Algebraic subgroups of $GL_2(\mathbb{C})$

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ABSTRACT

In this note we classify, up to conjugation, all algebraic subgroups of $GL_2(\mathbb{C})$.

1. INTRODUCTION

Although the classification, up to conjugation, of the algebraic subgroups of $SL_2(\mathbb{C})$ ([3, Theorem 4.12], [6, Theorem 4.29]), and the classification of subgroups of $GL_2$ over a finite field ([11], [8, Theorem 6.17]) are well known, it seems that the determination of all algebraic subgroups of $GL_2(\mathbb{C})$ is not presented well in the literature. In this paper we give this classification, including full proofs. The final result is Theorem 4. We note that $\mathbb{C}$ can be replaced everywhere by any algebraically closed field of characteristic zero.

Notation. $\mu_n \subset \mathbb{C}^*$ denotes the $n$th roots of unity and $\zeta_n$ denotes a primitive $n$th root of unity. Let $\beta : GL_2(\mathbb{C}) \to PGL_2(\mathbb{C}) = PSL_2(\mathbb{C})$, $\gamma : SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$ denote the canonical projections. For any algebraic subgroup $H \subset PSL_2(\mathbb{C})$ we write $H_{SL_2} = \gamma^{-1}(H) \subset SL_2(\mathbb{C})$. Further

$$B := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$$
and
\[ D_\infty := \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| c \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & -d \\ d & 0 \end{pmatrix} \middle| d \in \mathbb{C}^* \right\} \]
are the Borel subgroup and the infinite dihedral subgroup of \( \text{SL}_2(\mathbb{C}) \).

We first recall the classification of all algebraic subgroups of \( \text{PGL}_2(\mathbb{C}) \).

**Theorem 1.** Let \( H \) be an algebraic subgroup of \( \text{PGL}_2(\mathbb{C}) \). Then, up to conjugation, one of the following cases occurs:

1. \( H = \text{PGL}_2(\mathbb{C}) \);
2. \( H \) is a subgroup of the group \( \gamma(B) \);
3. \( H = \gamma(D_\infty) \);
4. \( H = D_n \) (the dihedral group of order \( 2n \)), \( A_4 \) (the tetrahedral group), \( S_4 \) (the octahedral group), or \( A_5 \) (the icosahedral group).

The above theorem reduces the problem to describing the algebraic groups in \( \text{GL}_2(\mathbb{C}) \) mapping to a given subgroup \( G \subset \text{PGL}_2(\mathbb{C}) \). Each example is therefore a central extension of \( G \) and corresponds to an element in \( H^2(G, \mu) \), where \( \mu \) is either \( \mathbb{C}^* \) or a finite cyclic subgroup of \( \mathbb{C}^* \). The first case defines the Schur multiplier of \( G \). In the interesting cases, \( \mu \) is a finite group and the Schur multiplier does not provide information because the canonical map \( H^2(G, \mu) \to H^2(G, \mathbb{C}^*) \) is not injective (see also Remark 3).

We note that Theorem 1 is a corollary of the following two well-known theorems.

**Theorem 2** (Klein [4]). A finite subgroup of \( \text{PGL}_2(\mathbb{C}) \) is isomorphic to one of the following polyhedral groups:

- a cyclic group \( C_n \);
- a dihedral group \( D_n \) of order \( 2n \), \( n \geq 2 \);
- the tetrahedral group \( A_4 \) of order 12;
- the octahedral group \( S_4 \) of order 24;
- the icosahedral group \( A_5 \) of order 60.

Up to conjugation, all of these groups occur as subgroups of \( \text{PGL}_2(\mathbb{C}) \) exactly once.

In Theorem 1, the cyclic groups \( C_n \) happen to be subgroups of \( \gamma(B) \).

**Theorem 3** ([3, Theorem 4.12]; [6, Theorem 4.29]). Suppose that \( G \) is an algebraic subgroup of \( \text{SL}_2(\mathbb{C}) \). Then, up to conjugation, one of the following cases occurs:

1. \( G = \text{SL}_2(\mathbb{C}) \);
2. \( G \) is a subgroup of the Borel group \( B \);

288
(3) \( G \) is not contained in the Borel group \( B \) and is a subgroup of the infinite dihedral group \( D_\infty \):

(4) \( G \) is one of the groups \( A_4^{SL_2}, S_4^{SL_2}, A_5^{SL_2} \).

