A note on nonlinear $H_\infty$ control of two-block interconnected systems

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Abstract

In this paper nonlinear $H_\infty$ control for a class of two-block interconnected systems is investigated. The situation where the regular nonlinear $H_\infty$ suboptimal control problem is solvable for one of the blocks is considered. An auxiliary nonlinear system is defined from the original two-block system, and it is shown that a feedback solution to the nonlinear $H_\infty$ suboptimal control problem for this auxiliary system, also applies to the nonlinear $H_\infty$ suboptimal control problem for the original two-block system. The advantage of the auxiliary problem to the original problem is that the auxiliary penalty variable has lower dimension than the original penalty variable.

1. Introduction

In [1] decomposition ideas from linear theory were used to solve the (singular) nonlinear $H_\infty$ problem. Also in [2] a decomposition idea was used. In this paper we will show that the results obtained in [3] for the totally singular case, can be extended to a more general situation.

The system under consideration is an affine nonlinear system, which we will denote by $\Sigma$, given by state space equations of the form

\[ \dot{z} = f(x) + g(x)u + k(x)d, \quad f(x_0) = 0 \]
\[ z = h(x,u), \quad h(x_0,0) = 0 \]  \hspace{1cm} (1)

where $x = [x^1, \ldots, x^n]^T$ are local coordinates for an $n$-dimensional state space manifold $X$, $u \in \mathbb{R}^m$ is the control input, $d \in \mathbb{R}^r$ the disturbance input and $z \in \mathbb{R}^s$ the penalty variable. $f$, $g$, $k$ and $h$ are all $C^k$ with $k \geq 2$. We will assume that there exist local coordinates $z = [z^1, \ldots, z^r]^T$, $1 \leq q < n$, such that $f$, $g$, $h$ and $k$ take the form

\[ f(z) = \begin{bmatrix} f_1(x_1) + g_1(x_1)z_2 \\ f_2(x_1, x_2) \end{bmatrix}, \quad g(z) = \begin{bmatrix} 0 \\ g_2(x_1, x_2) \end{bmatrix} \]

\[ k(z) = \begin{bmatrix} k_1(x_1) \\ k_2(x_1, x_2) \end{bmatrix}, \quad h(z, u) = \begin{bmatrix} h_1(x_1, x_2) \\ h_2(x_1, x_2) \\ h_3(x_1, x_2, u) \end{bmatrix} \]

for $x_1 = [x^1, \ldots, x^q]^T$ and $x_2 = [x^{q+1}, \ldots, x^n]^T$. Moreover, $z_1 \in \mathbb{R}^{r_1}$, $z_2 \in \mathbb{R}^{r_2}$ and $z_3 \in \mathbb{R}^{r_3}$, where $s_1 \geq 1$ and $s = s_1 + s_2 + s_3$. Thus, $\Sigma$ can be viewed as a two-block interconnected system where the system blocks are two general nonlinear systems: a system $\Sigma_1$ where $x_1$ is the state, $z_1$ is the output and $(z_2, d)$ are the inputs and a system $\Sigma_2$ where $x_2$ is the state, $z_2$ is the output and $(u, x_1, d)$ are the inputs.

We will now state the problem that we want to consider in this paper.

Problem 1.1 Let $\gamma$ be a fixed positive constant. Solve the state feedback nonlinear $H_\infty$ suboptimal control problem associated with the system $\Sigma$, i.e. find a nonlinear static state feedback

\[ u = \alpha(x), \quad \alpha(x_0) = 0 \]  \hspace{1cm} (2)

such that the closed loop system (1), (2) has $L_2$-gain $\leq \gamma$ from $d$ to $z$.

Now, consider the system $\Sigma$ and define the pre-Hamiltonian $K$: $T^*X \times \mathbb{R}^m \times \mathbb{R}^r \rightarrow \mathbb{R}$ as

\[ K(x,p,u,d) = p^T(f(x) + g(x)u + k(x)d) \]
\[ - \frac{1}{2} \gamma^2 p^T d + \frac{1}{2} z^T z \]  \hspace{1cm} (3)

where $z = [x^1, \ldots, x^n]^T$, $p = [p^1, \ldots, p^n]^T$ are natural coordinates for the cotangent bundle $T^*X$. For the sake of completeness, we will write down the following well-known result (cf. [4]).

