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A Hamiltonian approach to stabilization of nonholonomic mechanical systems

B.M. Maschke* A.J. van der Schaft

Abstract
A simple procedure is provided to write the equations of motion of controlled mechanical systems with constraints as Hamiltonian equations with respect to a "Poisson" bracket which does not necessarily satisfy the Jacobi-identity. Based on the Hamiltonian form a stabilization procedure is proposed.

1 Introduction
In a recent paper we have shown that (uncontrolled) mechanical systems with classical constraints can be written as Hamiltonian equations of motion with respect to a generalized type of Poisson bracket, and with respect to a Hamiltonian which is obtained by restricting the internal energy to the constrained state space. This bracket does not necessarily satisfy the Jacobi-identity, which is one of the defining properties of a true Poisson bracket. In fact, the Jacobi-identity is satisfied if and only if the constraints are holonomic. This work was motivated by a paper of Bates & Shanky [3] on the Hamiltonian formulation of nonholonomic systems, as well as by our previous work on the Hamiltonian formulation of non-resistive physical systems by network modelling [8], [9].

In the present paper we extend this set-up to controlled nonholonomic mechanical systems. Furthermore we show how the Hamiltonian form of the equations (the Jacobi-identity being satisfied or not) may be used for stabilization purposes. Indeed we show how the stabilization procedure for standard Hamiltonian control systems as proposed in [14], [11], see also [5], can be extended to this case. These considerations were very much motivated by the papers [2], [4] on stabilization of controlled nonholonomic systems. We close with our treatment of two well-known simple examples of nonholonomic systems, discussed before in [2].

2 The Hamiltonian formulation of systems with constraints
Let Q be an n-dimensional configuration manifold with local coordinates \( q = (q_1, \ldots, q_n) \). Consider a smooth Lagrangian function \( L : TQ \to \mathbb{R} \), denoted by \( L(q, \dot{q}) \), satisfying throughout the usual regularity condition

\[
\mathfrak{d}c t \left[ \frac{\partial L}{\partial \dot{q}_i \partial \dot{q}_j} \right] \neq 0
\]

The equations of motion for the mechanical system on \( Q \) with Lagrangian \( L(q, \dot{q}) \) and constraints (2) are given as (see e.g. [10], [19], [1])

\[
\dot{q}_i - \frac{\partial H}{\partial \dot{q}_i} = \frac{\partial}{\partial q_i} (A(q) \lambda + B(q)u),
\]

where \( B(q)u \) are the external forces (controls) applied to the system, with \( B(q) \) an \( n \times m \) matrix with entries depending smoothly on \( q \). Here \( \frac{\partial}{\partial q_i} \) denotes the column vector \( \left( \frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_n} \right)^T \), and similarly for \( \frac{\partial}{\partial \dot{q}_i} \) and subsequent expressions. The constraint forces \( A(q(t))\lambda(t) \) are determined by the requirement that the constraints \( A^T(q(t))\dot{q}(t) = 0 \) have to be satisfied for all \( t \).

Defining in the usual way the Hamiltonian \( H(q, p) \) by the Legendre transformation

\[
H(q, p) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q, \dot{q}), \quad p_i = \frac{\partial L}{\partial \dot{q}_i},
\]
the constrained Euler-Lagrange equations (6) transform into the constrained Hamiltonian equations on $T^*Q$

\[
\dot{q} = \frac{\partial H}{\partial p}(q, p)
\]

\[
\dot{p} = -\frac{\partial H}{\partial q}(q, p) + A(q)\lambda + B(q)u
\]

\[
A^T(q)\frac{\partial H}{\partial p}(q, p) = A^T(q)\dot{q} = 0
\]

An intrinsic definition of the constrained Hamiltonian equations may be given as follows. The cotangent bundle $T^*Q$ is equipped with its canonical Poisson bracket $\{,\}$, in natural coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_m)$ for $T^*Q$ expressed as (with $F$ and $G$ smooth functions on $T^*Q$)

\[
\{F, G\}(q, p) = \sum_{i=1}^m \left( \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right)(q, p) =
\]

with $J$ the standard Poisson structure matrix. Recall that for any smooth function $H: T^*Q \to \mathbb{R}$ its Hamiltonian vectorfield $\mathcal{L}_H$ on $T^*Q$ is defined in the local coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_m)$ as

\[
\mathcal{L}_H = \left( \begin{array}{c} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_n} \end{array} \right), \quad J = \left( \begin{array}{cc} 0_n & I_n \\ -I_n & 0_n \end{array} \right),
\]