2. ALGEBRAIC SUBGROUPS OF \( GL_2(\mathbb{C}) \)

Given a group \( H \subset PGL_2(\mathbb{C}) \) as in Theorem 1, we will determine all algebraic subgroups \( G \subset GL_2(\mathbb{C}) \) such that \( \beta(G) = H \). We first observe that there is only one maximal group with this property, namely \( H_{\text{max}} := \beta^{-1}(H) \). Any \( G \) with \( \beta(G) = H \) satisfies \( \mathbb{C}^* \cdot G = \mathbb{C}^* \cdot H \), \( SL_2 = H_{\text{max}} \).

By the Noetherian property, \( G \) contains a minimal algebraic subgroup with image \( H \). We will denote any such minimal subgroup by \( H_{\text{min}} \). Any \( G \) with \( \beta(G) = H \) has the form \( \mu_k \cdot H_{\text{min}} \) or \( \mathbb{C}^* \cdot H_{\text{min}} = H_{\text{max}} \). Our problem now remains to determine all minimal groups \( H_{\text{min}} \) (up to conjugation). We will proceed case by case based on Theorem 1.

2.1. \( H = PGL_2(\mathbb{C}) \)

**Proposition 1.** For \( H = PGL_2(\mathbb{C}) \) the only minimal group is \( SL_2(\mathbb{C}) \).

**Proof.** Clearly \( H_{\text{max}} = GL_2(\mathbb{C}) \). Let \( G \) be a minimal group with \( \beta(G) = PGL_2(\mathbb{C}) \). The latter group is equal to its commutator subgroup and therefore \( \beta([G, G]) = H \). Since \( G \) is minimal, one has \( G = [G, G] \) and \( G \subset SL_2(\mathbb{C}) \). By Theorem 3, \( G \) cannot be a proper subgroup of \( SL_2(\mathbb{C}) \). \( \square \)

2.2. \( H \) is a subgroup of the group \( \gamma(B) \)

Then \( H = \gamma(F) \) for some algebraic subgroup \( F \) of \( B \subset SL_2(\mathbb{C}) \). The algebraic subgroups of the Borel group \( B \subset SL_2(\mathbb{C}) \) are listed below:

\[
\begin{align*}
B; \quad G_m &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C}^* \right\}; \\
G_a &= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{C} \right\}; \\
F^k_1 &= \left\{ \begin{pmatrix} \xi & c \\ 0 & \xi^{-1} \end{pmatrix} \middle| \xi^k = 1, c \in \mathbb{C} \right\}, \quad \text{with } k \in \mathbb{Z}_{\geq 1}; \\
F^l_2 &= \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-l} \end{pmatrix} \middle| \xi^l = 1 \right\}, \quad \text{with } l \in \mathbb{Z}_{\geq 1}.
\end{align*}
\]

We note that \( \mu_l \cong F^l_2 \subset G_m \subset B \) and \( F^1_1 = G_a \subset F^k_1 \subset B \).

2.2.1. \( H = \gamma(B) \)

**Proposition 2.** For \( H = \gamma(B) \) the minimal groups are

\[
H_{k,l} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a^k c^l = 1 \right\}
\]

with \( k, l \in \mathbb{Z} \) satisfying \( k + l \neq 0 \) and \( \gcd(k, l) = 1 \).
Proof. Let \( G \subset \text{H}_{\text{max}} = \{(a \ b) \mid a, b, c \in \mathbb{C}, \ ac \neq 0\} \) be minimal with \( \beta(G) = H \).

Then \( G \) contains an element of the form \( A = \alpha \cdot (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \) with \( \alpha \in \mathbb{C}^{*} \). The unipotent component \( A_u = (\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}) \) of the multiplicative Jordan decomposition of \( A \) belongs to \( G \). Then \( G \) contains the normal subgroup \( N := (\begin{smallmatrix} 1 \\ b \\ 1 \end{smallmatrix}) \mid b \in \mathbb{C} \) and \( G/N \) is a proper subgroup of \( \text{H}_{\text{max}}/N \cong \mathbb{C} \times \mathbb{C} \). It follows that \( G = (\begin{smallmatrix} a \\ 0 \\ c \end{smallmatrix}) \mid a^k c^l = 1 \) for a certain pair \( (k, l) \neq (0, 0) \). This group has projective image \( \gamma(B) \) precisely when \( k + l \neq 0 \). By minimality \( \gcd(k, l) = 1 \). □

2.2.2. \( H = \gamma(\text{G}_m) \)

Proposition 3. In this case, the minimal groups are

\[
\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a^k b^l = 1 \right\}
\]

with \( k, l \in \mathbb{Z} \) satisfying \( k + l \neq 0 \) and \( \gcd(k, l) = 1 \).

