Algebraic subgroups of $GL_2(\mathbb{C})$

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**ABSTRACT**

In this note we classify, up to conjugation, all algebraic subgroups of $GL_2(\mathbb{C})$.

1. **INTRODUCTION**

Although the classification, up to conjugation, of the algebraic subgroups of $SL_2(\mathbb{C})$ ([3, Theorem 4.12], [6, Theorem 4.29]), and the classification of subgroups of $GL_2$ over a finite field ([11], [8, Theorem 6.17]) are well known, it seems that the determination of all algebraic subgroups of $GL_2(\mathbb{C})$ is not presented well in the literature. In this paper we give this classification, including full proofs. The final result is Theorem 4. We note that $\mathbb{C}$ can be replaced everywhere by any algebraically closed field of characteristic zero.

**Notation.** $\mu_n \subseteq \mathbb{C}^*$ denotes the $n$th roots of unity and $\zeta_n$ denotes a primitive $n$th root of unity. Let $\beta: GL_2(\mathbb{C}) \rightarrow PGL_2(\mathbb{C}) = PSL_2(\mathbb{C})$, $\gamma: SL_2(\mathbb{C}) \rightarrow PSL_2(\mathbb{C})$ denote the canonical projections. For any algebraic subgroup $H \subseteq PSL_2(\mathbb{C})$ we write $H^{SL_2} = \gamma^{-1}(H) \subseteq SL_2(\mathbb{C})$. Further

$$B := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \bigg| a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$$
and

$$D_\infty := \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \middle| c \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & -d \\ d^{-1} & 0 \end{pmatrix} \middle| d \in \mathbb{C}^* \right\}$$

are the Borel subgroup and the infinite dihedral subgroup of $\text{SL}_2(\mathbb{C})$.

We first recall the classification of all algebraic subgroups of $\text{PGL}_2(\mathbb{C})$.

**Theorem 1.** Let $H$ be an algebraic subgroup of $\text{PGL}_2(\mathbb{C})$. Then, up to conjugation, one of the following cases occurs:

1. $H = \text{PGL}_2(\mathbb{C})$;
2. $H$ is a subgroup of the group $\gamma(\mathcal{B})$;
3. $H = \gamma(D_\infty)$;
4. $H = D_n$ (the dihedral group of order $2n$), $A_4$ (the tetrahedral group), $S_4$ (the octahedral group), or $A_5$ (the icosahedral group).

The above theorem reduces the problem to describing the algebraic groups in $\text{GL}_2(\mathbb{C})$ mapping to a given subgroup $G \subset \text{PGL}_2(\mathbb{C})$. Each example is therefore a central extension of $G$ and corresponds to an element in $H^2(G, \mu)$, where $\mu$ is either $\mathbb{C}^*$ or a finite cyclic subgroup of $\mathbb{C}^*$. The first case defines the Schur multiplier of $G$. In the interesting cases, $\mu$ is a finite group and the Schur multiplier does not provide information because the canonical map $H^2(G, \mu) \to H^2(G, \mathbb{C}^*)$ is not injective (see also Remark 3).

We note that Theorem 1 is a corollary of the following two well-known theorems.

**Theorem 2** (Klein [4]). A finite subgroup of $\text{PGL}_2(\mathbb{C})$ is isomorphic to one of the following polyhedral groups:

- a cyclic group $C_n$;
- a dihedral group $D_n$ of order $2n$, $n \geq 2$;
- the tetrahedral group $A_4$ of order 12;
- the octahedral group $S_4$ of order 24;
- the icosahedral group $A_5$ of order 60.

Up to conjugation, all of these groups occur as subgroups of $\text{PGL}_2(\mathbb{C})$ exactly once.

In Theorem 1, the cyclic groups $C_n$ happen to be subgroups of $\gamma(\mathcal{B})$.

**Theorem 3** ([3, Theorem 4.12]; [6, Theorem 4.29]). Suppose that $G$ is an algebraic subgroup of $\text{SL}_2(\mathbb{C})$. Then, up to conjugation, one of the following cases occurs:

1. $G = \text{SL}_2(\mathbb{C})$;
2. $G$ is a subgroup of the Borel group $\mathcal{B}$;
(3) \( G \) is not contained in the Borel group \( B \) and is a subgroup of the infinite dihedral group \( D_\infty \);
(4) \( G \) is one of the groups \( A_4^{\text{SL}_2}, S_4^{\text{SL}_2}, A_5^{\text{SL}_2} \).

