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# Tracking Control of Nonlinear Systems with Disturbance Attenuation

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## Abstract

Sufficient geometric conditions are given which lead to the explicit construction of a state feedback tracking control for single–input single–output nonlinear systems with bounded unmodelled disturbances entering nonlinearly. For any initial condition the output asymptotically tracks a bounded reference signal with bounded time derivatives with an arbitrary attenuation of the influence of the disturbance. The sufficient conditions are weaker than those presented in [6] and the technique of proof is also different.

## 1 Introduction

This paper provides sufficient conditions for the explicit construction of a state feedback control capable of forcing the output $y$ to track a bounded reference signal $y_d(t)$ with an arbitrary attenuation of a bounded unmodelled time varying disturbance $\theta(t)$ for nonlinear systems

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + q(x,\theta(t)), \quad x \in R^n, u \in R, \\ y &= h(x), \qquad y \in R. \end{aligned} \tag{1}$$

In (1) $f : R^n \to R^n$, $g : R^n \to R^n$, $q : R^n \times \omega \to R^n$, $h : R^n \to R$ are smooth functions, $g(x) \neq 0$, $\forall x \in R^n$, $x$ is the state, $u$ is the input, $\theta : R^+ \to \Omega \in R^p$ is the disturbance, $y$ is the output which is required to track a reference signal $y_d(t)$. This problem is also called almost disturbance decoupling, following the terminology introduced in [11] for linear systems: it is posed when the well understood disturbance decoupling (or rejection) problem, that is the design of a state feedback control which makes the output insensitive to unmodelled disturbances, is not solvable.

Necessary and sufficient geometric conditions for the solvability of the disturbance decoupling problem are obtained in [4] and [5], generalizing the results established in [12] and [1] for linear systems. Equivalent conditions based on the notion of characteristic indexes are given in [2] for linear systems and in [8] for nonlinear ones. When the disturbance decoupling problem is not solvable, it is natural to look for conditions which guarantee the attenuation of the influence of the disturbance on the output with any desired degree of accuracy. This problem, called almost disturbance decoupling, was posed and solved in [11] for linear systems in terms of necessary and sufficient geometric conditions. In particular, the problem turns out to be always solvable for single–input, single–output linear systems of type

$$\begin{aligned} \dot{x} &= Fx + gu + Q\theta \\ y &= hx \end{aligned} \tag{2}$$

where $F$ and $Q$ are $n \times n$ and $n \times p$ constant matrices, $g$ and $h^T$ are $n \times 1$ constant vectors. As pointed out in [11] the almost disturbance decoupling problem is related to high–gain feedback design since, when it is not (exactly) solvable, the higher the gains are the higher the disturbance attenuation results. In fact in [7] a parameterized state feedback control is explicitly obtained when the almost disturbance decoupling is solvable; for square and minimum–phase systems a parameterized output–feedback control is given in [10].

At the moment it is not known whether, as in the linear case, the almost disturbance decoupling problem is always solvable for single–input, single–output nonlinear systems (1). Sufficient conditions are obtained in [6] using differential geometric tools and singular perturbation techniques. The example (given in [6])

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta_1(t) \\ \dot{x}_2 &= x_2^3\theta_2(t) + u(t) \\ y &= x_1 \end{aligned} \tag{3}$$

fails to satisfy the sufficient conditions in [6] and shows that the almost disturbance decoupling problem cannot be solved on the basis of linear approximations. The non–local nature of the problem is also pointed out by the example (also given in [6])

$$\begin{aligned} \dot{x}_1 &= \arctan x_2 + \theta(t) \\ \dot{x}_2 &= u(t) \\ y &= x_1 \end{aligned} \tag{4}$$

where disturbances $|\theta(t)| > \pi/2$ cannot be attenuated.

In this paper we provide sufficient conditions for disturbance attenuation which generalize those given

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in [6]: for instance, the example (3) satisfies the conditions given here. The control algorithm we develop generalizes the one given in [6] and coincides with it in special simpler cases even though the techniques of proof are entirely different. We do not use singular perturbation techniques; we determine special coordinates in which the control algorithm and a Lyapunov function are recursively built.