Proof. A minimal subgroup \( G \) is a proper subgroup of \( \text{H}_{\text{max}} = \{(a \ 0) \mid a, b \in \mathbb{C}^{*}\} \) with image \( \text{G}_m \) in \( \text{PGL}_2(\mathbb{C}) \). Therefore it is of dimension one, hence it has the form \( \{(a \ 0) \mid a^k b^l = 1\} \) for some pair of integers \( (k, l) \neq (0, 0) \). This group has image \( \text{G}_m \) in \( \text{PGL}_2(\mathbb{C}) \), if and only if \( k + l \neq 0 \). Since \( G \) is minimal one moreover has \( \gcd(k, l) = 1 \).

Remark 1. Two pairs \( (k, l) \) and \( (m, n) \) define conjugated minimal subgroups of \( \text{GL}_2(\mathbb{C}) \) for Proposition 2 if and only if \( (k, l) = \pm (m, n) \). For Proposition 3 the two pairs define conjugated groups if and only if \( (k, l) \in \{\pm (m, n), \pm (n, m)\} \).

2.2.3. \( H = \gamma(\text{G}_a) \)

In this case, we have \( H^{\text{SL}_2} = \{\pm 1\} \cdot \text{G}_a \) and \( \text{H}_{\text{max}} = \mathbb{C}^{*} \cdot \text{G}_a \).

Proposition 4. In this case, the only minimal group is \( \text{G}_a \).

Proof. Let \( G \) be minimal. Then \( G \) contains an element of the form \( A = \alpha \cdot (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \) with \( \alpha \in \mathbb{C}^{*} \). The unipotent component \( A_u = (\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}) \) of the multiplicative Jordan decomposition of \( A \) also belongs to \( G \) and thus \( G \supset \{(a \ b) \mid a \in \mathbb{C}\} = \text{G}_a \). By minimality \( G = \text{G}_a \).

2.2.4. \( H = \gamma(F_k^2) \)

The group \( H \) is topologically (for the Zariski topology) generated by the images of the elements \( (\begin{smallmatrix} \zeta_k & 0 \\ 0 & 1 \end{smallmatrix}) \) and \( (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \) in \( \text{PGL}_2(\mathbb{C}) \) (where \( \zeta_k \) is a primitive \( k \)th root of the identity). Let \( G \) denote a minimal subgroup with \( \beta(G) = H \). As before one concludes that \( G \supset \{(a \ b) \mid a \in \mathbb{C}\} = \text{G}_a \). Moreover, \( G \) is (topologically) generated by \( \text{G}_a \) and an element of the form \( A := \alpha \cdot (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \) with \( \alpha \in \mathbb{C}^{*} \). If \( \alpha \) is not a root of unity, then the group, topologically generated by \( A \) and \( \text{G}_a \), contains \( \mathbb{C}^{*} \) and is equal to \( \text{H}_{\text{max}} \). By the minimality of \( G \) we have that \( \alpha \) is some primitive \( n \)th root of
unity. We define $s$ by $s = k/2$ if $k$ is divisible by 2 and $s = k$ otherwise. For every prime number $p$, not dividing $s$, we may consider the subgroup of $G$ generated by $A^p$ and $Ga$. This group maps surjectively to $H$. Thus, by minimality, this group is equal to $G$ and $p$ does not divide the order $n$ of $\alpha$. We find that every prime divisor of $n$ is also a prime divisor of $s$. Define, for any positive integer $n$ with this property, and every primitive $n$th root of unity $\zeta_n$, the group $H(\zeta_n)$ as generated by $\zeta_n \cdot \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $Ga$. This group $H(\zeta_n)$ depends on the choice of the primitive $n$th root of unity $\zeta_n$. Further $\beta(H(\zeta_n)) = H$. The group $H(\zeta_n)$ is minimal since any proper subgroup of $H(\zeta_n)$, containing $Ga$, is contained in the group generated by $(\zeta_n \cdot \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right))^p$ and $Ga$, where the prime $p$ divides $s$. The latter group does not map surjectively to $H$. Moreover we found $G \supset H(\zeta_n)$ for some $n$. Thus we found all minimal groups, namely the groups $H(\zeta_n)$.

**Proposition 5.** For $H = \gamma(F^k_1)$ the minimal groups are the $H(\zeta_n)$, generated by $\zeta_n \cdot \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbb{C}\right\} = Ga$, where every prime divisor of the positive integer $n$ divides $k$ if $k$ is odd and divides $k/2$ if $k$ is even.

**Remark 2.** One has $H(\zeta_n)^0 = Ga$ and the order of the cyclic group $H(\zeta_n)/H(\zeta_n)^0$ is the smallest common multiple of $n$ and $k$ (for $k$ odd) and that of $n$ and $k/2$ (if $k$ is even). Moreover, if $H(\zeta_n)$ is conjugated to $H_m$, then $n = m$. However the converse is not true in general.