2. ALGEBRAIC SUBGROUPS OF \( \text{GL}_2(\mathbb{C}) \)

Given a group \( H \subset \text{PGL}_2(\mathbb{C}) \) as in Theorem 1, we will determine all algebraic subgroups \( G \subset \text{GL}_2(\mathbb{C}) \) such that \( \beta(G) = H \). We first observe that there is only one maximal group with this property, namely \( H_{\text{max}} := \beta^{-1}(H) \). Any \( G \) with \( \beta(G) = H \) satisfies \( \mathbb{C}^* \cdot G = \mathbb{C}^* \cdot H_{\text{SL}_2} = H_{\text{max}} \).

By the Noetherian property, \( G \) contains a minimal algebraic subgroup with image \( H \). We will denote any such minimal subgroup by \( H_{\text{min}} \). Any \( G \) with \( \beta(G) = H \) has the form \( \mu_k \cdot H_{\text{min}} \) or \( \mathbb{C}^* \cdot H_{\text{min}} = H_{\text{max}} \). Our problem now remains to determine all minimal groups \( H_{\text{min}} \) (up to conjugation). We will proceed case by case based on Theorem 1.

2.1. \( H = \text{PGL}_2(\mathbb{C}) \)

**Proposition 1.** For \( H = \text{PGL}_2(\mathbb{C}) \) the only minimal group is \( \text{SL}_2(\mathbb{C}) \).

**Proof.** Clearly \( H_{\text{max}} = \text{GL}_2(\mathbb{C}) \). Let \( G \) be a minimal group with \( \beta(G) = \text{PGL}_2(\mathbb{C}) \). The latter group is equal to its commutator subgroup and therefore \( \beta([G, G]) = H \). Since \( G \) is minimal, one has \( G = [G, G] \) and \( G \subset \text{SL}_2(\mathbb{C}) \). By Theorem 3, \( G \) cannot be a proper subgroup of \( \text{SL}_2(\mathbb{C}) \). \( \square \)

2.2. \( H \) is a subgroup of the group \( \gamma(B) \)

Then \( H = \gamma(F) \) for some algebraic subgroup \( F \) of \( B \subset \text{SL}_2(\mathbb{C}) \). The algebraic subgroups of the Borel group \( B \subset \text{SL}_2(\mathbb{C}) \) are listed below:

\[
B; \quad G_m = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^* \right\}; \quad G_a = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{C} \right\};
\]

\[
F_1^k = \left\{ \begin{pmatrix} \xi & c \\ 0 & \xi^{-1} \end{pmatrix} \mid \xi^k = 1, c \in \mathbb{C} \right\}, \quad \text{with } k \in \mathbb{Z}_{\geq 1};
\]

\[
F_2^l = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \mid \xi^l = 1 \right\}, \quad \text{with } l \in \mathbb{Z}_{\geq 1}.
\]

We note that \( \mu_l \cong F_2^l \subset G_m \subset B \) and \( F_1^l = G_a \subset F_1^k \subset B \).

2.2.1. \( H = \gamma(B) \)

**Proposition 2.** For \( H = \gamma(B) \) the minimal groups are

\[
H_{k,l} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a^k c^l = 1 \right\}
\]

with \( k, l \in \mathbb{Z} \) satisfying \( k + l \neq 0 \) and \( \gcd(k, l) = 1 \).
Proof. Let \( G \subset H_{\text{max}} = \{ (a, b, c) \mid a, b, c \in \mathbb{C}, \; ac \neq 0 \} \) be minimal with \( \beta(G) = H \). Then \( G \) contains an element of the form \( A = \alpha \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) with \( \alpha \in \mathbb{C}^* \). The unipotent component \( A_u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) of the multiplicative Jordan decomposition of \( A \) belongs to \( G \). Then \( G \) contains the normal subgroup \( N := \{ (1, b) \mid b \in \mathbb{C} \} \) and \( G/N \) is a proper subgroup of \( H_{\text{max}}/N \cong \mathbb{G}_m \times \mathbb{G}_m \). It follows that \( G = \{ (a, b^k) \mid a^k = 1 \} \) for a certain pair \( (k, l) \neq (0, 0) \). This group has projective image \( \gamma(B) \) precisely when \( k + l \neq 0 \). By minimality \( \gcd(k, l) = 1 \). \( \square \)

2.2.2. \( H = \gamma(G_m) \)

Proposition 3. In this case, the minimal groups are

\[
\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a^k b^l = 1 \right\}
\]

with \( k, l \in \mathbb{Z} \) satisfying \( k + l \neq 0 \) and \( \gcd(k, l) = 1 \).