## 2 Main Result

**Definition 2.1** *The control characteristic index of system (1) is defined as the integer $\rho$ such that*

$$\begin{aligned} L_gL_f^ih(x) &= 0, \qquad 0 \le i \le \rho-2, \forall x \in R^n, \\ L_gL_f^{\rho-1}h(x) &\neq 0, \qquad \forall x \in R^n. \end{aligned}$$

*If $L_gL_f^ih(x) = 0$, $\forall i, \forall x \in R^n$, then $\rho = \infty$.* □

**Definition 2.2** *The disturbance characteristic index $\nu$ of system (1) is defined as the integer such that*

$$\begin{aligned} L_qL_f^ih(x) &= 0, \qquad 0 \le i \le \nu-2, \forall x \in R^n, \\ L_qL_f^{\nu-1}h(x) &\neq 0, \qquad \text{for some } \theta \in \Omega, \text{some } x \in R^n. \end{aligned}$$

□

As shown in [5] and [8] the exact disturbance decoupling problem is locally solvable if and only if $\nu > \rho$.

**Assumption 1** We assume in the following that $\rho$ is well defined and that $\nu \le \rho < \infty$, that is that the exact disturbance decoupling problem is not solvable.

**Definition 2.3** *The tracking problem with disturbance attenuation is said to be solvable for system (1), if for any smooth bounded reference trajectory $y_d(t)$, with bounded time derivatives $y_d^{(1)}, \ldots, y_d^{(\rho)}$, for any bounded disturbance $\theta(t) \in \Omega \subset R^p$, and for any initial condition $x(0)$ a parameterized state feedback control law $u = u(x, k, t)$ exists such that $\|x(t)\|$ and the output error $e(t) = y(t) - y_d(t)$ are bounded $\forall t \ge 0$, and for every $\epsilon > 0$ and $T > 0$ there exists $k(\epsilon, T)$ such that*

$$\|e(t)\| \le \epsilon, \quad \forall t \ge T(\epsilon), \forall k > k(\epsilon, T).$$

□

Under Assumption 1 we can locally define a change of coordinates

$$\begin{aligned} z_1 &= h(x) \\ &\vdots \\ z_\rho &= L_f^{\rho-1}h(x) \\ z_{\rho+1} &= \phi_{\rho+1}(x) \\ &\vdots \\ z_n &= \phi_n(x) \end{aligned} \tag{5}$$

with $\phi_i(x)$, $\rho+1 \le i \le n$, such that

$$< d\phi_i, g >= 0.$$

In new coordinates we have

$$\begin{aligned} \dot{z}_1 &= z_2 + L_qh(x) \\ &\vdots \\ \dot{z}_{\rho-1} &= z_\rho + L_qL_f^{\rho-2}h(x) \\ \dot{z}_\rho &= L_f^\rho h(x) + L_qL_f^{\rho-1}h(x) + L_gL_f^{\rho-1}h(x)u \\ \dot{z}_{\rho+j} &= L_f\phi_{\rho+j}(z) + L_q\phi_{\rho+j}(z, \theta(t)) \\ &\triangleq \beta_j(z, \theta(t)), \quad 1 \le j \le n-\rho \\ y &= z_1 \end{aligned} \tag{6}$$

in which $L_gL_f^{\rho-1}h(x) \neq 0$, $\forall x \in R^n$, according to Assumption 1 and Definition 2.1. Denoting $z_r = (z_{\rho+1}, \ldots, z_n)$ and $\beta = (\beta_1, \ldots, \beta_{n-\rho})$ the dynamics

$$\dot{z}_r = \beta(z_r, z_1(t), \ldots, z_\rho(t), \theta_1(t), \ldots, \theta_p(t)) \tag{7}$$

is called the tracking dynamics where $z_1(t), \ldots, z_\rho(t), \theta_1(t), \ldots, \theta_p(t)$ are the inputs. When $z_1 = \cdots = z_\rho = 0$ and $\theta = 0$ the tracking dynamics is the zero dynamics.

