac{\partial}{\partial y_i} + u_i \frac{\partial}{\partial u_i} \right), \frac{\partial}{\partial y_j} \right] = -\sum_{i=1}^{m} \frac{\partial a_i}{\partial u_j}(y, u, \dot{u}) \frac{\partial}{\partial y_i} - \frac{\partial}{\partial u_j}.
$$

\begin{equation}
(4.69)
\end{equation}

Clearly, $S_2$ has constant dimension. Consider the Lie bracket of the following vector fields in $S_2$:

$$
\left[ -\sum_{i=1}^{m} \frac{\partial a_i}{\partial u_j}(y, u, \dot{u}) \frac{\partial}{\partial y_i}, \frac{\partial}{\partial u_j} \right] = \sum_{i=1}^{m} \frac{\partial^2 a_i}{\partial u_j \partial u_i}(y, u, \dot{u}) \frac{\partial}{\partial y_i}.
$$

\begin{equation}
(4.70)
\end{equation}

If $\partial^2 a_i/(\partial u_i \partial u_j) \neq 0$, then (4.70) is not contained in $S_2$. Hence $S_2$ is involutive only if the functions $a_i(y, u, \dot{u})$ are all affine in $\dot{u}$, i.e., of the form

$$
a(y, u, \dot{u}) = b(y, u) + \sum_{j=1}^{m} c_j(y, u) \dot{u}_j
$$

\begin{equation}
(4.71)
\end{equation}
Realization of Nonlinear Systems

for certain smooth mappings \( b, c_1, \ldots, c_m : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \). Then by (4.69)

\[
S_2 = \text{span} \left\{ \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}, \frac{\partial}{\partial u_1} + \sum_{i=1}^{m} c_i'(y, u) \frac{\partial}{\partial y_i}, \ldots, \frac{\partial}{\partial u_m} + \sum_{i=1}^{m} c_m'(y, u) \frac{\partial}{\partial y_i} \right\}
\]

(4.72)

with \( c_j(y, u) = \text{col}(c_j^1(y, u), \ldots, c_j^m(y, u)) \), \( j = 1, \ldots, m \).

Now consider the Lie bracket

\[
[ \frac{\partial}{\partial u_j} + \sum_{i=1}^{m} c_i'(y, u) \frac{\partial}{\partial y_i}^*, \frac{\partial}{\partial u_l} + \sum_{i=1}^{m} c_i'(y, u) \frac{\partial}{\partial y_i} ] = \sum_{i=1}^{m} \left( \frac{\partial c_i'}{\partial u_l} \frac{\partial c_i'}{\partial u_l}^* + \sum_{r=1}^{m} \left( \frac{\partial c_i'}{\partial y_r} - \frac{\partial c_i'}{\partial y_r} \right) \right) \frac{\partial}{\partial y_i}.
\]

(4.73)

It follows that \( S_2 \) is involutive if and only if all expressions (4.73) are zero. Then \( S_2 = S^* \), and so the conditions of Theorem 4.8 are satisfied. Hence there exists an input–output realization of (4.65). This realization is constructed by noting that the vanishing of (4.73) is the classical integrability condition for the existence of a smooth map \( k : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \) such that (see [20])

\[
\frac{\partial k}{\partial u_i}(x, u) = c_i(k(x, u), u), \quad i = 1, \ldots, m, \quad x \in \mathbb{R}^m,
\]

(4.74)

\((x, u) \mapsto (k(x, u), u)\) is a diffeomorphism.

Define now (see [15])

\[
l(x, u) := \left( \frac{\partial k}{\partial x}(x, u) \right)^{-1} b(k(x, u), u),
\]

(4.75)

then the input–state–output system

\[
\dot{x} = l(x, u), \quad y = k(x, u)
\]

(4.76)

is an input–output realization of (4.65).

Furthermore, since the map \((x, u) \mapsto c(k(x, u), u)\) is a diffeomorphism, the minimal locally controlled invariant codistribution containing \( dC \) equals the whole cotangent space to \((x, u) \in \mathbb{R}^m \times \mathbb{R}^m\), and hence (4.76) is a minimal input–output realization.

We note that for this example it can even be proved that the involutivity conditions consisting of (4.71) and the vanishing of (4.73) are also necessary for the existence of an input–output realization; we refer to the proof given in [4].