Proof. A minimal subgroup \( G \) is a proper subgroup of \( H_{\text{max}} = \{ (a, b), c \mid a, b \in \mathbb{C}^* \} \) with image \( G_m \) in \( \text{PGL}_2(\mathbb{C}) \). Therefore it is of dimension one, hence it has the form \( \{ (a, 0) \mid a^k = 1 \} \) for some pair of integers \( (k, l) \neq (0, 0) \). This group has image \( G_m \) in \( \text{PGL}_2(\mathbb{C}) \), if and only if \( k + l \neq 0 \). Since \( G \) is minimal one moreover has \( \gcd(k, l) = 1 \).

Remark 1. Two pairs \( (k, l) \) and \( (m, n) \) define conjugated minimal subgroups of \( \text{GL}_2(\mathbb{C}) \) for Proposition 2 if and only if \( (k, l) = \pm (m, n) \). For Proposition 3 the two pairs define conjugated groups if and only if \( (k, l) \in \{ \pm (m, n), \pm (n, m) \} \).

2.2.3. \( H = \gamma(G_a) \)

In this case, we have \( H^{\text{SL}_2} = \{ \pm 1 \} \cdot G_a \) and \( H_{\text{max}} = \mathbb{C}^* \cdot G_a \).

Proposition 4. In this case, the only minimal group is \( G_a \).

Proof. Let \( G \) be minimal. Then \( G \) contains an element of the form \( A = \alpha \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) with \( \alpha \in \mathbb{C}^* \). The unipotent component \( A_u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) of the multiplicative Jordan decomposition of \( A \) also belongs to \( G \) and thus \( G \supset \{ (a, 0) \mid a \in \mathbb{C} \} = G_a \). By minimality \( G = G_a \).

2.2.4. \( H = \gamma(F^k) \)

The group \( H \) is topologically (for the Zariski topology) generated by the images of the elements \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) in \( \text{PGL}_2(\mathbb{C}) \) (where \( \zeta_k \) is a primitive \( k \)th root of the identity). Let \( G \) denote a minimal subgroup with \( \beta(G) = H \). As before one concludes that \( G \supset \{ (a, 0) \mid a \in \mathbb{C} \} = G_a \). Moreover, \( G \) is (topologically) generated by \( G_a \) and an element of the form \( A := \alpha \cdot \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) with \( \alpha \in \mathbb{C}^* \). If \( \alpha \) is not a root of unity, then the group, topologically generated by \( A \) and \( G_a \), contains \( \mathbb{C}^* \) and is equal to \( H_{\text{max}} \). By the minimality of \( G \) we have that \( \alpha \) is some primitive \( n \)th root of
let's define $s$ by $s = k/2$ if $k$ is divisible by 2 and $s = k$ otherwise. For every prime number $p$, not dividing $s$, we may consider the subgroup of $G$ generated by $A^p$ and $G_a$. This group maps surjectively to $H$. Thus, by minimality, this group is equal to $G$ and $p$ does not divide the order $n$ of $\alpha$. We find that every prime divisor of $n$ is also a prime divisor of $s$. Define, for any positive integer $n$ with this property, and every primitive $n$th root of unity $\zeta$, the group $H(\zeta)$ as generated by $\zeta \cdot (\zeta_0^0_{0 \ 1})$ and $G_a$. This group $H(\zeta)$ depends on the choice of the primitive $n$th root of unity $\zeta$. Further $\beta(H(\zeta)) = H$. The group $H(\zeta)$ is minimal since any proper subgroup of $H(\zeta)$, containing $G_a$, is contained in the group generated by $(\zeta \cdot (\zeta_0^0_{0 \ 1}))^p$ and $G_a$, where the prime $p$ divides $s$. The latter group does not map surjectively to $H$. Moreover we found $G \supset H(\zeta)$ for some $n$. Thus we found all minimal groups, namely the groups $H(\zeta)$.

**Proposition 5.** For $H = \gamma(F_1^k)$ the minimal groups are the $H(\zeta)$, generated by $\zeta \cdot (\zeta_0^0_{0 \ 1})$ and $\{\zeta_{a 1} \mid a \in \mathbb{C}\} = G_a$, where every prime divisor of the positive integer $n$ divides $k$ if $k$ is odd and divides $k/2$ if $k$ is even.

**Remark 2.** One has $H(\zeta) = G_a$ and the order of the cyclic group $H(\zeta)/H(\zeta)^0$ is the smallest common multiple of $n$ and $k$ (if $k$ odd) and that of $n$ and $k/2$ (if $k$ is even). Moreover, if $H(\zeta)$ is conjugated to $H_m$, then $n = m$. However the converse is not true in general.

