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Nonlinear systems which have finite-dimensional $\mathcal{H}_\infty$ suboptimal central controllers

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Abstract

Following up the work of Başar and Bernhard [2], we have recently derived in [12] the nonlinear central controller solving the nonlinear (standard) $\mathcal{H}_\infty$ suboptimal control problem. This nonlinear central controller is an infinite-dimensional system, and resembles very much the solution in nonlinear stochastic filtering or nonlinear deterministic filtering. After showing that in the linear case the nonlinear central controller reduces to the finite-dimensional central controller as obtained in [4], we consider in the present note the question if there are truly nonlinear systems having finite-dimensional central controllers. Guided by similar considerations in nonlinear stochastic and deterministic filtering, see especially [5], we characterize a specific class of nonlinear systems having finite-dimensional central controllers. This class can be regarded as the deterministic $\mathcal{H}_\infty$ analogue of the class of nonlinear systems admitting finite dimensional filters as identified by Benes [3].

We consider nonlinear control systems of the form

\[ \begin{align*}
\dot{x} &= a(x) + b(x)u + g(x)d
\end{align*} \]

\[ \begin{align*}
y &= c(x) + d
\end{align*} \]

\[ \begin{align*}
z &= \begin{bmatrix} h(x) \\ u \end{bmatrix}
\end{align*} \]

(1)

where $x = (x_1, \ldots, x_n)$ are local coordinates for a smooth state space manifold $M$. Furthermore, $u \in \mathbb{R}^n$ denote the control inputs, $d = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \in \mathbb{R}^n$ the exogenous inputs (disturbances and/or references), $y \in \mathbb{R}^p$ the measured outputs, and $z \in \mathbb{R}^m$ the to-be-controlled outputs (tracking errors, cost variables). The maps $a(x), b(x), g(x), c(x), h(x)$ are all assumed to be $C^4$, with $k \geq 2$. Throughout we assume the existence of a fixed equilibrium $x_0$, i.e. $a(x_0) = 0$, and without loss of generality we set $x_0 = 0$, and also we let $c(0) = h(0) = 0$. Now let $\gamma$ be a fixed positive constant. The $\mathcal{H}_\infty$ suboptimal central control problem (for disturbance attenuation level $\gamma$) is to find a compensator

\[ \begin{align*}
\xi &= k(\xi, y)
\end{align*} \]

\[ \begin{align*}
u &= m(\xi, y)
\end{align*} \]

(2)

where $\xi = (\xi_1, \ldots, \xi_p)$ are local coordinates for a manifold $M_\xi$ (the state space of the compensator), with $k(0,0) = 0$ and $m(0,0) = 0$, such that the closed-loop system (1), (2) has $\mathcal{L}_2$-gain less than or equal to $\gamma$, in the sense that there exists a nonnegative constant $K$ depending on $x(0), y(0)$ and zero for $x(0) = 0, y(0) = 0$, such that

\[ \begin{align*}
\int_0^T \| d(t) \|^2 dt &\leq \gamma \int_0^T \| d(t) \|^2 dt + K
\end{align*} \]

(3)

for all $d(\cdot)$ and $T \geq 0$, with $z(\cdot)$ denoting the closed-loop response for initial condition $x(0), y(0)$. If some observability conditions are satisfied (with regard to the outputs $z(\cdot)$), then property (3) will also imply internal stability of the closed-loop system. For further motivation and details we refer e.g. to [1], [7], [9], [10], [12].

The state of the art for this problem is, very roughly, as follows.

The state feedback problem (i.e., $y = z$) is reasonably well-understood [9], [10], [7]. For the full (dynamic output feedback) problem appealing necessary conditions have been found generalizing the famous necessary and sufficient conditions for linear systems obtained in [4], see [1], [11], [12]. Much effort and ingenuity has been put in obtaining various sufficient conditions, see e.g. [1], [6], [7], but the problem is still largely open.