5. External Systems with Internal Variables and Interconnections of Input–State–Output Systems

As already mentioned in Section 1, a more general form of the external systems

\[
R_i(w, \dot{w}, \ldots, w^{(k)}) = 0, \quad i = 1, \ldots, l, \quad w \in \mathbb{R}^q,
\]

(5.1)
we have considered so far, are external systems described by higher-order differential equations in the external variables \(w\) and some internal variables \(\xi\):

\[
Pt(w, \dot{w}, \ldots, w^{(k)}, \xi, \dot{\xi}, \ldots, \xi^{(k)}) = 0, \quad i = 1, \ldots, l, \quad w \in \mathbb{R}^q, \quad \xi \in \mathbb{R}^s.
\] (5.2)

In modeling systems one is often naturally led to such a form (e.g., (nonlinear) electrical networks with \(w\) the currents and voltages at the external ports and \(\xi\) some internal currents and voltages). Note that state space systems are also of this form. Hence the realization of (5.2) can be interpreted as a procedure to bring (5.2), by equivalence operations, acting on the class of systems (5.2), into a particular form of (5.2), namely a state space system. (This point of view was stressed in [19].)

Remark. To be precise, one should allow \(\xi\) in (5.2) to belong to an arbitrary \(s\)-dimensional manifold. This generalization leaves the results of this section invariant. (Analogously, \(w\) can be allowed to be an element of an arbitrary \(q\)-dimensional manifold, see [15].)

In the linear case, (5.2) takes the form

\[
P \left( \frac{d}{dt} \right) w(t) = Q \left( \frac{d}{dt} \right) \xi(t),
\] (5.3)

where \(P\) and \(Q\) are polynomial matrices. This clearly encompasses the form proposed earlier in [13] and [25]:

\[
P \left( \frac{d}{dt} \right) z(t) = Q \left( \frac{d}{dt} \right) u(t); \quad y(t) = R \left( \frac{d}{dt} \right) z(t) + S \left( \frac{d}{dt} \right) u(t)
\] (5.4)

with external variables \(w = \text{col}(y, u)\) and internal variables \(z\). A realization procedure for (5.3) has been given in [19], while one can also transform (by operations on \(P\) and \(Q\)) equations (5.3) into the form \(R(d/ dt)w(t) = 0\) (see [23], [24]) and then apply the realization procedures for this case [19], [23], [24].

Let us now consider the nonlinear case (5.2). Our goal will be to show how the realization procedure of Section 4 can be immediately generalized to this more general case. Denote \(z = \text{col}(w, \xi) \in \mathbb{R}^{q+s}\), and associate with (5.2) the driven state space system

\[
\frac{d}{dt}\begin{pmatrix}
z \\
\dot{z} \\
\vdots \\
\z^{(k)}
\end{pmatrix} = \begin{pmatrix} 0 & I_{q+s} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_{q+s} & \vdots \\ 0 & \cdots & \cdots & 0 \\
\end{pmatrix}
\begin{pmatrix}
z \\
\dot{z} \\
\vdots \\
\z^{(k)}
\end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}
\] (5.5a)

with output functions

\[
P_t(z, \dot{z}, \ldots, z^{(k)}), \quad i = 1, \ldots, l
\] (5.5b)
Assume, in analogy with Assumptions 3 and 4:

**Assumption 10.** The functions $P_i, i = 1, \ldots, l$, are not constant.

**Assumption 11.** For any $i = 1, \ldots, l$ let $\rho_i$ be the smallest integer such that for some $j \in \{1, \ldots, q + s\}$ and some $(z, \dot{z}, \ldots, z^{(k)}) \in \mathbb{R}^{(k+1)(q+s)}$

$$\frac{\partial P_i}{\partial z^{(k-\rho_i)}}(z, \dot{z}, \ldots, z^{(k)}) \neq 0. \quad (5.6)$$

Then the $l \times (q+s)$ matrix $A$ with $(i,j)$th element (5.6) has rank $l$.

Under these assumptions there exists, by Theorems 4.2 and 4.3, a maximal controlled invariant submanifold $N^*$ for (5.5a) contained in the set $P_1 = \cdots = P_l = 0$ and a driven state space system living on $N^*$ (see (4.33))

$$\dot{x}^1 = g_0^1(x^1) + \sum_{j=1}^{m} v_j g_j^1(x^1), \quad x^1 \in M^1 := N^*. \quad (5.7a)$$

Since $N^* \subseteq \mathbb{R}^{(k+1)(q+s)}$ the projection of $N^*$ onto the first $q$ components of $\mathbb{R}^{(k+1)(q+s)}$ is a smooth mapping. Hence we obtain smooth output functions

$$w_j = G_j^1(x^1), \quad j = 1, \ldots, q. \quad (5.7b)$$

As in Section 4, the driven state space system (5.7) is by construction a driven realization of the external system (5.2). In order to obtain a (minimal) input-output realization we have to proceed as in steps 2 and 3 of Section 4. We have therefore reduced the realization of external systems (5.2) to the realization of external systems (5.1).