#### 2.2.5. $H = \gamma(D_{\infty})$

Similarly to Section 2.2.4 one finds the following proposition:

**Proposition 6.** For $H = \gamma(F_2^1)$ the minimal groups are the cyclic groups generated by $\zeta \cdot (\zeta_0^0_{0 \ 1})$ where $n$ is a positive integer such that every prime divisor of $n$ is a prime divisor of $l$ if $l$ is odd or of $l/2$ if $l$ is even.

#### 2.3. $H = \gamma(D_{\infty})$

Let $G$ be minimal with $\beta(G) = H$. Then $G$ is a proper subgroup of $H_{\max} = \mathbb{C}^\ast \cdot D_{\infty}$. The component of the identity $G^0 \subset G$ has the form $\{(a_{0 \ 0}^0) \mid a_k b_l = 1\}$ for some $(k, l)$ with $\gcd(k, l) = 1$. Consider an element $B \in G$ with image (the class of) $\left(\begin{array}{cc}0 & 1 \\ 1 & 0 \end{array}\right) \in H$. Thus $B = \beta \cdot \left(\begin{array}{cc}0 & 1 \\ 1 & 0 \end{array}\right)$ for some $\beta \in \mathbb{C}^\ast$. From $BG^0 B^{-1} = G^0$ it follows that $k = l$ and thus $G^0 = \{(a_{0 \ 0}^0) \mid a b = 1\}$. By the minimality of $G$ one has that $B^2 = \beta^2$ is a root of unity. The subgroup of $G$, generated by $G^0$ and $B_k$, where $k$ is any odd integer, is also mapped surjectively to $H$. The minimality of $G$ implies that $\beta^2$ is a primitive $2^n$th root of unity for some $n \geq 0$. Let $H_n$ be the group generated by $\{(a_{0 \ 0}^0) \mid a b = 1\}$ and $B_n := \zeta_{2^{n+1}} \cdot \left(\begin{array}{cc}1 & 0 \\ 0 & 1 \end{array}\right)$. This group does not depend on the choice of $\zeta_{2^{n+1}}$ since one may replace $B_n$ by any odd power of $B_n$. Further $G \subset H_n$ for some $n$. The group $G$ must contain $\{(a_{0 \ 0}^0) \mid a b = 1\}$ and some element $\lambda \cdot \left(\begin{array}{cc}0 & 1 \\ 1 & 0 \end{array}\right)$. The latter element has the form $\left(\begin{array}{cc}0 & 1 \\ 0 & 1 \end{array}\right) \cdot (\zeta_{2^{n+1}} \cdot \left(\begin{array}{cc}0 & 1 \\ 1 & 0 \end{array}\right))^p$ with $a b = 1$ and $p \in \mathbb{Z}$. One concludes that $a = b = \pm 1$ and $p$ is odd. It follows that $G = H_n$ and we conclude: $\{H_n \mid n \geq 0\}$ is the collection of the minimal groups.
2.4. $H = D_n, A_4, S_4$ or $A_5$

We first note that if $H \subset \text{PGL}_2(\mathbb{C})$ is a finite subgroup, then every $H_{\text{min}} \subset \text{GL}_2(\mathbb{C})$ is also finite. Indeed, it is clear that $H^{\text{SL}_2}$ is finite. Because $H_{\text{min}} \not\subseteq \mathbb{C}^* \cdot H^{\text{SL}_2}$, we see that $H_{\text{min}}$ is finite.

2.4.1. $H = D_n$

We write $D_n = \langle a, b \mid a^n = b^2 = 1, ba = a^{-1}b \rangle \subset \text{PGL}_2(\mathbb{C})$.

(i) $n$ odd and $n \geq 3$. In this case, we may choose for $a$ and $b$ the images in $\text{PGL}_2(\mathbb{C})$ of the matrices $\begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with $\zeta_n$ a primitive $n$th root of unity.

Let $G$ be a minimal group. As $G$ is finite and generated by preimages of $a, b \in D_n$, one has that

$$G = \left\langle A = \lambda \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, B = \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

for certain roots of unity $\lambda, \mu$. We have $A^n = \lambda^n, B^2 = \mu^2, BA = \lambda^2 A^{-1} B$. Every element of $G$ has the form $tA^k, tA^kB, k = 0, 1, \ldots, n - 1$, with $t \in (\lambda^2, \lambda^n, \mu^2) = (\lambda, \mu^2)$. Hence $G \cap \mathbb{C}^* = (\lambda, \mu^2)$. Since both $\lambda(\zeta_n^{n-1}) \in G$ and $\lambda \in G$, we can write

$$G = \left\langle A = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, B = \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

The subgroup of $G$ generated by $A$ and $B^m$, where $m \geq 1$ is odd, also maps surjectively to $D_n$. By the minimality of $G$, this implies that the order of $\mu$ is $2^k$ for some $k \geq 0$. Now define

$$H_k := \left\langle \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \zeta_{2^k} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

for $k \geq 0$. This group $H_k$ does not depend on the choice of the primitive $2^k$th root of unity because one can replace the second generator by any odd power of itself. The groups $H_k$ are the only candidates for minimal groups.