Similarly, for any one-form $\alpha$ on $T^*Q$ we may define the "Hamiltonian" vectorfield $\mathcal{L}_\alpha$ as

\[
\mathcal{L}_\alpha = \left( \begin{array}{c} \alpha_1(q, p) \\ \vdots \\ \alpha_m(q, p) \end{array} \right), \quad J = \left( \begin{array}{cc} 0_m & I_m \\ -I_m & 0_m \end{array} \right)
\]

where $(\alpha_1(q, p), \cdots, \alpha_m(q, p))$ is the local coordinate expression of the one-form $\alpha$. (Note that $Z_{2n} H = X_{H_2}$.) Now the columns of $A(q)$ define in local coordinates $k$ one-forms $\alpha^1, \cdots, \alpha^k$ on $Q$. Similarly, the columns of $B(q)$ define $m$ one-forms $\beta^1, \cdots, \beta^m$ on $Q$. Since any one-form on $Q$ may be regarded as a one-form on $T^*Q$, we can thus define the vectorfields $Z_{\alpha^i}, Z_{\beta^j}, \cdots, Z_{\beta^m}$ on $T^*Q$. It can now be readily seen that a coordinate-free description of the first part of (8) is given as (see also [2])

\[
\dot{x} = X_{H_2}(x) + a(x)\lambda + b(x)u, \quad x \in T^*Q,
\]

where $a(x)$ is the matrix with columns $Z_{\alpha^1}, \cdots, Z_{\alpha^k}$, and $b(x)$ is the matrix with columns $Z_{\beta^1}, \cdots, Z_{\beta^m}$.

The Lagrange multipliers $\lambda$ may be computed by differentiating $A^T(q)\frac{\partial H}{\partial p}(q, p) = 0$ along (8), i.e.

\[
\left[ A^T(q)\frac{\partial H}{\partial p}(q, p) \right]^T \frac{\partial H}{\partial q}(q, p) + A^T(q)\frac{\partial^2 H}{\partial p^2}(q, p)\cdot
\]

\[
-\frac{\partial H}{\partial p}(q, p) + B(q)u + A^T(q)\frac{\partial^2 H}{\partial p^2}(q, p)A(q)\lambda = 0
\]

with $\frac{\partial^2 H}{\partial p^2}$ the Hessian matrix with respect to $p$. This equation may be solved for $\lambda$ (as function of $q, p, u$) as long as

\[
\det A^T(q)\frac{\partial^2 H}{\partial p^2}(q, p)A(q)\neq 0, \quad (q, p) \in T^*Q,
\]

which condition is obviously satisfied because of our standing assumptions (1) and rank $A(q) = k$. Expressing $\lambda$ as a function of $(q, p, u)$ and substituting in (8) then leads to the dynamical equations of motion on the constrained state space

\[
X_r = \{(q, p, u) \in T^*Q | A^T(q)\frac{\partial H}{\partial p}(q, p) = 0\}
\]

As shown in [15] a much more efficient and insightful way of obtaining the equations of motion on $X_\lambda$ is however the following. Since rank $A(q) = k$, there exists locally a smooth $n \times (n-k)$ matrix $S(q)$ of rank $n-k$ such that

\[
A^T(q)S(q) = 0
\]

(Equivalently, $S(q)$ is such that $D(q) = \text{Im}(S(q))$. Now define $\dot{\tilde{p}} = (\tilde{p}^1, \tilde{p}^2) = (p_1, \cdots, p_{n-k}, \tilde{p}_{n+1}, \cdots, \tilde{p}_n)$ as

\[
\tilde{p}^1 := S^T(q)p, \quad \tilde{p}^2 \in \mathbb{R}^{n-k}
\]