Let us now apply the above theory to the problem of well-posedness of interconnections of systems. First let us consider a feedback connection. (This example was treated for the linear case in [19].) Consider two input-state-output systems

$$\begin{align*}
\dot{x}_i &= f_i(x_i, u_i), \\
y_i &= h_i(x_i, u_i),
\end{align*} \quad i = 1, 2. \quad (5.8)$$

The second system is placed in a feedback loop for the first system, and so we have the additional interconnection equations

$$\begin{align*}
u_2 &= y_1, \\
y &= y_1, \\
u_1 &= u + y_2.
\end{align*} \quad (5.9)$$

Clearly, (5.8) together with (5.9) defines an external system (5.2) with external variables $w = \text{col}(y, u) \in W = Y \times U$, and internal variables $\xi = \text{col}(x_1, x_2, y_1, y_2, u_1, u_2)$. Substitution of (5.9) into (5.8) yields the driven
state space realization

\[ \begin{align*}
  \dot{x}_1 &= f_1(x_1, u_1), \\
  \dot{x}_2 &= f_2(x_2, h_1(x_1, u_1)), \\
  y &= h_1(x_1, u_1), \\
  u &= u_1 - h_2(x_2, h_1(x_1, u_1))
\end{align*} \]  

(5.10)

with state \( \mathbf{col}(x_1, x_2) \) and driving variables \( u_1 \). (This same realization can also be formally obtained using the procedure given above.) Write

\[ F(x_1, x_2, u_1) = \mathbf{col}(h_1(x_1, u_1), u_1 - h_2(x_2, h_1(x_1, u_1))). \]  

(5.11)

It follows that the Jacobian matrix

\[ \frac{\partial F}{\partial u_1} = \begin{pmatrix} \frac{\partial h_1}{\partial u_1}(x_1, u_1) \\ 1 - \frac{\partial h_2}{\partial u_2}(x_2, h_1(x_1, u_1)) \frac{\partial h_1}{\partial u_1}(x_1, u_1) \end{pmatrix} \]  

(5.12)

is always injective. Hence, by the implicit function theorem we can (locally) define coordinates \( (y', u') \) for \( W \) in which coordinates \( F \) takes the form (see [14])

\[ F(x_1, x_2, u_1) = \left( \begin{array}{c} \bar{F}(x_1, x_2, u_1) = y' \\ u_1 = u' \end{array} \right). \]  

(5.13)

Hence we obtain (locally) the input-output realization

\[ \begin{align*}
  \dot{x}_1 &= f_1(x_1, u'), \\
  \dot{x}_2 &= f_2(x_2, h_1(x_1, u')), \\
  y' &= \bar{F}(x_1, x_2, u'),
\end{align*} \]  

(5.14)

which may already be minimal, or otherwise can be reduced to a minimal one as in Section 4. In (5.14) \( (y', u') \) are the new coordinates for \( W = Y \times U \) and so \( u' \) is a function of the old inputs \( u \) and the old outputs \( y \). It follows from (5.12) that \( u' \) can be taken \textit{equal} to \( u \) if and only if the matrix

\[ I - \frac{\partial h_2}{\partial u_2}(x_2, h_1(x_1, u_1)) \frac{\partial h_1}{\partial u_1}(x_1, u_1) \]  

(5.15)

is \textit{injective}. This condition, at least in the linear case, is usually imposed as a requisite for "well-posedness" of the connection. However, as was already noted in [19], the connection is \textit{always} well-posed in the sense of (5.14), and the extra condition (5.15) only ensures that we can take \( u \) as input in the input-output realization (5.14).

Finally, let us consider interconnections given by constraints on the outputs. (This kind of interconnection is often encountered in mechanics.) Let

\[ \begin{align*}
  x_i &= f_i(x_i, u_i), & y_i &\in \mathbb{R}^{p_i}, & u_i &\in \mathbb{R}^{m_i} \quad i = 1, 2, \\
  y_i &= h_i(x_i, u_i), & x_i &\in M^i, & \dim M^i &= n_i
\end{align*} \]  

(5.16a, b)
be two input-state-output systems, with additional constraints of the form

\[ H_j(y_1, y_2) = 0, \quad j = 1, \ldots, r. \]  

(Eq. 5.17)

Equations (5.16) together with (5.17) form an external system (5.2) with external variables \( w = \text{col}(y_1, y_2, u_1, u_2) \) and internal variables \( x_1 \) and \( x_2 \). Define the map \( H = \text{col}(H_1, \ldots, H_r) \). Then the matrix \( A \) as in (5.6) is given as (under the assumption that the functions \( H_i \) in (5.17) are not constant, see Assumption 3)

\[ A = \begin{pmatrix} 0 & 0 & 0 & 0 & -I_n_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_n_2 \\ -I_p_1 & 0 & \frac{\partial h_1}{\partial u_1} & 0 & \frac{\partial h_1}{\partial x_1} & 0 \\ 0 & -I_p_2 & 0 & \frac{\partial h_2}{\partial u_2} & 0 & \frac{\partial h_2}{\partial x_2} \\ \frac{\partial H}{\partial y_1} & \frac{\partial H}{\partial y_2} & 0 & 0 & 0 & 0 \end{pmatrix} (y_1, y_2, u_1, u_2, x_1, x_2). \]  