Following up the work of Başar and Bernhard [2], and using older work on nonlinear deterministic filtering [8], [5], we have recently taken another approach to the full $\mathcal{H}_\infty$ suboptimal control problem. In fact, in [12] we have shown that under suitable technical conditions the $\mathcal{H}_\infty$ suboptimal control problem is solved by the controller

\[ \begin{align*}
\dot{x} &= a(x) - b(x)u + g(x)d
\end{align*} \]

\[ \begin{align*}
y &= c(x) + d
\end{align*} \]

\[ \begin{align*}
u &= -B^T \hat{P} x
\end{align*} \]

(4)

with $P \geq 0$ being the minimal solution to the Hamilton-Jacobi equation

\[ \begin{align*}
P_d(x)u(x) + \frac{1}{2} P_d(x) \left[ \frac{1}{2} g(x)p^T(x) - b(x)B^T(x) \right] P_d(x)
\end{align*} \]

\[ \begin{align*}
+ \frac{1}{2} h^T(x)h(x) = 0, \quad P(0) = 0,
\end{align*} \]

(5)

(implying the solvability of the state feedback $\mathcal{H}_\infty$ suboptimal control problem, see [9], [10]), and with $R$ being a solution to the nonstationary Hamilton-Jacobi equation

\[ \begin{align*}
R_l(x, t) + R_x(x, t)m(x) + \frac{1}{2} \frac{3}{2} R_x(x, t)g(x)p^T(x)R_x(x, t)
\end{align*} \]

\[ \begin{align*}
+ \frac{1}{2} h^T(x)h(x) - \frac{1}{2} \gamma^2 c^T(x)c(x) + \gamma^2 c^T(x)p(t)
\end{align*} \]

(6)

(with $u(t)$ given as the output of (4)), such that $S(t, 0) := R(t, 0) - P(x)$ has a unique minimum $\hat{x}(t)$ for every $t$ with invertible Hessian matrix $S_{xx}(\hat{x}(t), t)$.

This controller is obtained as the solution of a certain min-max optimization problem with imperfect state measurements, see [2], and thus is called the nonlinear central controller, since in the linear case this optimization problem is known [2] to yield the central controller of [4]. In fact, in the sequel we will directly demonstrate how in the linear case the nonlinear central controller reduces to the central controller of [4].

Although the first part of the nonlinear central controller, i.e. (4), has an appealing "worst-case disturbance" observer structure (see [2], [12]), in general the gain matrix $[S_{xx}(\hat{x}(t), t)]^{-1}$ cannot be computed off-line, since the partial differential equation (6) for $R(x, t)$ is directly driven by the measured outputs $y(t)$, as well as by $w(t)$. Thus the nonlinear central controller is infinite-dimensional.

Now suppose (1) is linear, i.e.

\[ \begin{align*}
\dot{x} &= Ax + Bu + Gd
\end{align*} \]

\[ \begin{align*}
y &= Cx + d
\end{align*} \]

\[ \begin{align*}
z &= \begin{bmatrix} I \\ z \end{bmatrix}
\end{align*} \]

(7)

It is readily checked (see [9] for the same argument in the context of deterministic filtering) that the solution $R(x, t)$ of (6) in this case has the form $R(x, t) = c(t) + \hat{c}^T(t)x + \frac{1}{2} x^T \hat{Q}(t)x$ with $R_{xx}(x, t) = \hat{Q}(t) \geq 0$ satisfying

\[ \begin{align*}
\hat{Q}(t) + \hat{A}^T \hat{Q}(t) + \hat{Q}(t) \hat{A} + \frac{1}{2} \gamma^2 \hat{Q}(t)GC^T \hat{Q}(t) + H^T H - \gamma^2 C^T C = 0
\end{align*} \]

(8)
It follows that $R_{ac}(x,t) = Q(t)$ can be computed off-line, and in fact can be taken to be the maximal constant matrix $Q > 0$ solving the algebraic Riccati equation

$$A^TQ + QA + \frac{1}{2}GG^TQ + H^TH - \gamma C^TC = 0,$$

(9)

(Notice that the dual Riccati equation (FARE) of (9) is obtained from (9) by dividing by $\gamma^2$ and pre- and post-multiplication by $\gamma^2Q^{-1}$).

Furthermore the stationary Hamilton-Jacobi equation (5) reduces to an algebraic Riccati equation, and it is immediately seen that the resulting linear controller (4) is precisely the central controller of [4].