It immediately follows from (16) that $(q, p) \mapsto (\tilde{q}, \tilde{p}^1, \tilde{p}^2)$ is a coordinate transformation. The constrained Hamiltonian dynamics (8) in the new coordinates $(\tilde{q}, \tilde{p}^1, \tilde{p}^2)$ take the following form. In the new coordinates $(\tilde{q}, \tilde{p})$ the Poisson structure matrix transforms from (9) into

\[
\tilde{J}(\tilde{q}, \tilde{p}) = \left( \begin{array}{cc} \{ (\tilde{q}_i, \tilde{q}_j) \}_{i,j} & \{ (\tilde{q}_i, \tilde{p}_j) \}_{i,j} \\ \{ (\tilde{p}_i, \tilde{q}_j) \}_{i,j} & \{ (\tilde{p}_i, \tilde{p}_j) \}_{i,j} \end{array} \right), \quad i,j = 1, \cdots, n
\]

and the constrained Hamiltonian dynamics (8) transform into

\[
\begin{bmatrix} \dot{\tilde{q}} \\ \dot{\tilde{p}} \\ \dot{\tilde{p}}^2 \end{bmatrix} = \tilde{J}(\tilde{q}, \tilde{p}) \begin{bmatrix} \frac{\partial \tilde{H}}{\partial \tilde{q}} \\ \frac{\partial \tilde{H}}{\partial \tilde{p}} \end{bmatrix} + \begin{bmatrix} 0 \\ \lambda + B(q)u \end{bmatrix} \tilde{\Lambda}(q) \tilde{\hat{H}}(q)
\]

\[
A^T(q)\frac{\partial \tilde{H}}{\partial \tilde{p}} = 0
\]

with $\tilde{\Lambda}(q) := A^T(q)A(q)$ an invertible matrix, and $\tilde{\hat{H}}(q, \tilde{p})$ the Hamiltonian $H(q, p)$ expressed in the new coordinates $\tilde{q}, \tilde{p}$. Now truncate the transformed Poisson structure matrix $\tilde{J}$ in (18) by leaving out the last $k$ columns and last $k$ rows, and let $\tilde{\dot{p}}$ satisfy the constraint equation $\frac{\partial \tilde{H}}{\partial \tilde{p}} = 0$. This defines a $(2n-k) \times (2n-k)$ skew-symmetric matrix $J_r$ on $X_r$. An explicit expression for $J_r$ is obtained as follows [15]. Denote the $i$-th column of $\tilde{S}(q)$ by $S_i(q)$, then

\[
J_r = \left( \begin{array}{ccc} 0_n & S(q) & -S^T(q) \\ -S^T(q) & (S_i(q))_{i=1, \cdots, n} \end{array} \right)
\]

where $p$ is expressed as function of $q, \tilde{p}$, with $\tilde{p}$ satisfying $\frac{\partial \tilde{H}}{\partial \tilde{p}} = 0$. Note that rank $J_r = 2(n-k)$ everywhere on $X_r$. Furthermore, define the reduced Hamiltonian $H_r : X_r \to \mathbb{R}$ as $\tilde{H}(q, \tilde{p})$ with $\tilde{p}$ satisfying $\frac{\partial \tilde{H}}{\partial \tilde{p}} = 0$. 

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Clearly, \((q, ̇p)\) serve as local coordinates for \(X\). It immediately follows from (19) by disregarding the last equations involving \(λ\) and noting that \(\frac{∂}{∂q}(p, 0) = 0\) that the dynamics on \(X\) in coordinates \((q, ̇p)\) are described as

\[
\begin{pmatrix}
  \dot{q} \\
  ̇p
\end{pmatrix} = J(q, ̇p) \begin{pmatrix}
  \frac{∂F_2}{∂q}(q, ̇p) \\
  \frac{∂F_2}{∂p}(q, ̇p)
\end{pmatrix} + \begin{pmatrix}
  0 \\
  B(q)
\end{pmatrix} u \tag{21}
\]