(Eq. 5.18)

It is clear that \( A \) is surjective if and only if

\[ \begin{pmatrix} -I_p_1 & 0 & \frac{\partial h_1}{\partial u_1} & 0 \\ 0 & -I_p_2 & 0 & \frac{\partial h_2}{\partial u_2} \\ \frac{\partial H}{\partial y_1} & \frac{\partial H}{\partial y_2} & 0 & 0 \end{pmatrix} \]  

is surjective. Some elementary matrix operations show that this is the case if and only if the matrix

\[ \begin{bmatrix} \frac{\partial H}{\partial y_1}(y_1, y_2) & \frac{\partial h_1}{\partial u_1}(x_1, u_1) & \frac{\partial h_1}{\partial x_1} \\ \frac{\partial h_2}{\partial u_2}(x_2, u_2) & \frac{\partial H}{\partial y_2}(y_1, y_2) \end{bmatrix} \]  

is surjective. This can be alternatively seen by substituting (5.16b) into (5.17) and defining

\[ G(x_1, x_2, u_1, u_2) = H(h_1(x_1, u_1), h_2(x_2, u_2)). \]  

(Eq. 5.21)

Then the external system (5.16) with (5.17) is equivalent to (5.16) together with \( G(x_1, x_2, u_1, u_2) = 0 \). The matrix \( A \) for this second description is equal to the matrix \( A \) of (5.18), except for its last block row which equals

\[ \begin{bmatrix} 0 & 0 & \frac{\partial G}{\partial u_1} & \frac{\partial G}{\partial u_2} & 0 & 0 \end{bmatrix}. \]  

(Eq. 5.22)

This matrix \( A \) is surjective if and only if the matrix

\[ \begin{bmatrix} \frac{\partial G}{\partial u_1} & \frac{\partial G}{\partial u_2} \end{bmatrix} \]  

is surjective. Clearly, (5.23) equals (5.20).
Concluding, if (5.20) (or (5.23)) is surjective then there exists a driven state space realization of the external system defined by the interconnection (5.17). Using the fact that the A-matrix only depends on \((x_1, x_2, u_1, u_2)\) it is easy to see that this driven state space realization satisfies the conditions of Theorem 4.8 and hence an input-output realization also exists.

6. Equivalence Transformations on External Systems

In this section we discuss the generality or restrictiveness of Assumption 4 as stated in Section 4. This assumption was needed in order to obtain a driven state space realization of an external system

\[
R_i(w, \dot{w}, \ldots, w^{(k)}) = 0, \quad i = 1, \ldots, l, \quad w \in \mathbb{R}^q.
\]

Recall that for any \(i\) the characteristic number \(\rho_i\) is the smallest integer for which there exists a \(j \in \{1, \ldots, q\}\) such that

\[
\frac{\partial R_i}{\partial w_j^{(k-\rho_i)}}(w, \dot{w}, \ldots, w^{(k)}) \neq 0
\]

for some \((w, \dot{w}, \ldots, w^{(k)}) \in \mathbb{R}^{(k+1)q}\). (By Assumption 3 the functions \(R_i\) are not constant, and so \(\rho_i \leq k, \quad i = 1, \ldots, l\).) In Assumption 4 it is assumed that the rank of the matrix \(A(w, \dot{w}, \ldots, w^{(k)})\) with the \((i, s)\)th element defined as

\[
\frac{\partial R_i}{\partial w_s^{(k-\rho_i)}}(w, \dot{w}, \ldots, w^{(k)})
\]

equals \(l\) for all \((w, \dot{w}, \ldots, w^{(k)})\).

Let us first investigate this assumption in the linear case (see Section 2)

\[
R \left( \frac{d}{dt} \right) w(t) = 0,
\]

where \(R(s)\) is an arbitrary \(l \times q\) polynomial matrix. It is clear that for \(i = 1, \ldots, l\), the number \((k - \rho_i)\) equals the degree of the \(i\)th row (i.e., the highest power of \(s\) appearing in the \(i\)th row) of \(R(s)\). Furthermore, we have

\[
R(s) = \text{diag}(s^{k-\rho_i}) A + L(s),
\]