**Assumption 2** The tracking dynamics (7) is bounded input bounded state.

**Theorem 2.1** *Assume in addition to Assumptions 1 and 2 that the following conditions are satisfied for system (1):*

*(i) there exist $\rho - \nu + 1$ smooth functions $\alpha_i(x)$, $\nu - 1, \le i \le \rho - 1$, satisfying $d\alpha_i \in span\{dh, dL_fh, \ldots, dL_f^ih\}$, such that $\forall x \in R^n$, $\forall \theta \in \Omega$,*

$$|L_qL_f^ih| \le \alpha_i, \qquad \nu-1 \le i \le \rho-1;$$

*(ii) the vector fields*

$$\tilde{f} = f - \frac{1}{L_gL_f^{\rho-1}h}L_f^\rho h, \qquad \tilde{g} = \frac{1}{L_gL_f^{\rho-1}h}g$$

*are complete.*

*Then, the problem of tracking with disturbance attenuation is solvable.*

*Proof.* We consider the general case in which $\nu = 1$. By virtue of Assumption 1 and the additional condition (ii) the change of coordinates (5) is globally defined (see [6] and [3]) and system (1) can be globally transformed into (6). We introduce a new control variable $v$, defined as

$$v = L_gL_f^{\rho-1}h(x)u + L_f^\rho h(x) - y_d^{(\rho)} \tag{8}$$

which substituted in (6) gives

$$\begin{aligned} \dot{e}_1 &= e_2 + L_qh(x) \\ &\vdots \\ \dot{e}_{\rho-1} &= e_\rho + L_qL_f^{\rho-2}h(x) \\ \dot{e}_\rho &= L_qL_f^{\rho-1}h(x) + v \\ \dot{z}_r &= \beta(z_r, z_1(t), \ldots, z_\rho(t), \theta_1(t), \ldots, \theta_p(t)) \end{aligned} \tag{9}$$

where $e_i = z_i - y_d^{(i-1)}$, $1 \le i \le \rho$. Note that if $\nu > \rho$ the control (8) solves the exact disturbance decoupling problem. Define

$$e_2^* = -ke_1 - e_1\mu_1(t) \tag{10}$$

where $\mu_1$ is a smooth function yet to be defined and $k > 0$. Consider the function

$$V_1 = \frac{1}{2}e_1^2 \tag{11}$$

The time derivative of $V_1$, with $e_2 = e_2^*$ in (9), is given by

$$\dot{V}_1 = -ke_1^2 - e_1^2\mu_1 + e_1 L_q h(x) \tag{12}$$

and, according to the inequality in (i), we have

$$\dot{V}_1 \le -ke_1^2 - e_1^2\mu_1 + |e_1|\alpha_0(z_1) \tag{13}$$

Since $\alpha_0$ is a smooth function and $y_d$ is bounded, we can write

$$\alpha_0(z_1) = \alpha_0(e_1 + y_d) = \alpha_0(y_d) + \bar{\alpha}_0(e_1, y_d)e_1 \tag{14}$$

in which

$$\bar{\alpha}_0(e_1, y_d) = \frac{\alpha_0(z_1) - \alpha_0(y_d)}{e_1}. \tag{15}$$

Hence, we choose $\mu_1$ as a smooth function satisfying

$$\mu_1 \ge \bar{\alpha}_0(e_1, y_d). \tag{16}$$

From (13), we obtain

$$\dot{V}_1 \le -ke_1^2 + |e_1||\alpha_0(y_d)| \tag{17}$$

Therefore, if $\rho = 1$ the thesis is proved with

$$v = e_2^*. \tag{18}$$

In fact, we can write

$$\frac{\dot{V}_1}{V_1} \le -2k + \frac{|\alpha_0(y_d)|}{|e_1|}.$$

Recalling that $y_d$ is bounded, $|\alpha_0(y_d)| < \gamma$, $\gamma > 0$; for every $|e_1| \ge \epsilon/2$, we obtain

$$\frac{\dot{V}_1}{V_1} \le -2k + 2\gamma/\epsilon,$$

which implies

$$V_1(t) \le V_1(0)e^{(-2k+2\gamma/\epsilon)t}.$$

For any $\epsilon > 0$, $T > 0$ there exists $k$ which solves the problem.

If $\rho > 1$, we prove the following Claim.