A logical question is now: are there any other systems, apart from the linear ones, for which $R_{ac}(x,t)$ and thus $S_{ac}(x,t)$ can be computed off-line, and therefore the nonlinear central controller reduces to the finite-dimensional controller (4)? In order to study this problem we take the same approach as used in [5] for the deterministic nonlinear filtering problem, i.e. we consider the Hamiltonian function $H(x,p)$ corresponding to the Hamilton-Jacobi equation (6):

$$H(x,p) = p^T a(x) + \frac{1}{2} p^T \gamma^2 g(x) g^T(x) + \frac{1}{2} b^T(x) h(x)$$

$$- \frac{1}{2} \gamma^2 c^T(x) c(x) + \gamma^2 T(x) g(x)$$

$$+ \frac{1}{2} p^T h(x) g(x) - \frac{1}{2} \gamma^2 || g(t) ||^2 + \frac{1}{2} || u(t) ||^2,$$

(10)

with $p \in \mathbb{R}^n$ denoting the co-state. (Notice that (6) can be rewritten as $R_{ac}(x,t) = 0$.) It is natural (see [5]) to consider the following type of canonical transformations $(x,p) \mapsto (x,\tilde{p})$, where $p = \tilde{p} + V^T(x)$

(11)

for some $C^4$ function $V(x)$. Then the Hamiltonian $H(x,p)$ transforms into $\tilde{H}(x,\tilde{p}) := H(x,\tilde{p} + V^T(x))$, leading to the transformed Hamilton-Jacobi equation $\tilde{R}(x,t) + \tilde{R}(x,\tilde{p}) = 0$ given as

$$\tilde{R}_a(x,t) + \tilde{R}_g(x,t) g(x) + \frac{1}{2} \gamma^2 g(x) g^T(x)$$

$$+ \frac{1}{2} \gamma^2 c^T(x) c(x) + \frac{1}{2} \gamma^2 T(x) g(x)$$

$$- \frac{1}{2} \gamma^2 c^T(x) c(x) + \frac{1}{2} \gamma^2 T(x) g(x) + \frac{1}{2} b^T(x) h(x)$$

$$+ \frac{1}{2} \gamma^2 c^T(x) c(x) + \frac{1}{2} \gamma^2 T(x) g(x) + \frac{1}{2} b^T(x) h(x)$$

(12)

$$+ \frac{1}{2} \gamma^2 || g(t) ||^2 + \frac{1}{2} || u(t) ||^2 = 0.$$

It immediately follows that $\tilde{R}$ is a solution of (12) if and only if $R + V$ is a solution of (6). Now assume that $b(x)$ and $g(x)$ are constant, $c(x)$ is linear in $x$, and suppose that the nonlinear system (1) has the property that $V(x)$ can be found such that $a(x) + \frac{1}{2} \gamma^2 g(x) g^T(x)$ is at most linear in $x$,

$$\frac{1}{2} \gamma^2 c^T(x) c(x) + \frac{1}{2} \gamma^2 T(x) g(x) + \frac{1}{2} b^T(x) h(x)$$

(13)

is at most quadratic in $x$,

$$V(x) h(x)$$

(14)

is at most linear in $x$.

Then, as in the linear case, it follows that the solution $\tilde{R}(x,t)$ of (12) can be written as $\tilde{R}(x,t) = \tilde{R}(t) + \tilde{R}(t)x + \frac{1}{2} \gamma^2 Q(t)x$, where $Q(t) = \tilde{R}_q(x,t)$ is the solution of a differential Riccati equation of the same type as (8), and thus can be computed off-line (without knowing $y(t)$ and $u(t)$). Therefore also the Hessian $R_{ac}(x,t)$ of the solution $R(x,t) = \tilde{R}(t) + V(x)$ of (6) can be computed off-line, and thus in this case the nonlinear central controller reduces to the finite-dimensional controller (4)! Summarizing:

**Theorem:** Consider the nonlinear system (1). Suppose the $H_{ac}$ sub-optimal control problem is solvable by the nonlinear central controller (4), (6). Assume that $b(x), g(x)$ and $c(x)$ are as in (13), while $V(x)$ can be found such that (14) holds. Then there exists a solution $R(x,t)$ of (6) for which $R_{ac}(x,t)$ can be computed off-line, and the nonlinear central controller reduces to the finite-dimensional controller (4).

**Example:** Consider the almost linear system

$$\dot{x}_1 = x_2 + u + d_1, \quad y = x_2 + d_2$$

$$\dot{x}_2 = -x_2$$

Then $V(x_1, x_2) = \frac{1}{2} x_2^2$ satisfies (12), and thus the nonlinear central controller reduces to a finite dimensional controller.

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References