We now show that $H_k$ is indeed minimal. For $k = 0, 1$, the groups

$$H_0 = \left\langle \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad H_1 = \left\langle \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

are minimal since they have order $2n$. The two groups are conjugated by the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. We note that $H_2 = D_n^{\text{SL}_2}$. For $k \geq 2$, we see that $H_k \cap \mathbb{C}^* = (\zeta_{2^k})$. Suppose that $D$ is a subgroup of $H_k$ which maps surjectively to $D_n$, then

$$D = \left\langle \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, t\zeta_{2^k} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

for some $t \in (\zeta_{2^k})$. Since the order of $t\zeta_{2^k}$ is also $2^k$, one has $D = H_k$ and thus $H_k$ is minimal. For $k \geq 1$, the order of $H_k$ is $2^k \cdot n$. Thus two minimal groups $H_k$ and $H_l$ with $k, l \geq 1$ are conjugated only if $k = l$. 292
(ii) $n$ even and $n > 2$. A minimal $G$ can be written as
\[ G = \begin{pmatrix} \lambda (\zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \quad B = \mu \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \]
for certain roots of unity $\lambda, \mu$. We have
\[ A^n = -\lambda^n, \quad B^2 = -\mu^2, \quad BA = \lambda^2 A^{-1} B. \]
As before, this implies that $G \cap \mathbb{C}^* = \langle \lambda^2, -\lambda^n, -\mu^2 \rangle = \langle -1, \lambda^2, \mu^2 \rangle$. One can replace $A$ and $B$ by $c_1 A$ and $c_2 B$ with $c_1, c_2 \in \langle -1, \lambda^2, \mu^2 \rangle$. For a good choice of $c_1, c_2$, the group $(c_1 A, c_2 B)$ will be a proper subgroup unless there exists an integer $N$ with $\lambda, \mu \in \mu_{2N}$. Thus the latter holds by the minimality of $G$. Then $\langle -1, \lambda, \mu \rangle = \mu_{2m+1}$ for some $m \geq 0$.

For $m = 0$, we have $G \cap \mathbb{C}^* = \mu_2$ and this leads to only one group, namely
\[ \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \rangle = D_{n \text{SL}_2}. \]
This group is clearly minimal. For $m \geq 1$, one has $G \cap \mathbb{C}^* = \mu_{2^m}$ and this leads to the three groups given by the table:

<table>
<thead>
<tr>
<th></th>
<th>$\lambda$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_{1,m}$</td>
<td>$\zeta_{2m+1}$</td>
<td>1</td>
</tr>
<tr>
<td>$H_{2,m}$</td>
<td>$\zeta_{2m+1}$</td>
<td>$\zeta_{2m+1}$</td>
</tr>
<tr>
<td>$H_{3,m}$</td>
<td>1</td>
<td>$\zeta_{2m+1}$</td>
</tr>
</tbody>
</table>

They all are minimal and have order $2^m \cdot 2$. However $H_{1,m}$ and $H_{2,m}$ are conjugated. Indeed, $(\zeta_{2n}^0 0) H_{1,m} (\zeta_{2n}^0 0) = H_{2,m}$ because
\[
\begin{pmatrix} \zeta_{2n} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \zeta_{2n}^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \zeta_{2m+1}^{-2} \cdot \begin{pmatrix} \zeta_{2m+1} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix} \cdot \begin{pmatrix} \zeta_{2m+1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} .
\]

(iii) $n = 2$. As in (ii). In this case also $H_{1,m}$ and $H_{3,m}$ are also conjugated, namely by a matrix of the form $\begin{pmatrix} 0 & a \\ 1 & -1 \end{pmatrix}$.