These equations are in Hamiltonian format! Indeed, the matrix \(J\) defines a bracket \(\{\cdot, \cdot\}\) on \(X\) by setting

\[
\{F_r, G_s\}(q, ̇p) := \left( \frac{∂F_r}{∂q}(q, ̇p) \right) J(q, ̇p) \left( \frac{∂G_s}{∂p}(q, ̇p) \right)
\]

for any two smooth functions \(F_r, G_s : X \to \mathbb{R}\). Clearly, this bracket satisfies the first two defining properties of a Poisson bracket (see e.g. [6],[12],[7]),

(i) \(\{F_r, G_s\} = -\{G_s, F_r\}\) (skew-symmetry)

(ii) \(\{F_r, [G_s, H_t]\} = \left\{ \{F_r, G_s\}, H_t \right\} + \left\{ G_s, \{F_r, H_t\} \right\}\) (Leibniz rule)

for every \(F_r, G_s, H_t : X \to \mathbb{R}\). However, for \(\{\cdot, \cdot\}\), to be a Poisson bracket also the following property

(iii) \(\{F_r, [G_s, H_t]\} + \left\{ G_s, \{F_r, H_t\} \right\} = 0\) (Jacobi-identity)

needs to be satisfied. If (24) is satisfied then (21) with \(u = 0\) defines a generalized Hamiltonian system with respect to a Poisson bracket, see e.g. [12],[7],[8]. In this case local coordinates \((q, ̇p, ̇q)\) for \(X\) may be found such that the system for \(u = 0\) takes the form [6],[12],[8]

\[
\begin{pmatrix}
  \dot{q} \\
  ̇p \\
  ̇q
\end{pmatrix} = \begin{pmatrix}
  \frac{∂F_2}{∂q}(q, ̇p) \\
  \frac{∂F_2}{∂p}(q, ̇p) \\
  \frac{∂F_2}{∂q}(q, ̇p)
\end{pmatrix} + \begin{pmatrix}
  0 \\
  B(q)
\end{pmatrix} ̇q \tag{25}
\]

However, in [15] it has been shown that \(\{\cdot, \cdot\}\) satisfies the Jacobi-identity (and thus is a true Poisson bracket) if and only if the constraints \(S^T(q) \dot{q} = 0\) are holonomic! This underscores the difficulties of nonholonomic constraints. On the other hand, even if the Jacobi-identity is not satisfied (as in the case for nonholonomic systems), the (pseudo-)Hamiltonian format (21) may still be useful, as we wish to indicate in the next section.

Note that our approach is not unrelated to the approach taken in [4]. Here the Lagrange multipliers \(λ\) in the Euler-Lagrange equations (6) are eliminated by premultiplying the equations (6) by the matrix \(S^T(q)\), and it is shown that the thus reduced equations can be written as a set of first-order differential equations in \(q\) and \(p \in \mathbb{R}^{n-k}\) with \(q = S(q)\dot{q}\) parametrizing the admissible velocities \(\dot{q}\). This can be regarded as the “Lagrangian counterpart” of our Hamiltonian approach.

### 3 Stabilization

We note that the dynamics (21) are energy preserving. In fact, by skew-symmetry of \(J\), we immediately obtain

\[
\frac{d}{dt} H_r = \frac{∂H_r}{∂p}(q, ̇p) B_r(q) u \tag{26}
\]

with \(\frac{∂}{dt}\) denoting differentiation along (21). Suppose now that \((q_0, ̇p_0)\) is a stationary point of the Hamiltonian \(H_r\), i.e. \(\frac{∂H_r}{∂q}(q_0, ̇p_0) = 0\), \(\frac{∂H_r}{∂p}(q_0, ̇p_0) = 0\), implying that \((q_0, ̇p_0)\) is an equilibrium of the uncontrolled constrained dynamics \((u = 0)\)

\[
\begin{pmatrix}
  \dot{q} \\
  ̇p
\end{pmatrix} = J(q, ̇p) \begin{pmatrix}
  \frac{∂F_2}{∂q}(q, ̇p) \\
  \frac{∂F_2}{∂p}(q, ̇p)
\end{pmatrix}
\]

(27)