\(^{o)}\) is the \(l \times l\) diagonal matrix with \(s^{(k-\rho_i)}\) on the \((i, i)\)th place, \(A \in q\) matrix, and \(L(s)\) is an \(l \times q\) polynomial matrix such that the \(i\)th row of \(L(s)\) is less than \(k - \rho_i\). Hence, in the linear case the \(A\) matrix (not depending on \((w, \dot{w}, \ldots, w^{(k)})\)) and Assumption 4.8 that rank \(A = l\). This last condition is actually well known in the polynomial matrices; \(R(s)\) is called a row proper polynomial matrix \(^{1}\). In the linear case (6.3) Assumption 4 is satisfied if and only if proper.

investigate the restrictiveness of the assumption of row-properness and the notion of equivalence transformation of a linear external
system (6.3) (see [19]). Recall that an \( l \times l \) polynomial matrix \( U(s) \) is called \textit{unimodular} if

\[
\det U(s) = \text{constant} \neq 0 \tag{6.5}
\]
or, equivalently, if \( U(s) \) has a polynomial inverse. Now consider instead of (6.3) the external system

\[
\bar{R} \left( \frac{d}{dt} \right) w(t) = 0, \tag{6.6}
\]

with \( \bar{R}(s) := U(s)R(s) \). We claim that the external behavior of (6.6) equals the external behavior of (6.3). Indeed, let \( w(t) \) satisfy (6.3). Denote the \( i \)th row of \( R(s) \) by

\[
R^i(s) = (R^i_1(s), \ldots, R^i_q(s)). \tag{6.7}
\]

Then \( w(t) \) satisfies the \( l \) higher-order equations

\[
R^i_1 \left( \frac{d}{dt} \right) w_1(t) + R^i_2 \left( \frac{d}{dt} \right) w_2(t) + \cdots + R^i_q \left( \frac{d}{dt} \right) w_q(t) = 0, \quad i = 1, \ldots, l. \tag{6.8}
\]

It follows that \( w(t) \) also satisfies the prolonged equations

\[
\frac{d^{\nu_i}}{dt^{\nu_i}} \left( R^i_1 \left( \frac{d}{dt} \right) w_1(t) + R^i_2 \left( \frac{d}{dt} \right) w_2(t) + \cdots + R^i_q \left( \frac{d}{dt} \right) w_q(t) \right) = 0 \tag{6.9}
\]

for any integer \( \nu_i \geq 0 \). Premultiplication of \( R(s) \) by a polynomial matrix \( U(s) \) means that the \( l \) equations given by \( \bar{R} \left( \frac{d}{dt} \right) w(t) = 0 \), with \( \bar{R}(s) = U(s)R(s) \), are just linear combinations of equations (6.8) and (6.9). Hence \( w(t) \) also satisfies \( \bar{R} \left( \frac{d}{dt} \right) w(t) = 0 \). Conversely, if \( w(t) \) satisfies \( \bar{R} \left( \frac{d}{dt} \right) w(t) = 0 \), then we prove that \( R \left( \frac{d}{dt} \right) w(t) = 0 \), since \( R(s) = U^{-1}(s) \bar{R}(s) \). (Here we use the fact that \( U(s) \) is unimodular.) A transformation from (6.3) to (6.6) leaving the external behaviour invariant is called an \textit{equivalence transformation}. Hence, to every \( l \times l \) unimodular matrix there corresponds an equivalence transformation. Conversely, it is easy to show that if (6.3) and (6.6) define the same external behavior, then there exists a unimodular matrix \( U(s) \) with \( \bar{R}(s) = U(s)R(s) \) (see [19]).

Returning now to the assumption of row-properness we have the following basic proposition (see, for instance, [25]). Since we wish to generalize this proposition to the nonlinear case we also include a full proof.

**Proposition 6.1.** Let \( R(s) \) be an \( l \times q \) polynomial matrix. Then there exists an \( l \times l \) unimodular matrix \( U(s) \) such that

\[
U(s)R(s) = \begin{pmatrix} \bar{R}(s) \\ 0 \end{pmatrix}, \tag{6.10}
\]

where \( \bar{R}(s) \) is an \( l' \times q \) row-proper matrix (with \( l' \leq l \)) and 0 denotes the \( (l - l') \times q \) zero matrix.
Proof. Denote the degree of the $i$th row $R(s)$ by $\sigma_i$. Write, as in (6.4),

$$R(s) = \text{diag}(s^{\sigma_i})A + L(s),$$

(6.11)

with the degree of the $i$th row of $L(s)$ strictly less than $\sigma_i$. If rank $A = l$ we take $U(s)$ to be the identity matrix and we are done. Suppose rank $A < l$. Denote the rows of $R(s)$ by $R_i(s)$ and the rows of $A$ by $A_i$, $i = 1, \ldots, l$. Then there exist nontrivial constants $\alpha_i$ such that

$$\sum_{i=1}^{l} \alpha_i A_i = 0.$$  

(6.12)