Proposition 1.1 Consider the closed loop system (1), (2). Suppose that there exists a non-negative $C^1$ solution $V: X \rightarrow \mathbb{R}^+$ to the differential dissipation inequality

\[ K(x, V_z(x), \alpha(x), d) \leq 0, \quad V(x_0) = 0, \quad \text{for all } d \in \mathbb{R}^r \]  \hspace{1cm} (4)

Then the state feedback (2) solves Problem 1.1. Moreover, if the closed loop system (1), (2) is zero-state observable with $x_0$ as the zero-state (i.e. $z(t) \equiv 0$, $d(t) \equiv 0$ implies that $z(t) \equiv x_0$), then $V(x) > 0$ for all $x \neq x_0$, and $x_0$ is a locally asymptotically stable equilibrium when $d(t) \equiv 0$. If $V(x)$ is proper, then $x_0$ is a globally asymptotically stable equilibrium when $d(t) \equiv 0$.

2. Main Results

For the derivation of our main results in this section the following assumption will be instrumental.

Assumption 2.1 Consider the two-block interconnected system $\Sigma$ given by (1). Let $X_1$ be the submanifold of $X$ with local coordinates $[x^1, \ldots, x^q]^T$. Then there exists a non-negative $C^0 \ (k \geq v \geq 2)$ solution $P : X_1 \rightarrow \mathbb{R}^+$ to the Hamilton-Jacobi inequality

\[ P_{x_1}(x_1) f_1(x_1) + \]
\[ \frac{1}{2} P_{x_1}(x_1) \left[ \frac{1}{\gamma^2} k_1(x_1) k_1^T(x_1) - g_1(x_1) g_1^T(x_1) \right] P_{x_1}(x_1) \]
\[ + \frac{1}{2} h_1^T(x_1) h_1(x_1) \leq 0, \quad P(x_1,0) = 0 \]  \hspace{1cm} (5)
where \(z_0 = [x_1^T, x_2^T]^T\).

In consequence of Assumption 2.1, the \(\text{regular} \ \mathcal{H}_\infty\) suboptimal control problem associated with the subsystem

\[
\dot{z}_1 = f_1(x_1) + g_1(x_1)z_2 + k_1(x_1)d, \quad f_1(x_1, 0) = 0
\]

\[z = \left[ \begin{array}{c} h_1(x_1) \\ z_2 \end{array} \right], \quad h_1(x_1, 0) = 0 \tag{6}
\]

is solvable when \(z_2\) is regarded as the control input. The \(C^{\alpha-1}\) state feedback solution is then given by [4]

\[z^*_2 = g_1^T(x_1)p_{z_1}^+(x_1) \tag{7}
\]

Motivated by the results in [1], we will now use this fact to define an auxiliary nonlinear system from the original two-block interconnected system \(\Sigma\). The auxiliary nonlinear system, which we will denote by \(\tilde{\Sigma}\), is again an affine nonlinear system given by state space equations of the form

\[
\tilde{\Sigma} : \begin{cases}
\dot{z} = \tilde{f}(x) + g(x)u + k(x)d, \quad \tilde{f}(x_0) = 0 \\
\tilde{z}_0 = h(x, u), \quad h(x_0, 0) = 0
\end{cases} \tag{8}
\]

where

\[
d = d - d^*, \quad d^* = \frac{1}{\gamma^2}k_1^T(x_1)p_{z_1}^+(x_1) \tag{9}
\]

\[
\tilde{f}(x) = \left[ \begin{array}{c} f_1(x_1) + g_1(x_1)z_2 + k_1(x_1)d^* \\ f_2(x_1, x_2) + k_2(x_1, x_2)d^* \end{array} \right] \tag{10}
\]

\[
\tilde{h}(x, u) = \left[ \begin{array}{c} h_2(x_1, x_2) + g_2^T(x_1)p_{z_1}^+(x_1) \\ h_3(x_1, x_2, u) \end{array} \right] \tag{11}
\]

Using the Hamilton-Jacobi inequality (5) the following lemma can be easily proven.