**Claim.** Assume that for a given index $i$, $1 \le i \le \rho$, for the system

$$\begin{aligned} \dot{e}_1 &= e_2 + L_q h(x) \\ &\vdots \\ \dot{e}_i &= e_{i+1} + L_q L_f^{i-1} h(x) \end{aligned} \tag{19}$$

there exist $i$ functions

$$\begin{aligned} &e_2^*(e_1, y_d), e_3^*(e_1, e_2, y_d, y_d^{(1)}), \ldots, \\ &e_{i+1}^*(e_1, \ldots, e_i, y_d, \ldots, y_d^{(i-1)}), \\ &e_j^*(0, \ldots, 0, y_d, \ldots, y_d^{(j-2)}) = 0, \quad 2 \le j \le i+1, \end{aligned} \tag{20}$$

such that in new coordinates ($M_j > 0$, $1 \le j \le i$)

$$\begin{aligned} \tilde{e}_1 &= e_1 \\ \tilde{e}_2 &= \frac{e_2 - e_2^*}{M_2} \\ &\vdots \\ \tilde{e}_i &= \frac{e_i - e_i^*}{M_i} \end{aligned} \tag{21}$$

the function

$$V_i = \frac{1}{2}\sum_{j=1}^{i} \tilde{e}_j^2 \tag{22}$$

has time derivative, with $e_{i+1} = e_{i+1}^*$ in (19), satisfying the inequality

$$\dot{V}_i \le -k \left\| \begin{bmatrix} \tilde{e}_1 \\ \vdots \\ \tilde{e}_i \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} \tilde{e}_1 \\ \vdots \\ \tilde{e}_i \end{bmatrix} \right\| \eta_i(y_d, \ldots, y_d^{(i-1)}, k) \tag{23}$$

with $\eta_i$ a suitable smooth function such that

$$\lim_{k\to\infty} \frac{\eta_i(y_d, \ldots, y_d^{(i-1)}, k)}{k} = 0. \tag{24}$$

Then, for the system

$$\begin{aligned} \dot{e}_1 &= e_2 + L_q h(x) \\ &\vdots \\ \dot{e}_{i+1} &= e_{i+2} + L_q L_f^{i} h(x) \end{aligned} \tag{25}$$

there exists a function

$$\begin{aligned} &e_{i+2}^*(e_1, \ldots, e_{i+1}, y_d, \ldots, y_d^{(i)}), \\ &e_{i+2}^*(0, \ldots, 0, y_d, \ldots, y_d^{(i)}) = 0 \end{aligned} \tag{26}$$

such that in new coordinates ($M_{i+1} > 0$)

$$\tilde{e}_j, \qquad 1 \le j \le i, \qquad \tilde{e}_{i+1} = \frac{e_{i+1} - e_{i+1}^*}{M_{i+1}} \tag{27}$$

the function

$$V_{i+1} = \frac{1}{2}\sum_{j=1}^{i+1} \tilde{e}_j^2 \tag{28}$$

has time derivative, with $e_{i+2} = e_{i+2}^*$ in (25), satisfying the inequality

$$\begin{aligned} \dot{V}_{i+1} \le & -\frac{k}{2} \left\| \begin{bmatrix} \tilde{e}_1 \\ \vdots \\ \tilde{e}_{i+1} \end{bmatrix} \right\|^2 \\ & + \left\| \begin{bmatrix} \tilde{e}_1 \\ \vdots \\ \tilde{e}_{i+1} \end{bmatrix} \right\| \eta_{i+1}(y_d, \ldots, y_d^{(i)}, k) \end{aligned} \tag{29}$$

where $\eta_{i+1}$ is a suitable smooth function such that

$$\lim_{k\to\infty} \frac{\eta_{i+1}(y_d,\dots,y_d^{(i)},k)}{k} = 0. \tag{30}$$

*Proof of the Claim.* For convenience we adopt the following notations:

$$\begin{aligned} z_{(i)} &= [z_1,\dots,z_i]^T, \qquad z_{(i)}^d = [y_d,\dots,y_d^{(i-1)}]^T \\ e_{(i)} &= [e_1,\dots,e_i]^T, \qquad \tilde{e}_{(i)} = [\tilde{e}_1,\dots,\tilde{e}_i]^T \end{aligned}$$

From (21), (22) and (25) we obtain

$$\begin{aligned} \dot{V}_{i+1} &= -k\|\tilde{e}_{(i)}\|^2 + \|\tilde{e}_{(i)}\|\eta_i(z_{(i)}^d,k) + M_{i+1}\tilde{e}_i\tilde{e}_{i+1} \\ &\quad + \tilde{e}_{i+1}\dot{\tilde{e}}_{i+1} \end{aligned} \tag{31}$$