If \(H_r\) happens to have a strict minimum in \((q_0, ̇p_0)\), then it follows from (26) with \(u = 0\) that \((q_0, ̇p_0)\) is a Lyapunov stable equilibrium of (27). On the other hand, as in the case of ordinary Hamiltonian control systems (see e.g. [14],[11]), equation (26) suggests for improved stabilization the smooth state feedback

\[
u = -B_r(q, ̇p) \frac{∂H_r}{∂p}(q, ̇p) \tag{28}
\]

which results in the monotonous energy decrease

\[
\frac{d}{dt} H_r = -\frac{∂H_r}{∂p}(q, ̇p) B_r(q) \frac{∂H_r}{∂p}(q, ̇p) \leq 0 \tag{29}
\]

(29)

(Note that (28) can be written as \(u = -γp\) with the conjugated effort corresponding to the generalizd flow \(u\) [8].) If \(H_r\) has a strict minimum in \((q_0, ̇p_0)\), then \((q_0, ̇p_0)\) will thus be at least a Lyapunov stable equilibrium of the closed-loop system (21), (28), and moreover the trajectories will converge to the largest invariant (with respect to (27)) set contained in

\[
\{ (q, ̇p) \in X \mid \frac{∂H_r}{∂p}(q, ̇p) B_r(q) = 0 \} \tag{30}
\]

However it can be shown, as in [2],[4], that (21) does not satisfy Brockett’s necessary condition, and thus cannot be asymptotically stabilized by a smooth state feedback. Hence this largest invariant set will be always larger than the singleton \(\{(q_0, ̇p_0)\}\).

If \(H_r\) does not have a strict minimum in \((q_0, ̇p_0)\) then, as in the case of ordinary Hamiltonian control systems ([11],[11]), we may try to shape by preliminary feedback the internal energy \(E_r\), if possible, to a function which does have a strict minimum in \((q_0, ̇p_0)\). Indeed, let \(H_r\) be of the form, as usually encountered in applications,

\[
H_r(q, ̇p) = V(q) + \frac{1}{2}p^T G(q)p \tag{31}
\]

(potential energy plus kinetic energy). Necessarily \(p_0 = 0\), and \(\frac{∂}{∂q}(q_0) = 0\). Now consider the equation

\[
- S^T(q) \frac{∂V}{∂q}(q) = B_r(q) u \tag{32}
\]
For every smooth function $V$ such that $S^T(q)\frac{\partial V}{\partial q}(q) \in \text{Im} B_r(q)$, for all $q$, we can determine a smooth feedback $u = \dot{u}(q)$ which solves (32). Application of the feedback $u = \dot{u}(q) + u$, with $u$ the new control variables, will result in a modified system (instead of (21))

$$\left( \frac{\dot{q}}{\dot{p}^i} \right) = J_r(q, p^i) \frac{\partial H_r}{\partial q}(q, p^i) + 0 \frac{\partial B_r(q)}{\partial q}(q, p^i) \left( \frac{\partial V}{\partial q}(q) \right)$$

with $H_r(q, p^i) = H_r(q, p^i) + \dot{V}(q)$. (This results from the special form of $J_r$ given in (20).) It is possible to find in this manner a function $\dot{V}$ such that $V + \dot{V}$ has a strict minimum in $q_0$, then $H_r$ will have a strict minimum in $q_0$, and thus the additional feedback (28), with $u$ replaced by $v$, will further stabilize the system. The resulting combined feedback is then given as

$$u = \dot{u}(q) - B_r(q) \frac{\partial H_r}{\partial q}(q, p^i)$$

with $\dot{u}(q)$ solving (32).