**Lemma 2.1** Let \(z \in \mathbb{R}^n\) and \(\tilde{z} \in \mathbb{R}^{n-z_1}\) denote the penalty variable of \(\Sigma\) and \(\tilde{\Sigma}\) respectively. Then the following inequality is satisfied

\[
\frac{1}{2}d^T \tilde{z} \geq p_{z_1}(x_1)(f_1(x_1) + g_1(x_1)z_2)
\]

\[
+ \frac{1}{2\gamma^2}\|k_1^T(x_1)p_{z_1}^+(x_1)\|^2 + \frac{1}{2}z^Tz \tag{12}
\]

Our main result relies on the next lemma.

**Lemma 2.2** Let \(K_\gamma\) and \(\tilde{K}_\gamma\) denote the pre-Hamiltonian of \(\Sigma\) and \(\tilde{\Sigma}\) respectively. Then it follows that

\[
K_\gamma(x, p + p_{z_1}^+(x_1), u, d) \leq \tilde{K}_\gamma(x, p, u, d) \tag{13}
\]

**Proof:** Using Lemma 2.1 and the fact that \(\tilde{f}(x) + k(x)d = f(x) + k(x)d\) it follows that

\[
\tilde{K}_\gamma(x, p, u, d) \geq p^T(\tilde{f}(x) + g(x)u + k(x)d)
\]

\[
+ p_{z_1}(x_1)(f_1(x_1) + g_1(x_1)z_2) + \frac{1}{2\gamma^2}\|k_1^T(x_1)p_{z_1}^+(x_1)\|^2
\]

\[
+ \frac{1}{2}z^Tz - \frac{1}{2}\gamma^2d^Td \tag{14}
\]

Moreover, from (9) it follows that

\[
\frac{1}{2}\gamma^2d^Td = \frac{1}{2}\gamma^2d^Td + \frac{1}{2\gamma^2}\|k_1^T(x_1)p_{z_1}^+(x_1)\|^2
\]

\[
- p_{z_1}(x_1)k_1(x_1)d \tag{15}
\]

Inserting (15) in (14) then gives (13) since \(p_{z_1}(x_1) = 0\).

We are now ready to state our main result which is an immediate consequence of the last lemma.

**Theorem 2.1** Suppose that the static state feedback

\[u = \alpha(x), \quad \alpha(x_0) = 0 \tag{16}\]

solves the nonlinear \(\mathcal{H}_\infty\) suboptimal control problem for the auxiliary system \(\tilde{\Sigma}\) in the sense that there exists a non-negative \(C^1\) solution \(W : \mathcal{X} \rightarrow \mathbb{R}^+\) to the differential dissipation inequality

\[K_\gamma(x, W^T(x), \alpha(x), d) \leq 0, \quad W(x_0) = 0, \quad \text{for all } d \in \mathbb{R}^n \tag{17}\]

Then the state feedback (16) also solves Problem 1.1 and the differential dissipation inequality

\[K_\gamma(x, \tilde{W}^T(x), \alpha(x), d) \leq 0, \quad \tilde{W}(x_0) = 0, \quad \text{for all } d \in \mathbb{R}^n \tag{18}\]

holds for the non-negative \(C^1\) function \(V : \mathcal{X} \rightarrow \mathbb{R}^+\) given by \(V = W + P\).

**Remark 2.1** It is easy to see that systems that admit the following decomposition

\[
\dot{z}_1 = f_1(x_1) + g_1(x_1)z_2 + k_1(x_1)d
\]

\[
\dot{z}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)z_3 + k_2(x_1, x_2)d
\]

\[
\vdots
\]

\[
\dot{z}_p = f_p(x_1, \ldots, x_p) + g_p(x_1, \ldots, x_p)u + k_p(x_1, \ldots, x_p)d
\]

\[
z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_p \\ z_{p+1} \end{bmatrix} = \begin{bmatrix} h_1(x_1) \\ h_2(x_1, x_2) \\ \vdots \\ h_p(x_1, \ldots, x_p) \\ h_{p+1}(x_1, \ldots, x_p, u) \end{bmatrix}
\]

are candidates for recursive use of the results in Theorem 2.1.

**References**