2.2.5. $H = \gamma(F^k_2)$

Similarly to Section 2.2.4 one finds the following proposition:

**Proposition 6.** For $H = \gamma(F^k_2)$ the minimal groups are the cyclic groups generated by $\zeta_n \cdot \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $n$ is a positive integer such that every prime divisor of $n$ is a prime divisor of $l$ if $l$ is odd or of $l/2$ if $l$ is even.

2.3. $H = \gamma(D_{\infty})$

Let $G$ be minimal with $\beta(G) = H$. Then $G$ is a proper subgroup of $H_{\max} = \mathbb{C}^* \cdot D_{\infty}$. The component of the identity $G^0 \subset G$ has the form $\left\{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mid a^k b^l = 1 \right\}$ for some $(k, l)$ with $\gcd(k, l) = 1$. Consider an element $B \in G$ with image (the class of) $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in H$. Thus $B = \beta \cdot \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ for some $\beta \in \mathbb{C}^*$. From $BG^0 B^{-1} = G^0$ it follows that $k = l$ and thus $G^0 = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid ab = 1 \right\}$. By the minimality of $G$ one has that $B^2 = \beta^2$ is a root of unity. The subgroup of $G$, generated by $G^0$ and $B^k$, where $k$ is any odd integer, is also mapped surjectively to $H$. The minimality of $G$ implies that $\beta^2$ is a primitive $2^n$th root of unity for some $n \geq 0$. Let $H_n$ be the group generated by $\left\{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid ab = 1 \right\}$ and $B_n := \zeta_{2n+1} \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This group does not depend on the choice of $\zeta_{2n+1}$ since one may replace $B_n$ by any odd power of $B_n$. Further $G \subset H_n$ for some $n$. The group $G$ must contain $\left\{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid ab = 1 \right\}$ and some element $\lambda \cdot \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The latter element has the form $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) (\zeta_{2n+1} \cdot \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))^p$ with $ab = 1$ and $p \in \mathbb{Z}$. One concludes that $a = b = \pm 1$ and $p$ is odd. It follows that $G = H_n$ and we conclude: $\{H_n \mid n \geq 0\}$ is the collection of the minimal groups.
2.4. \( H = D_n, A_4, S_4 \) or \( A_5 \)

We first note that if \( H \subset \text{PGL}_2(\mathbb{C}) \) is a finite subgroup, then every \( H_{\text{min}} \subset \text{GL}_2(\mathbb{C}) \) is also finite. Indeed, it is clear that \( H_{\text{SL}_2} \) is finite. Because \( H_{\text{min}} \subset \mathbb{C}^* \cdot H_{\text{SL}_2} \), we see that \( H_{\text{min}} \) is finite.

2.4.1. \( H = D_n \)

We write \( D_n = \langle a, b \mid a^n = b^2 = 1, ba = a^{-1}b \rangle \subset \text{PGL}_2(\mathbb{C}) \).

(i) \( n \) odd and \( n \geq 3 \). In this case, we may choose for \( a \) and \( b \) the images in \( \text{PGL}_2(\mathbb{C}) \) of the matrices \( \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), with \( \zeta \) a primitive \( n \)th root of unity.

Let \( G \) be a minimal group. As \( G \) is finite and generated by preimages of \( a, b \in D_n \), one has that

\[
G = \left\{ A = \lambda \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, B = \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}
\]

for certain roots of unity \( \lambda, \mu \). We have \( A^n = \lambda^n, B^2 = \mu^2, BA = \lambda^2 A^{-1} B \). Every element of \( G \) has the form \( t \lambda^k \), or \( t \lambda^k B, k = 0, 1, \ldots, n - 1 \), with \( t \in \langle \lambda^2, \lambda^n, \mu^2 \rangle = \langle \lambda, \mu^2 \rangle \). Hence \( G \cap \mathbb{C}^* = \langle \lambda, \mu^2 \rangle \). Since both \( \lambda \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix} \in G \) and \( \lambda \in G \), we can write

\[
G = \left\{ A = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, B = \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}
\]

The subgroup of \( G \) generated by \( A \) and \( B^m, \) where \( m \geq 1 \) is odd, also maps surjectively to \( D_n \). By the minimality of \( G \), this implies that the order of \( \mu \) is \( 2^k \) for some \( k \geq 0 \). Now define

\[
H_k := \left\{ \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \zeta_2^k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}
\]

for \( k \geq 0 \). This group \( H_k \) does not depend on the choice of the primitive \( 2^k \)th root of unity because one can replace the second generator by any odd power of itself. The groups \( H_k \) are the only candidates for minimal groups.