2.4.2. $H = A_4$
Let $G \subset H_{\text{max}} = \mathbb{C}^* \cdot A_4^{\text{SL}_2}$ be a minimal group. Consider $G^+ \subset \mathbb{C}^* \times A_4^{\text{SL}_2}$, the preimage of $G$ under the obvious map $\alpha: \mathbb{C}^* \times A_4^{\text{SL}_2} \to \mathbb{C}^* \cdot A_4^{\text{SL}_2}$. We note that the kernel of $\alpha$ is $\{(1, 0), (-1, 0)\}$. Since $\beta(G) = A_4$, there exists for every $a \in A_4^{\text{SL}_2}$ an element $(\lambda, a) \in G^+$. Let $\mu_k := \{\lambda \in \mathbb{C}^* | (\lambda, 1) \in G^+\}$. Then we obtain a homomorphism $h: A_4^{\text{SL}_2} \to \mathbb{C}^*/\mu_k$ given by $h(a) = \lambda \mod \mu_k$ if $(\lambda, a) \in G^+$. 293
This homomorphism factors as $A_4^{SL_2} \to C_3 \xrightarrow{h_1} C^*/\mu_k$, where $C_3 = \{1, \sigma, \sigma^2\}$ is the quotient of $A_4^{SL_2}$ by its commutator subgroup. If $h_1$ is trivial, then $G^+$ contains $\{(1, a) \mid a \in A_4^{SL_2}\}$ and by minimality $G = A_4^{SL_2}$. By Theorem 3, the latter group of order 24 is minimal.

Now we suppose that $h_1$ is not trivial. Write $k = 3^{r_1} \ell$ with $\gcd(\ell, 3) = 1$. For any $a \in A_4^{SL_2}$ there exists an element $(\lambda, a) \in G^+$ with $\lambda^3 \in \mu_{3^{r_1} \ell}$ and $\lambda$ can be multiplied by any element in $\mu_{3^{r_1} \ell}$. Thus there exist a pair $(\lambda, a) \in G^+$ with $\lambda \in \mu_{3^{r_1} \ell}$.

Now $G^+ \cap (\mu_{3^{r_1} \ell} \times A_4^{SL_2})$ is a subgroup of $G^+$ mapping surjectively to $A_4$. The minimality of $G$ implies that $\ell = 1$ and $G^+ \subset \mu_{3^{r_1} \ell} \times A_4^{SL_2}$. Moreover, $\mu_{3^{r_1} \ell} \subset G^+$ and the map $G^+ \to G$ is bijective. Then $G$ has the form $\mu_{3^{r_1} \ell} \cdot \{\delta(a)a \mid a \in A_4^{SL_2}\}$, where $\delta = A_4^{SL_2} \to C_3 \to A_4^{SL_2}$ for some map $\delta$ which lifts the homomorphism $h_1 : C_3 \to \mu_{3^{r_1} \ell} \subset C^*/\mu_{3^{r_1} \ell}$. There are two possibilities for nontrivial homomorphism $h_1$ (and thus for $\delta_1$ and $\delta$) and we find therefore two subgroups of $GL_2(\mathbb{C})$, lying in $\mu_{3^{r_1} \ell} \times A_4^{SL_2}$. The last group is contained in $\mu_{3^{r_1} \ell} \cdot \xi^{SL_2}$. Conjugation by an element $\tau \in S_4 \setminus A_4$ induced on $C_3 = A_4/[A_4, A_4]$ the only non trivial automorphism and permutes the two possibilities for $h_1$. One lifts $\tau$ to an element $\tau' \in S_4^{SL_2}$. Conjugation by $\tau'$ permutes the two possibilities for $h_1$ and therefore the above two groups are conjugated. It suffices to consider the group $H_r := \mu_{3^{r_1} \ell} \cdot \{\delta(a)a \mid a \in A_4^{SL_2}\}$ with $\delta_1$ given by $\delta_1(1) = 1$, $\delta_1(\sigma) = \zeta_{3^{r_1} + 1}$, $\delta_1(\sigma^2) = \zeta_{3^{r_1} + 1}^2$. The order of $H_r$ is $3^{r_1} \cdot 24$. The group $H_0$ is isomorphic to $A_4^{SL_2}$, but not conjugated to $A_4^{SL_2}$. The minimality of $H_0$ follows from the fact that $A_4$ does not have a faithful two-dimensional representation.

Finally, we will show that $H_r$ is minimal for $r \geq 1$. Suppose that $D$ is a subgroup of $H_r$ with $\beta(D) = A_4$. Let $\tau \in A_4^{SL_2}$ be an element of order 3. Then $D$ contains an element $d = \lambda \delta(\tau) \tau$ for some $\lambda \in \{\pm 1\} \times \mu_{3^{r_1} \ell}$. Now $\delta(\tau) \in \{\zeta_{3^{r_1} + 1}, \zeta_{3^{r_1} + 1}^2\}$ and $d^3 \in D \cap C^*$ has order $3^{r_1}$ or $2 \cdot 3^{r_1}$. Thus $D$ contains $\mu_{3^{r_1} \ell}$ and it follows that $D = H_r$. Thus we found:

**There are two minimal groups for $A_4$ with order 24 and for every $r \geq 1$ there is one minimal group of order $3^{r_1} \cdot 24$.**