Since by (21) and (25),

$$\begin{aligned} \dot{\tilde{e}}_{i+1} &= \frac{1}{M_{i+1}}\frac{d}{dt}(e_{i+1} - e_{i+1}^*) \\ &= \frac{1}{M_{i+1}}\Bigg(e_{i+2} + L_qL_f^ih \\ &\quad - \sum_{j=1}^{i}\frac{\partial e_{i+1}^*}{\partial e_j}(e_{j+1} + L_qL_f^{j-1}h) \\ &\quad - \sum_{j=1}^{i}\frac{\partial e_{i+1}^*}{\partial y_d^{(j-1)}}y_d^{(j)}\Bigg) \end{aligned} \tag{32}$$

we define

$$\begin{aligned} e_{i+2}^* &= -M_{i+1}(k\tilde{e}_{i+1} + \tilde{e}_{i+1}\mu_{i+1}(t) + M_{i+1}\tilde{e}_i) \\ &\quad + \textstyle\sum_{j=1}^{i}\left(\frac{\partial e_{i+1}^*}{\partial e_j}e_{j+1} + \frac{\partial e_{i+1}^*}{\partial y_d^{(j-1)}}y_d^{(j)}\right) \end{aligned} \tag{33}$$

so that (32) with $e_{i+2} = e_{i+2}^*$ becomes

$$\begin{aligned} \dot{\tilde{e}}_{i+1} &= -k\tilde{e}_{i+1} + \mu_{i+1}\tilde{e}_{i+1} - \tilde{e}_iM_{i+1} \\ &\quad + \frac{1}{M_{i+1}}\left(L_qL_f^ih - \textstyle\sum_{j=1}^{i}\frac{\partial e_{i+1}^*}{\partial e_j}L_qL_f^{j-1}h\right) \end{aligned} \tag{34}$$

Substituting (34) into (31), we have (with $e_{i+2} = e_{i+2}^*$)

$$\begin{aligned} \dot{V}_{i+1} &= -k\|\tilde{e}_{(i)}\|^2 + \|\tilde{e}_{(i)}\|\eta_i - k\tilde{e}_{i+1}^2 - \mu_{i+1}\tilde{e}_{i+1}^2 \\ &\quad + \frac{\tilde{e}_{i+1}}{M_{i+1}}\left(L_qL_f^ih - \textstyle\sum_{j=1}^{i}\frac{\partial e_{i+1}^*}{\partial e_j}L_qL_f^{j-1}h\right) \end{aligned} \tag{35}$$

By assumption (ii), we have

$$|L_qL_f^jh| \le \alpha_j(z_{(j+1)}), \qquad 0 \le j \le i, \tag{36}$$

Therefore, we can write

$$\begin{aligned} & L_qL_f^ih - \sum_{j=1}^{i}\frac{\partial e_{i+1}^*}{\partial e_j}L_qL_f^{j-1}h \le \alpha_i(z_{(i+1)}) \\ & + \sum_{j=1}^{i}\left|\frac{\partial e_{i+1}^*}{\partial e_j}\alpha_{j-1}(z_{(j)})\right| \\ & = \alpha_i(z_{(i+1)}) - \alpha_i(z_{(i+1)}^d) + \alpha_i(z_{(i+1)}^d) \\ & + \sum_{j=1}^{i}\left|\frac{\partial e_{i+1}^*}{\partial e_j}\alpha_{j-1}(z_{(j)}) - \frac{\partial e_{i+1}^*}{\partial e_j}\bigg|_{e_{(i)}=0}\alpha_{j-1}(z_{(j)}^d)\right. \\ & \left. + \frac{\partial e_{i+1}^*}{\partial e_j}\bigg|_{e_{(i)}=0}\alpha_{j-1}(z_{(j)}^d)\right| \end{aligned} \tag{37}$$