We now show that \( H_k \) is indeed minimal. For \( k = 0, 1 \), the groups

\[
H_0 = \left\{ \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \quad H_1 = \left\{ \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}
\]

are minimal since they have order \( 2n \). The two groups are conjugated by the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). We note that \( H_2 = D_{2n}^{\text{SL}_2} \). For \( k \geq 2 \), we see that \( H_k \cap \mathbb{C}^* = \langle \zeta_2^2 \rangle \). Suppose that \( D \) is a subgroup of \( H_k \) which maps surjectively to \( D_n \), then

\[
D = \left\{ \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, t \zeta_2^k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}
\]

for some \( t \in \langle \zeta_2^k \rangle \). Since the order of \( t \zeta_2^k \) is also \( 2^k \), one has \( D = H_k \) and thus \( H_k \) is minimal. For \( k \geq 1 \), the order of \( H_k \) is \( 2^k \cdot n \). Thus two minimal groups \( H_k \) and \( H_l \) with \( k, l \geq 1 \) are conjugated only if \( k = l \).
(ii) \( n \) even and \( n > 2 \). A minimal \( G \) can be written as
\[ G = \begin{pmatrix} A = \lambda \begin{pmatrix} \zeta_{2n} & 0 \\
0 & \zeta_{2n}^{-1} \end{pmatrix}, & B = \mu \begin{pmatrix} 0 & i \\
i & 0 \end{pmatrix} \end{pmatrix}, \]
for certain roots of unity \( \lambda, \mu \). We have
\[ A^n = -\lambda^n, \quad B^2 = -\mu^2, \quad BA = \lambda^2 A^{-1} B. \]
As before, this implies that \( G \cap \mathbb{C}^* = \langle \lambda^2, -\lambda^n, -\mu^2 \rangle = \langle -1, \lambda^2, \mu^2 \rangle \). One can replace \( A \) and \( B \) by \( c_1 A \) and \( c_2 B \) with \( c_1, c_2 \in \langle -1, \lambda^2, \mu^2 \rangle \). For a good choice of \( c_1, c_2 \), the group \( \langle c_1 A, c_2 B \rangle \) will be a proper subgroup unless there exists an integer \( N \) with \( \lambda, \mu \in \mu_{2N} \). Thus the latter holds by the minimality of \( G \). Then \( \langle -1, \lambda, \mu \rangle = \mu_{2m+1} \) for some \( m \geq 0 \).
For \( m = 0 \), we have \( G \cap \mathbb{C}^* = \mu_2 \) and this leads to only one group, namely
\[ \left( \begin{pmatrix} \zeta_{2n} & 0 \\
0 & \zeta_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\
i & 0 \end{pmatrix} \right) = D_{n}^{SL_2}. \]
This group is clearly minimal. For \( m \geq 1 \), one has \( G \cap \mathbb{C}^* = \mu_{2m} \) and this leads to the three groups given by the table:

<table>
<thead>
<tr>
<th></th>
<th>( \lambda )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_{1,m} )</td>
<td>( \zeta_{2m+1} )</td>
<td>1</td>
</tr>
<tr>
<td>( H_{2,m} )</td>
<td>( \zeta_{2m+1} )</td>
<td>( \zeta_{2m+1} )</td>
</tr>
<tr>
<td>( H_{3,m} )</td>
<td>1</td>
<td>( \zeta_{2m+1} )</td>
</tr>
</tbody>
</table>

They all are minimal and have order \( 2^m \cdot 2n \). However \( H_{1,m} \) and \( H_{2,m} \) are conjugated. Indeed, \( \begin{pmatrix} \zeta_{2n} & 0 \\
0 & 1 \end{pmatrix} H_{1,m} \begin{pmatrix} \zeta_{2n}^{-1} & 0 \\
0 & 1 \end{pmatrix} = H_{2,m} \) because
\[ \begin{pmatrix} \zeta_{2n} & 0 \\
0 & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\
i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\
i & 0 \end{pmatrix}, \quad \begin{pmatrix} \zeta_{2n} & 0 \\
0 & 1 \end{pmatrix} \begin{pmatrix} \zeta_{2n}^{-1} & 0 \\
0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & i \\
i & 0 \end{pmatrix}. \]

(iii) \( n = 2 \). As in (ii). In this case also \( H_{1,m} \) and \( H_{3,m} \) are also conjugated, namely by a matrix of the form \( \begin{pmatrix} 0 & a \\
a & -1 \end{pmatrix} \).