Now let $\sigma_j$ be the largest integer for which $\alpha_j$ in (6.12) is not zero ($\sigma_j$ need not be unique). Then the expression

$$\sum_{i=1}^{l} \alpha_i s^{\sigma_j - \sigma_i} R_i(s)$$

(6.13)

is a row vector of polynomials. Furthermore, by (6.12) the degree of this row vector is strictly less than $\sigma_j$. Define the matrix $U(s)$ as the $l \times l$ identity matrix with the $j$th row replaced by

$$(\alpha_1 s^{\sigma_j - \sigma_1}, \alpha_2 s^{\sigma_j - \sigma_2}, \ldots, \alpha_l s^{\sigma_j - \sigma_l}).$$

(6.14)

$U(s)$ is unimodular since $\alpha_j \neq 0$. It follows that the row degree of the $j$th row of $U(s)R(s)$ is strictly less than $\sigma_j$. If $U(s)R(s)$ is row proper we are done. Otherwise we repeat the above procedure for $U(s)R(s)$. Since in every step the degree of one row of the polynomial matrix $R(s)$ decreases, we obtain after a finite number of steps a row-proper matrix, or a row-proper matrix stacked with some zero rows as in (6.10). \qed

It immediately follows from this proposition that for a linear external system (6.3) there always exists an equivalence transformation (given by a unimodular matrix $U(s)$) which transforms (6.3) into

$$\tilde{R} \left( \frac{d}{dt} \right) w(t) = 0,$$

(6.15)

with $\tilde{R}(s)$ and $l' \times q$ row-proper matrix ($l' \leq l$). (Clearly, we may leave out the $(l - l')$ zero equations.) Hence Assumption 4 is no loss of generality in the linear case.

Let us now try to generalize this to nonlinear systems (6.1). First we shall single out a class of equivalence transformations, acting on the defining equations (6.1), which is big enough for our purposes. Denote the time-derivatives of $R_i(w, \dot{w}, \ldots, w^{(k)})$ up to an arbitrary order $\nu_i$ by

$$R_i^{(\nu_i)}(w, \dot{w}, \ldots, w^{(k)}) = \frac{d^{\nu_i} R_i}{dt^{\nu_i}}(w, \dot{w}, \ldots, w^{(k)}).$$

(6.16)

It is clear that if a time-function $w(t)$ satisfies, for a certain $j$, $R_j(w, \dot{w}, \ldots, w^{(k)}) = 0$, then $w(t)$ also satisfies, for any $\nu_j \geq 0$,

$$R_j^{(\nu_j)}(w, \dot{w}, \ldots, w^{(k)}) = 0.$$  

(6.17)
Now consider smooth functions $\phi: \mathbb{R}^l \to \mathbb{R}$ such that

$$\phi(0, 0, \ldots, 0) = 0, \quad (6.18a)$$

$$\phi(\cdot, 0, \ldots, 0): \mathbb{R} \to \mathbb{R} \text{ is a diffeomorphism.} \quad (6.18b)$$

Let $s \in \{1, \ldots, l\}$. Replace the set of equations (6.1) by a set of equations of the form

$$R_i(w, \dot{w}, \ldots, \dot{w}^{(k)}) = 0, \quad i = 1, \ldots, l, \quad i \neq s, \quad (6.19a)$$

$$\phi(R_s, R_1^{(v_1)}, \ldots, R_s^{(v_s-1)}, R_s^{(v_s-2)}, \ldots, R_1^{(v_1)})(w, \dot{w}, \ldots, \dot{w}^{(k)}) = 0. \quad (6.19b)$$

It immediately follows that the external behavior defined by (6.19) equals the external behaviour of (6.1). Indeed, let $w(t)$ satisfy (6.1). Then, trivially, $w(t)$ satisfies (6.19a). Furthermore, $w(t)$ satisfies (6.17) for any $j = 1, \ldots, l$ and $v_j \geq 0$. Hence, by (6.18a), $w(t)$ satisfies (6.19b). Conversely, let $w(t)$ satisfy (6.19). Then, by (6.19a) and (6.17) for $j \neq s$, we see that $w(t)$ satisfies

$$\phi(R_s(w, \dot{w}, \ldots, \dot{w}^{(k)}), 0, \ldots, 0) = 0. \quad (6.20)$$

By (6.18b) it follows that $R_s(w, \dot{w}, \ldots, \dot{w}^{(k)}) = 0$, and so $w(t)$ satisfies (6.1).

Concluding, any $\phi$ satisfying (6.18) defines an equivalence transformation from (6.1) to (6.19).