The treatment of [2], [4] corresponds to the special case that $B_r(q)$ has rank $m = n - k$. In this case equation (32) is solvable for every function $V(q)$, and thus the potential energy can be shaped in an arbitrary fashion. Therefore, for $H_r$ given by (31), every point $(q_0, 0) \in X$, can be rendered a Lyapunov stable equilibrium by a feedback (34). Note furthermore that in this case the largest invariant (with respect to (27)) set contained in (30) is actually given as

$$\{ (q, 0) \in X, | S^T(q) \frac{\partial (V + \dot{V})}{\partial q}(q) = 0 \}$$

(as follows from the form of $I_r$, given in (20)), where $\dot{V}$ is taken such that $V + \dot{V}$ has a strict minimum in $q_0$. A similar result has been obtained before in [4] (in the reduced Lagrangian framework) using a different Lyapunov function, and a different feedback control based on this. The main difference is that in our approach the Lyapunov function $H_r$ is directly based on the internal energy of the constrained dynamics, and consequently that $u$ given in (34) has a direct physical interpretation. Furthermore, contrary to [4], we consider the stabilization problem for arbitrary $B_r$ and an arbitrary number of controls.

We now treat within our approach two examples of nonholonomic control systems, both of which have been studied before in [2].

**Example 3.1 (Knife edge)** Consider the control of a knife edge moving in point contact on a plane surface. The constrained Lagrangian equations are given as (all numerical constants are set to unity)

$$\ddot{x} = \lambda \sin \varphi + u_1 \cos \varphi$$
$$\ddot{y} = -\lambda \cos \varphi + u_1 \sin \varphi$$
$$\ddot{\varphi} = u_2$$

with $(x, y)$ Cartesian coordinates of the contact point, $\varphi$ the heading angle of the knife-edge, $u_1$ the control in the direction of the heading angle, and $u_2$ the control torque about the vertical axis. The nonholonomic constraint is

$$\dot{x} \sin \varphi - \dot{y} \cos \varphi = 0$$

The total energy $H$ is given as $p_x^2 + 12 p_y^2 + 12 p_\varphi^2$, with $p_x, p_y, p_\varphi$ the corresponding generalized momenta. The constraint (37) can be written as $p_x \sin \varphi - p_y \cos \varphi = 0$. Define as in (17) new coordinates

$$p_1 = p_x$$
$$p_2 = p_x \cos \varphi + p_y \sin \varphi$$
$$p_3 = p_x \sin \varphi - p_y \cos \varphi$$

Then $(\varphi, x, y, p_1, p_2)$ are coordinates for $X_r$, and the dynamics (21) is computed as

$$\begin{bmatrix}
\dot{x} \\
\dot{y} \\
p_1 \\
p_2 
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos \varphi \\
-1 & 0 & 0 & 0 & 0 \\
0 & -\cos \varphi & -\sin \varphi & 0 & 0 
\end{bmatrix}
\begin{bmatrix}
\varphi \\
x \\
y \\
p_1 \\
p_2 
\end{bmatrix}$$

with $H_r(\varphi, x, y, p_1, p_2) = p_x^2 + p_y^2$. Take $V(\varphi, x, y) = \frac{1}{2} \varphi^2 + \frac{1}{2} x^2 + \frac{1}{2} y^2$, then the preliminary feedback $\dot{u}(\varphi, x, y)$ is determined by (see (32))

$$\begin{bmatrix}
\dot{\varphi} \\
x \\
y 
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -\cos \varphi & -\sin \varphi \\
0 & 1 & 0 
\end{bmatrix}
\begin{bmatrix}
\dot{\varphi} \\
x \\
y 
\end{bmatrix}$$

and the resulting combined feedback (34) is

$$u_1 = -x \sin \varphi - y \cos \varphi - p_2$$
$$u_2 = -\varphi$$

The trajectories will converge to the invariant set $\varphi = 0$, $x = 0$, $p_1 = 0$, $p_2 = 0$. A different $V$, however, will generally yield a different invariant set.

**Example 3.2 (Rolling vertical wheel)** Let $x, y$ be the Cartesian coordinates of the point of contact of the wheel with the plane, $\varphi$ denotes heading angle, and $\theta$ rotation angle. With all constants set to unity, the Lagrangian equations of motion are

$$\ddot{x} = \lambda_1$$
$$\ddot{y} = \lambda_2$$
$$\ddot{\varphi} = -\lambda_1 \cos \varphi - \lambda_2 \sin \varphi + u_1$$

with $u_1$ the control torque about the rolling axis and $u_2$ the control torque about the vertical axis. The nonholonomic constraints are (rolling without slipping)