2.4.2. \( H = A_4 \)

Let \( G \subset H_{\text{max}} = \mathbb{C}^* \cdot A_4^{SL_2} \) be a minimal group. Consider \( G^+ \subset \mathbb{C}^* \times A_4^{SL_2} \), the preimage of \( G \) under the obvious map \( \alpha : \mathbb{C}^* \times A_4^{SL_2} \rightarrow \mathbb{C}^* \cdot A_4^{SL_2} \). We note that the kernel of \( \alpha \) is \( \{(1, 0), (-1, 0)\} \). Since \( \beta(G) = A_4 \), there exists for every \( a \in A_4^{SL_2} \) an element \( (\lambda, a) \in G^+ \). Let \( \mu_k := \{\lambda \in \mathbb{C}^* \mid (\lambda, 1) \in G^+\} \). Then we obtain a homomorphism \( h : A_4^{SL_2} \rightarrow \mathbb{C}^*/\mu_k \) given by \( h(a) = \lambda \mod \mu_k \) if \( (\lambda, a) \in G^+ \).
This homomorphism factors as $A_4^{SL_2} \to C_3 \xrightarrow{h_1} \mathbb{C}^* / \mu_k$, where $C_3 = \{1, \sigma, \sigma^2\}$ is the quotient of $A_3^{SL_2}$ by its commutator subgroup. If $h_1$ is trivial, then $G^+$ contains $\{(1,a) \mid a \in A_3^{SL_2}\}$ and by minimality $G = A_4^{SL_2}$. By Theorem 3, the latter group of order 24 is minimal.

Now we suppose that $h_1$ is not trivial. Write $k = 3^r \ell$ with $\gcd(\ell, 3) = 1$. For any $a \in A_4^{SL_2}$ there exists an element $(\lambda, a) \in G^+$ with $\lambda^3 \in \mu_3^\ell$ and $\lambda$ can be multiplied by any element in $\mu_3^\ell$. Thus there exist a pair $(\lambda, a) \in G^+$ with $\lambda \in \mu_3^\ell$.

Now $G^+ \cap (\mu_3^{r+1} \times A_4^{SL_2})$ is a subgroup of $G^+$ mapping surjectively to $A_4$. The minimality of $G$ implies that $\ell = 1$ and $G^+ \subset \mu_3^{r+1} \times A_4^{SL_2}$. Moreover, $\mu_3^r \subset G^+$ and the map $G^+ \to G$ is bijective. Then $G$ has the form $\mu_3^r \cdot \{\delta(a)a \mid a \in A_4^{SL_2}\}$, where $\delta = A_4^{SL_2} \to C_3 \xrightarrow{\delta_1} \{1, \zeta_3^{r+1}, \zeta_3^{2r+1}\}$ for some map $\delta_1$ which lifts the homomorphism $h_1 : C_3 \to \mu_3^{r+1}/\mu_3^r \subset \mathbb{C}^*/\mu_3^r$. There are two possibilities for nontrivial homomorphism $h_1$ (and thus for $\delta_1$ and $\delta$) and we find therefore two subgroups of $GL_2(\mathbb{C})$, lying in $\mu_3^{r+1} \cdot A_4^{SL_2}$. The last group is contained in $\mu_3^{r+1} \cdot \xi_4^{SL_2}$. Conjugation by an element $\tau \in S_4 \setminus A_4$ induced on $C_3 = A_4/[A_4, A_4]$ the only non trivial automorphism and permutes the two possibilities for $h_1$. One lifts $\tau$ to an element $\tau' \in S_4^{SL_2}$. Conjugation by $\tau'$ permutes the two possibilities for $h_1$ and therefore the above two groups are conjugated. It suffices to consider the group $H_r := \mu_3^r \cdot \{\delta(a)a \mid a \in A_4^{SL_2}\}$ with $\delta_1$ given by $\delta_1(1) = 1$, $\delta_1(\sigma) = \zeta_3^{r+1}$, $\delta_1(\sigma^2) = \zeta_3^{2r+1}$. The order of $H_r$ is $3^r \cdot 24$. The group $H_0$ is isomorphic to $A_4^{SL_2}$, but not conjugated to $A_4^{SL_2}$. The minimality of $H_0$ follows from the fact that $A_4$ does not have a faithful two-dimensional representation.

Finally, we will show that $H_r$ is minimal for $r \geq 1$. Suppose that $D$ is a subgroup of $H_r$ with $\beta(D) = A_4$. Let $\tau \in A_4^{SL_2}$ be an element of order 3. Then $D$ contains an element $d = \lambda \delta(\tau)\tau$ for some $\lambda \in \{1, 3\} \times \mu_3^r$. Now $\delta(\tau) \in \{\zeta_3^{r+1}, \zeta_3^{2r+1}\}$ and $d^3 \in D \cap \mathbb{C}^*$ has order $3^r$ or $2 \cdot 3^r$. Thus $D$ contains $\mu_3^r$ and it follows that $D = H_r$. Thus we found:

There are two minimal groups for $A_4$ with order 24 and for every $r \geq 1$ there is one minimal group of order $3^r \cdot 24$.