**Remark 3.** A minimal subgroup $G$ for $H = A_4$ yields a central extension $1 \to \mu_k \to G \to A_4 \to 1$ for some $k$. The corresponding element $\xi$ of $H^2(A_4, \mu_k)$ has, by the minimality of $G$, the property that $\xi$ does not lie in the image of $H^2(A_4, \mu_d)$ for a proper divisor $d$ of $k$. Since the order of $A_4$ in 12, we only have to consider the groups $H^2(A_4, \mu_{3^{2r} \cdot b})$. The minimal groups that we found above correspond to all the cases $(a, b) = (1, r)$. The central extensions with $a \neq 1$ produce, apparently, groups which do not have a faithful representation of degree two.

### 2.4.3. $H = S_4$

Let $G \subset H_{\text{max}} = \mathbb{C}^* \cdot S_4^{SL_2}$ be a minimal group. Consider $G^+ \subset \mathbb{C}^* \times S_4^{SL_2}$, the preimage of $G$ under the obvious map $\alpha : \mathbb{C}^* \times S_4^{SL_2} \to \mathbb{C}^* \cdot S_4^{SL_2}$. The kernel of $\alpha$ is $\{(1, (1, 0)), (-1, (0, -1))\}$. Since $\beta(G) = S_4$, there exists for every $a \in S_4^{SL_2}$
an element \((\lambda, a) \in G^+\). Let \(\mu_k \coloneqq \{ \lambda \in \mathbb{C}^* \mid (\lambda, 1) \in G^+ \}\). Then we obtain a homomorphism \(h : S^{{SL}_2} \to \mathbb{C}^*/\mu_k\) given by \(h(a) = \lambda \mod \mu_k\) if \((\lambda, a) \in G^+\). This homomorphism factors as \(S^{{SL}_2} \to C_2 \xrightarrow{h_1} \mathbb{C}^*/\mu_k\), where \(C_2 = \{1, \sigma\}\) is the quotient of \(S^{{SL}_2}\) by its commutator subgroup. If \(h_1\) is trivial, then \(G^+\) contains \(\{(1, a) \mid a \in S^{{SL}_2}\}\) and by minimality \(G = S^{{SL}_2}\). According to Theorem 3, the latter group of order 48 is minimal.

Now we suppose that \(h_1\) is not trivial. Write \(k = 2\ell\) with \(\ell\) odd. For any \(a \in S^{{SL}_2}\) there exists an element \((\lambda, a) \in G^+\) with \(\lambda \in \mu_{2\ell+1}\). Now \(G^+ \cap (\mu_{2\ell+1} \times S^{{SL}_2})\) is a subgroup of \(G^+\) mapping surjectively to \(S_4\). The minimality of \(G\) implies that \(\ell = 1\) and \(G^+ \subseteq \mu_{2\ell+1} \times S^{{SL}_2}\). Define \(\delta : S^{{SL}_2} \to C_2 \xrightarrow{\delta_1} \{1, \zeta_{2\ell+1}\}\) by \(\delta_1(1) = 1\) and \(\delta_1(\sigma) = \zeta_{2\ell+1}\). All the elements of \(\mu_{2\ell+1}\) have the form \(\zeta_{2\ell+1}^e \cdot \lambda\) with \(e \in \{0, 1\}\) and \(\lambda \in \mu_{2\ell}\). From this it follows that \(G^+ = \{(\delta(a)\lambda, a) \mid a \in S^{{SL}_2}, \lambda \in \mu_{2\ell}\}\) and one concludes that \(G = H_r := \mu_{2\ell} \cdot \{\delta(a)a \mid a \in S^{{SL}_2}\}\). The group \(H_r\) has order \(2\ell \cdot 48\). We note that \(H_0\) is equal to \(S^{{SL}_2}\) and is minimal.

Let \(r > 0\) and let \(D \subseteq H_r\) be a subgroup satisfying \(\beta(D) = S_4\). Let \(\tau \in S^{{SL}_2}\) be an element with image the permutation \((1, 2) \in S_4\). Then \(D\) contains an element of the form \(d = \pm \lambda \delta(\tau) \tau\) with \(\lambda \in \mu_{2\ell}\). Then \(d^2 = \lambda^2 \zeta_{2\ell} \in D \cap \mathbb{C}^*\) has order \(2\ell\). Thus \(D\) contains \(\mu_{2\ell}\) and it follows easily that \(D = H_r\). Hence every \(H_r\) is minimal and we conclude that: There is for every \(r \geq 0\) a unique minimal group of order \(2\ell \cdot 48\).