$$\ddot{\theta} = \dot{\theta} \sin \varphi$$

The total energy $H$ is $\frac{1}{2} p_x^2 + \frac{1}{2} p_y^2 + \frac{1}{2} p_\varphi^2$, and the constraints can thus be rewritten as $p_x \cos \varphi, p_y = p_y \sin \varphi$. Define according to (17) new coordinates

$$p_1 = p_x \cos \varphi$$
$$p_2 = p_x \sin \varphi$$
$$p_3 = p_y$$

Then $(\varphi, x, y, p_1, p_2)$ are coordinates for $X_r$, and the dynamics (21) is computed as

$$\begin{bmatrix}
\dot{x} \\
\dot{y} \\
p_1 \\
p_2 
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos \varphi \\
-1 & 0 & 0 & 0 & 0 \\
0 & -\cos \varphi & -\sin \varphi & 0 & 0 
\end{bmatrix}
\begin{bmatrix}
\varphi \\
x \\
y \\
p_1 \\
p_2 
\end{bmatrix}$$

with $H_r(\varphi, x, y, p_1, p_2) = p_x^2 + p_y^2$. Take $V(\varphi, x, y) = \frac{1}{2} \varphi^2 + \frac{1}{2} x^2 + \frac{1}{2} y^2$, then the preliminary feedback $\dot{u}(\varphi, x, y)$ is determined by (see (32))

$$\begin{bmatrix}
\dot{\varphi} \\
x \\
y 
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -\cos \varphi & -\sin \varphi \\
0 & 1 & 0 
\end{bmatrix}
\begin{bmatrix}
\dot{\varphi} \\
x \\
y 
\end{bmatrix}$$

and the resulting combined feedback (34) is

$$u_1 = -x \sin \varphi - y \cos \varphi - p_2$$
$$u_2 = -\varphi$$

The trajectories will converge to the invariant set $\varphi = 0$, $x = 0$, $p_1 = 0$, $p_2 = 0$. A different $V$, however, will generally yield a different invariant set.

**Example 3.2 (Rolling vertical wheel)**
\[ p_1 = p_2 \]
\[ p_2 = p_y + p_x \cos \varphi + p_y \sin \varphi \]
\[ p_3 = p_x - p_y \cos \varphi \]
\[ p_4 = p_y - p_x \sin \varphi \]

Then \( \{x, y, \tilde{\varphi}, \varphi, p_x, p_y \} \) are coordinates for \( \mathcal{X}_r \), and the dynamics (21) is computed as (see also [15])

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\varphi} \\
\dot{\mathbf{\theta}} \\
\dot{p}_1 \\
\dot{p}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & \cos \varphi & 0 & 0 & 0 \\
0 & 0 & 0 & \sin \varphi & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\cos \varphi & -\sin \varphi & -1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
p_x \\
p_y \\
p_x \sin \varphi \\
p_y \cos \varphi \\
p_x \sin \varphi & -p_y & 1 & 0 & 0 & 0
\end{bmatrix}
\]

with \( H = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 \). A feedback (24) can be computed as in the preceding example. Note that Example 2 (as well as Example 1) can be easily generalized to a knife edge or rolling wheel on any surface. This corresponds to adding a potential energy to \( H \) (and to \( H_r \)). Furthermore, in both examples one control torque instead of two control torques can be considered.

4 Conclusions

We have shown, as an extension to [15], that the equations of motion of controlled mechanical systems with constraints may be directly formulated as Hamiltonian equations of motion with respect to a bracket which for nonholonomic constraints does not satisfy the Jacobi-identity, and with respect to a reduced Hamiltonian which is obtained by restricting the total energy to the constrained state space.

Like for ordinary Hamiltonian control systems a stabilization procedure has been proposed, based on the use of the reduced Hamiltonian as a candidate Lyapunov function. However, since Brockett's necessary condition is not satisfied, this will only result in Lyapunov stability, whereas asymptotic convergence is to a non-trivial invariant set. The main challenge is to investigate how the Hamiltonian structure may be used for asymptotic stabilization, in which case discontinuous or time-varying feedback is needed ([2], [4]).

References