Remark 3. A minimal subgroup $G$ for $H = A_4$ yields a central extension $1 \to \mu_k \to G \to A_4 \to 1$ for some $k$. The corresponding element $\xi$ of $H^2(A_4, \mu_k)$ has, by the minimality of $G$, the property that $\xi$ does not lie in the image of $H^2(A_4, \mu_d)$ for a proper divisor $d$ of $k$. Since the order of $A_4$ in 12, we only have to consider the groups $H^2(A_4, \mu_{2a^2b})$. The minimal groups that we found above correspond to all the cases $(a, b) = (1, r)$. The central extensions with $a \neq 1$ produce, apparently, groups which do not have a faithful representation of degree two.

2.4.3. $H = S_4$

Let $G \subset H_{\text{max}} = \mathbb{C}^* \cdot \xi_4^{SL_2}$ be a minimal group. Consider $G^+ \subset \mathbb{C}^* \times \xi_4^{SL_2}$, the preimage of $G$ under the obvious map $\alpha : \mathbb{C}^* \times \xi_4^{SL_2} \to \mathbb{C}^* \cdot \xi_4^{SL_2}$. The kernel of $\alpha$ is $\{(1, (1, 0)), (-1, (0, -1))\}$. Since $\beta(G) = S_4$, there exists for every $a \in \xi_4^{SL_2}$
an element \((\lambda, a) \in G^+\). Let \(\mu_k := \{\lambda \in \mathbb{C}^* \mid (\lambda, 1) \in G^+\}\). Then we obtain a homomorphism \(h : S_4 \rightarrow \mathbb{C}^*/\mu_k\) given by \(h(a) = \lambda \mod \mu_k\) if \((\lambda, a) \in G^+\). This homomorphism factors as \(S_4 \rightarrow C_2 \rightarrow \mathbb{C}^*/\mu_k\), where \(C_2 = \{1, \sigma\}\) is the quotient of \(S_4\) by its commutator subgroup. If \(h\) is trivial, then \(G^+\) contains \(\{(1, a) \mid a \in S_4\}\) and by minimality \(G = S_4\). According to Theorem 3, the latter group of order 48 is minimal.

Now we suppose that \(h\) is not trivial. Write \(k = ye\) with \(e\) odd. For any \(a \in S_4\) there exists an element \((\lambda, a) \in G^+\) with \(\lambda \in \mu_{2^{r+1}}\). Now \(G^+ \cap (\mu_{2^{r+1}} \times S_4)\) is a subgroup of \(G^+\) mapping surjectively to \(S_4\). The minimality of \(G\) implies that \(\ell = 1\) and \(G^+ \subset \mu_{2^{r+1}} \times S_4\). Define \(\delta : S_4 \rightarrow C_2 \rightarrow \{1, \zeta_{2^{r+1}}\}\) by \(\delta_1(1) = 1\) and \(\delta_1(\sigma) = \zeta_{2^{r+1}}\). All the elements of \(\mu_{2^{r+1}}\) have the form \(\zeta_{2^{r+1}}^e \cdot \lambda\) with \(e \in \{0, 1\}\) and \(\lambda \in \mu_{2^{r}}\). From this it follows that \(G^+ = \{(\delta(a)\lambda, a) \mid a \in S_4, \lambda \in \mu_{2^{r}}\}\) and one concludes that \(G = H_r := \mu_{2^r} \cdot \delta(a)a \in S_4\). The group \(H_r\) has order \(2^r \cdot 48\).

We note that \(H_0\) is equal to \(S_4\) and is minimal.

Let \(r > 0\) and let \(D \subset H_r\) be a subgroup satisfying \(\beta(D) = S_4\). Let \(\tau \in S_4\) be an element with image the permutation \((1, 2) \in S_4\). Then \(D\) contains an element of the form \(d = \pm \lambda \delta(\tau)\tau\) with \(\lambda \in \mu_{2^{r}}\). Then \(d^2 = \lambda^2 \zeta_{2^{r}} \in D \cap \mathbb{C}^*\) has order \(2^r\). Thus \(D\) contains \(\mu_{2^{r}}\) and it follows easily that \(D = H_r\). Hence every \(H_r\) is minimal and we conclude that: There is for every \(r \geq 0\) a unique minimal group of order \(2^r \cdot 48\).