For technical reasons we actually need in the sequel a somewhat weaker concept of equivalence transformation, called a local equivalence transformation. These are obtained by considering smooth functions $\phi$ satisfying (6.18) which are only defined on an open neighborhood of $0 \in \mathbb{R}^l$, i.e., smooth functions $\phi: U \to \mathbb{R}$, with $U$ an open neighborhood of $0 \in \mathbb{R}^l$, such that

$$\phi(0, 0, \ldots, 0) = 0, \quad (6.21a)$$

$$\phi(\cdot, 0, \ldots, 0) \text{ is a local diffeomorphism around } 0 \in \mathbb{R}. \quad (6.21b)$$

Notice that by the inverse function theorem, (6.21b) can be replaced by

$$\frac{\partial \phi}{\partial y_1}(0, 0, \ldots, 0) \neq 0. \quad (6.22)$$

In order to generalize Proposition 6.1 using this class of local equivalence transformations, we need the following technical lemma.

**Lemma 6.2.** Let $f_1, \ldots, f_l$ be functions on a manifold $M$ such that the dimension of the codistribution $\text{span}\{df_1, \ldots, df_l\}$ is constant. Suppose there exist functions $\lambda_2, \ldots, \lambda_l$ on $M$ such that

$$df_1 = \lambda_2 df_2 + \cdots + \lambda_l df_l. \quad (6.23)$$

Then there exists a smooth function $\phi: U \to \mathbb{R}$, with $U \subset \mathbb{R}^l$ an open neighbourhood of $0 \in \mathbb{R}^l$, such that

$$d(\phi \circ (f_1, \ldots, f_l)) = 0. \quad (6.24)$$
\[ \phi(0, \ldots, 0) = 0, \quad (6.25a) \]
\[ \frac{\partial \phi}{\partial y_1}(0, 0, \ldots, 0) \neq 0. \quad (6.25b) \]

Proof. Define the map \( F: M \rightarrow \mathbb{R}^l \) by \( F = (f_1, \ldots, f_l) \). It follows that the distribution \( D = F_*(TM) \) on \( \mathbb{R}^l \) has constant dimension less than \( l \). Hence, by the Frobenius theorem, there exists a function \( \phi \) with a nonvanishing differential on a neighborhood \( U \) of \( 0 \in \mathbb{R}^l \) such that \( d\phi(X) = 0 \), for any vector field \( X \in D \). We may choose \( \phi \) to satisfy (6.25a). Furthermore, by (6.23) we may choose \( \phi \) also to satisfy (6.25b). It follows that

\[ 0 = F^*(d\phi) = d(F^*\phi) \quad (6.26) \]

and because \( F^*\phi = \phi \circ (f_1, \ldots, f_l) \) we obtain (6.24).

Now let us consider the case that for a given external system (6.1) the matrix \( A(w, \dot{w}, \ldots, w^{(k)}) \) does not have rank \( l \) everywhere, and so Assumption 4 is not satisfied. In order to deal with this case we add the following regularity:

**Assumption 12.** The rank of the matrix \( A(w, \dot{w}, \ldots, w^{(k)}) \) is constant.

Denote the rows of the matrix \( A \) by \( A_i(w, \ldots, w^{(k)}) \). Since \( \operatorname{rank} A(w, \dot{w}, \ldots, w^{(k)}) < l \), there exist nontrivial functions \( \alpha_i(w, \dot{w}, \ldots, w^{(k)}) \) such that

\[ \sum_{i=1}^l \alpha_i A_i = 0. \quad (6.27) \]

Now let \( \rho_j \) be the smallest integer for which the function \( \alpha_j \) in (6.27) is not identically zero (\( \rho_j \) need not be unique). The following additional regularity assumption is needed.

**Assumption 13.** The function \( \alpha_j \) is nowhere zero.

Then there exist functions \( \lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_l \) such that

\[ A_j = \lambda_1 A_1 + \cdots + \lambda_{j-1} A_{j-1} + \lambda_{j+1} A_{j+1} + \cdots + \lambda_l A_l. \quad (6.28) \]

Recall from the proof of Theorem 4.3 that for \( 0 \leq \nu_i \leq \rho_i, \ i = 1, \ldots, l \), we have

\[ R_i^{(\nu_i)} = \frac{d^{\nu_i}}{dt^{\nu_i}} R_i = L_{\nu_i}^\nu R_i, \quad (6.29) \]