2.4.4. \(H = A_5\)

Let \(G \subseteq \text{GL}_2(\mathbb{C})\) be a minimal for \(H\). Since \(A_5 = [A_5, A_5]\), the group \([G, G]\) also satisfies \(\beta([G, G]) = H\). By minimality \(G = [G, G]\) and thus \(G \subseteq \text{SL}_2(\mathbb{C})\). This implies that \(G \subseteq A_5^{{SL}_2}\). Since, by Theorem 3, the latter group is minimal, we find that \(A_5^{{SL}_2}\) is the only minimal group.

In summary, we obtain the following result.

Theorem 4. The list of all minimal groups, up to conjugation, for each algebraic subgroup \(H \subseteq \text{PGL}_2(\mathbb{C})\) (see Theorem 1 and Section 2.2) is:

(1) \(H = \text{PGL}_2(\mathbb{C})\): the only minimal group is \(\text{SL}_2(\mathbb{C})\).

(2) \(H\) is a subgroup of the group \(B\):
   (a) \(H = \gamma(B)\): for each pair of integers \((k, l)\) with \(k + l \neq 0\) and gcd\((k, l) = 1\) there is a minimal one, namely \(\{(a^b_0) \mid a^b e^c l = 1\}\).
   (b) \(H = \gamma(G_m)\): for each pair of integers \((k, l)\) with \(k + l \neq 0\) and gcd\((k, l) = 1\) there is a minimal group, namely \(\{(a^0_b) \mid a^b b^l = 1\}\). Further \((k, l)\) and \((l, k)\) define conjugated groups.
   (c) \(H = \gamma(G_a)\): there is only one minimal group, namely \(G_a\).
   (d) \(H = \gamma(F^k)\): the minimal ones are the \(H(\zeta_n)\), generated by \(\zeta_n \cdot (\zeta_n^2 0)\) and \((1^a 0^b) \mid a \in \mathbb{C}\), where every prime divisor of the positive integer \(n\) divides \(k\) if \(k\) is odd and divides \(k/2\) if \(k\) is even.

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(e) \( H = \gamma(F_2^n) \): the minimal ones are the groups generated by \( \zeta_n \cdot \begin{pmatrix} \zeta_n^2 & 0 \\ 0 & 1 \end{pmatrix} \), where every prime divisor of the positive integer \( n \) divides \( l \) if \( l \) is odd and divides \( l/2 \) if \( l \) is even.

\( H = \gamma(D_{\infty}) \): the minimal groups are \( H_n \) with \( n \geq 0 \), where \( H_n \) is generated by \({\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} | ab = 1}\) and \( \zeta_{2n+1} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) with \( \zeta_{2n+1} \) a primitive \( 2^{n+1} \)th root of unity.

(4) \( H \) finite:

(a) \( H = D_n \):

(i) \( n \geq 3 \) odd: For every \( k \geq 1 \), there is one minimal group

\[ \left\{ \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \zeta_{2k} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \]

with \( \zeta_{2k} \) a primitive \( 2^k \)th root of unity;

(ii) \( n > 2 \) even: For \( k \geq 1 \), the minimal ones \( H_{1,k}, H_{2,k}, H_{3,k} \) have the form

\[ \begin{align*}
A &= \lambda \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \\
B &= \mu \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
\end{align*} \]

for certain roots of unity \( \lambda, \mu \) which are given in the table:

<table>
<thead>
<tr>
<th></th>
<th>( \lambda )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_{1,k} )</td>
<td>( \zeta_{2k+1} )</td>
<td>1</td>
</tr>
<tr>
<td>( H_{2,k} )</td>
<td>( \zeta_{2k+1} )</td>
<td>( \zeta_{2k+1} )</td>
</tr>
<tr>
<td>( H_{3,k} )</td>
<td>1</td>
<td>( \zeta_{2k+1} )</td>
</tr>
</tbody>
</table>

They all have order \( 2^k \cdot 2n \). Further \( H_{1,k} \) and \( H_{2,k} \) are conjugated.

For \( k = 0 \), there is only minimal group, namely the group

\[ \left\{ \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\} = D_n^{SL_2} \text{ of order } 4n; \]

(iii) \( n = 2 \): As in (ii), but now \( H_{1,k}, H_{2,k}, H_{3,k} \) are all conjugated.

(b) \( H = A_4 \): there are two minimal groups of order 24. For every \( n > 0 \) there is one minimal group of order \( 3^n \cdot 24 \).

(c) \( H = S_4 \): For every \( n \geq 0 \) there is a minimal group of order \( 2^n \cdot 48 \).

(d) \( H = A_5 \): There is only minimal group, namely \( A_5^{SL_2} \).

REFERENCES


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