### 2.4.4. \(H = A_5\)

Let \(G \subset GL_2(\mathbb{C})\) be a minimal for \(H\). Since \(A_5 = [A_5, A_5]\), the group \([G, G]\) also satisfies \(\beta([G, G]) = H\). By minimality \(G = [G, G]\) and thus \(G \subset SL_2(\mathbb{C})\). This implies that \(G \subset A_5^{SL_2}\). Since, by Theorem 3, the latter group is minimal, we find that \(A_5^{SL_2}\) is the only minimal group.

In summary, we obtain the following result.

**Theorem 4.** The list of all minimal groups, up to conjugation, for each algebraic subgroup \(H \subset PGL_2(\mathbb{C})\) (see Theorem 1 and Section 2.2) is:

1. \(H = PGL_2(\mathbb{C})\): the only minimal group is \(SL_2(\mathbb{C})\).
2. \(H\) is a subgroup of the group \(B\):
   a. \(H = \gamma(B)\): for each pair of integers \((k, l)\) with \(k + l \neq 0\) and \(\gcd(k, l) = 1\) there is a minimal one, namely \(\{(a^kb^l) \mid a^b = 1\}\).
   b. \(H = \gamma(G_m)\): for each pair of integers \((k, l)\) with \(k + l \neq 0\) and \(\gcd(k, l) = 1\) there is a minimal group, namely \(\{(a^kb^l) \mid a^b = 1\}\). Further \((k, l)\) and \((l, k)\) define conjugated groups.
   c. \(H = \gamma(G_a)\): there is only one minimal group, namely \(G_a\).
   d. \(H = \gamma(F_n^r)\): the minimal ones are the \(H(\zeta_n)\), generated by \(\zeta_n \cdot \{(a^2)^b \mid a \in \mathbb{C}\}\) and \(\{(a^2)^b \mid a \in \mathbb{C}\}\), where every prime divisor of the positive integer \(n\) divides \(k\) if \(k\) is odd and divides \(k/2\) if \(k\) is even.
(e) \( H = \gamma (F_2^3) \): the minimal ones are the groups generated by \( \xi_n \cdot (\xi_0^3 0 1) \), where every prime divisor of the positive integer \( n \) divides \( l \) if \( l \) is odd and divides \( l/2 \) if \( l \) is even.

(3) \( H = \gamma (D_{\infty}) \): the minimal groups are \( H_n \) with \( n \geq 0 \), where \( H_n \) is generated by \( \{ (a b) \mid ab = 1 \} \) and \( \xi_{2n+1} \cdot (0 1 0) \) with \( \xi_{2n+1} \) a primitive \( 2^{n+1} \)th root of unity.

(4) \( H \) finite:
   
   (a) \( H = D_n \):
      
      (i) \( n \geq 3 \) odd: For every \( k \geq 1 \), there is one minimal group
      \[
      \left( \begin{pmatrix} \xi_n & 0 \\ 0 & \xi_n^{-1} \end{pmatrix}, \xi_{2k} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),
      \]
      with \( \xi_{2k} \) a primitive \( 2^k \)th root of unity;
      
      (ii) \( n > 2 \) even: For \( k \geq 1 \), the minimal ones \( H_{1,k}, H_{2,k}, H_{3,k} \) have the form
      \[
      A = \lambda \begin{pmatrix} \xi_{2n} & 0 \\ 0 & \xi_{2n}^{-1} \end{pmatrix}, \quad B = \mu \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
      \]
      for certain roots of unity \( \lambda, \mu \) which are given in the table:

      | \( H_{1,k} \)  | \( \lambda \) | \( \mu \) |
      |-------------|-------------|-------------|
      | \( \xi_{2k+1} \) | 1           | \( \xi_{2k+1} \) |
      | \( \xi_{2k+1} \) | \( \xi_{2k+1} \) | 1           |
      | \( \xi_{2k+1} \) | \( \xi_{2k+1} \) | \( \xi_{2k+1} \) |

      They all have order \( 2^k \cdot 2n \). Further \( H_{1,k} \) and \( H_{2,k} \) are conjugated. For \( k = 0 \), there is only minimal group, namely the group
      \[
      \left( \begin{pmatrix} \xi_{2n} & 0 \\ 0 & \xi_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right) = D_{n}^{SL_2} \text{ of order } 4n;
      \]
      
      (iii) \( n = 2 \): As in (ii), but now \( H_{1,k}, H_{2,k}, H_{3,k} \) are all conjugated.

   (b) \( H = A_4 \): there are two minimal groups of order 24. For every \( n > 0 \) there is one minimal group of order \( 3^n \cdot 24 \).

   (c) \( H = S_4 \): For every \( n \geq 0 \) there is a minimal group of order \( 2^n \cdot 48 \).

   (d) \( H = A_5 \): There is only minimal group, namely \( A_5^{SL_2} \).

REFERENCES


296

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