where \( L_{\nu_i}^\nu R_i \) is of the form of (4.27), and that

\[ A_i = \left( \frac{\partial}{\partial w^{(k)}_1} L_{\nu_i}^\nu R_i, \ldots, \frac{\partial}{\partial w^{(k)}_q} L_{\nu_i}^\nu R_i \right). \quad (6.30) \]
Furthermore, a closer inspection shows that for \( i = 1, \ldots, l \)
\[
A_i = \left( \frac{\partial}{\partial w_i^{(k)}} L_{s_0}^{(\rho_i - \rho_j)} R_i, \ldots, \frac{\partial}{\partial w_q^{(k)}} L_{s_0}^{(\rho_i - \rho_j)} R_i \right) 
\] (6.31)
since \( \rho_i \geq \rho_j \) for \( i = 1, \ldots, l \). Now consider the function \( R_j \) together with the \( l - 1 \) functions
\[
L_{s_0}^{(\rho_i - \rho_j)} R_i, \quad i = 1, \ldots, j - 1, j + 1, \ldots, l, 
\] (6.32)
as functions of \( (w_1^{(k-J)}, \ldots, w_q^{(k-J)}) \). Application of Lemma 6.1 yields the existence of a smooth function \( \phi: U \subset \mathbb{R}^l \to \mathbb{R} \), with \( U \) an open neighborhood of \( 0 \in \mathbb{R}^l \), such that
\[
\phi(0, \ldots, 0) = 0, \quad \frac{\partial \phi}{\partial y_1}(0, 0, \ldots, 0) \neq 0, 
\] (6.33)
\[
\frac{\partial}{\partial w_i^{(k-J)}} \phi(R_j, L_{s_0}^{(\rho_i - \rho_j)} R_1, \ldots, L_{s_0}^{(\rho_i - \rho_j)} R_{j-1}, L_{s_0}^{(\rho_i - \rho_j)} R_{j+1}, \ldots, L_{s_0}^{(\rho_i)} R_l) 
\times (w, \dot{w}, \ldots, w^{(k)}) = 0, \quad s = 1, \ldots, q. 
\] (6.34)
As we saw above, \( \phi \) defines a local equivalence transformation from (6.1) to the external system obtained from (6.1) by replacing the equation
\[
R_i(w, \dot{w}, \ldots, w^{(k)}) = 0 \quad \text{in (6.1)} 
\] by
\[
\phi(R_j, L_{s_0}^{(\rho_i - \rho_j)} R_1, \ldots, L_{s_0}^{(\rho_i - \rho_j)} R_{j-1}, L_{s_0}^{(\rho_i - \rho_j)} R_{j+1}, \ldots, L_{s_0}^{(\rho_i)} R_l) 
\times (w, \dot{w}, \ldots, w^{(k)}) = 0. 
\] (6.35)
For this newly defined external system the characteristic numbers are the same, except for the \( j \)th one which is, by (6.34), strictly less than \( \rho_j \). If the \( A \)-matrix of the newly defined system equals \( 1 \) (and so satisfies Assumption 4), then we may replace (6.1) by this newly defined system (which is locally equivalent to it). If not, then we can repeat the above procedure (under Assumptions 12 and 13 for the newly defined system). Repeating this there are two possibilities:

1. After a finite number of steps we obtain an \( A \)-matrix which satisfies Assumption 4.

2. After a finite number of steps we obtain an \( A \)-matrix for which (possibly after a permutation of the rows) the first \( l' \) (\( l' < l \)) rows are independent while the last \( (l - l') \) rows are zero. Let the defining equations of this final external system be
\[
\tilde{R}_i(w, \dot{w}, \ldots, w^{(k)}) = 0, \quad i = 1, \ldots, l. 
\] (6.36)
Necessarily the characteristic numbers \( \rho_i, i = l - l' + 1, \ldots, l \), are equal to \( k \). Hence the last \( l - l' \) functions \( \tilde{R}_i, i = l - l' + 1, \ldots, l \), are constant functions.

Clearly, in case (1) we can transform (6.1) to an external system satisfying Assumption 4. In case (2) we obtain an external system satisfying Assumption 4:
\[
\tilde{R}_i(w, \dot{w}, \ldots, w^{(k)}) = 0, \quad i = 1, \ldots, l'. 
\] (6.37)
together with some equations

\[
\tilde{R}_i = 0, \quad i = l - l' + 1, \ldots, l,
\]  

(6.38)

where \(\tilde{R}_i\) are constants. If these constants are all zero we may leave out equations (6.38) and if not then equations (6.38) are inconsistent, and a solution does not exist. Summarizing, we have obtained

Theorem 6.3. Consider an external system (6.1). Under Assumptions 12 and 13 there exists a local equivalence transformation transforming (6.1) into an external system satisfying Assumption 4, possibly together with some equations (6.38) which are trivially satisfied or inconsistent.

Remark. Note that the external system (6.37) together with (6.38) does not satisfy Assumption 3.

Concluding, apart from the fact that the equations defining an external system may be inconsistent (take, for instance, the external system \(\dot{w} = 0, \dot{w} - 1 = 0, w \in \mathbb{R}\)), we have obtained (under the constant rank assumptions, Assumptions 12 and 13) a complete nonlinear analogue of Proposition 6.1. Notice, however, that we did not obtain a complete nonlinear generalization of the operation of premultiplication of a polynomial matrix \(R(s)\) with a unimodular matrix.

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References


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