Since $\alpha_j$, $0 \le j \le i$, and $e_{i+1}^*$ are smooth functions and $y_d^{(j)}(t)$, $0 \le j \le i$, are bounded, there exist functions $\alpha_{i,j}(e_{(i+1)}, z_{(i+1)}^d)$ (see [9], p. 39) such that

$$\alpha_i(z_{(i+1)}) - \alpha_i(z_{(i+1)}^d) = \sum_{j=1}^{i+1}\alpha_{i,j}(e_{(i+1)}, z_{(i+1)}^d)e_j \tag{38}$$

and functions $\beta_{j,l}(e_{(i)}, z_{(i)}^d)$ such that

$$\begin{aligned} & \frac{\partial e_{i+1}^*}{\partial e_j}\alpha_{j-1}(z_{(j)}) - \frac{\partial e_{i+1}^*}{\partial e_j}\bigg|_{e_{(i)}=0}\alpha_{j-1}(z_{(j)}^d) \\ & = \sum_{l=1}^{i}\beta_{j,l}(e_{(i)}, z_{(i)}^d)e_l, \quad 1 \le j \le i. \end{aligned} \tag{39}$$

From (37)–(39), we obtain

$$\begin{aligned} & L_qL_f^ih - \sum_{j=1}^{i}\frac{\partial e_{i+1}^*}{\partial e_j}L_qL_f^{j-1}h \\ & \le \alpha_i + \sum_{j=1}^{i}\left|\frac{\partial e_{i+1}^*}{\partial e_j}\bigg|_{e_{(i)}=0}\alpha_{j-1}\right| \\ & + \sum_{j=1}^{i+1}\alpha_{i,j}e_j + \sum_{j=1}^{i}\left|\sum_{l=1}^{i}\beta_{j,l}e_l\right| \end{aligned} \tag{40}$$

Since $e_i$ are related to $\tilde{e}_i$ by the change of coordinates (21), (27) which preserves the origin, we can write for $1 \le j \le i$

$$\begin{aligned} \sum_{j=1}^{i+1}\alpha_{i,j}(e_{(i+1)}, z_{(i+1)}^d)e_j &= \sum_{j=1}^{i+1}\bar{\alpha}_{i,j}(e_{(i+1)}, z_{(i+1)}^d)\tilde{e}_j \\ \sum_{l=1}^{i}\beta_{j,l}(e_{(i)}, z_{(i)}^d)e_l &= \sum_{l=1}^{i}\bar{\beta}_{j,l}(e_{(i)}, z_{(i)}^d)\tilde{e}_l. \end{aligned} \tag{41}$$

Therefore, defining

$$\begin{aligned} \Gamma_0(z_{(i+1)}^d) &= \alpha_i(z_{(i+1)}^d) \\ &\quad + \sum_{j=1}^{i}\left|\frac{\partial e_{i+1}^*}{\partial e_j}\bigg|_{e_{(i)}=0}\alpha_{j-1}(z_{(j)}^d)\right| \\ \Gamma_1(e_{(i+1)}, z_{(i+1)}^d) &= \frac{1}{M_{i+1}}\Bigg(\sum_{j=1}^{i+1}|\bar{\alpha}_{i,j}(e_{(i+1)}, z_{(i+1)}^d)| \\ &\quad + \sum_{j=1}^{i}\sum_{l=1}^{i}|\bar{\beta}_{j,l}(e_{(i)}, z_{(i)}^d)|\Bigg) \end{aligned} \tag{42}$$

from (35), we have

$$\begin{aligned}\dot{V}_{i+1} \quad \leq \quad & -k\|\tilde{e}_{(i)}\|^2 + \|\tilde{e}_{(i)}\|\eta_i - k\tilde{e}_{i+1}^2 - \mu_{i+1}\tilde{e}_{i+1}^2 \\ & + \frac{\Gamma_0}{M_{i+1}}|\tilde{e}_{i+1}| + |\tilde{e}_{i+1}|\Gamma_1\|\tilde{e}_{(i+1)}\|\end{aligned} \tag{43}$$

Let $M_{i+1}$ be defined as a constant such that

$$\left| \frac{\partial e_{i+1}^*}{\partial e_j}\bigg|_{e_{(i)}=0} \right| \leq M_{i+1}, \quad 1 \leq j \leq i, \tag{44}$$

which exists since $y_d$ and its *derivatives* are bounded, so that

$$\lim_{k\to\infty} \frac{\Gamma_0(z_{(i+1)}^d)}{kM_{i+1}} = 0. \tag{45}$$

From (43), we obtain

$$\dot{V}_{i+1} \leq -\frac{k}{2}\|\tilde{e}_{(i+1)}\|^2 + \|\tilde{e}_{(i)}\|\eta_i + \frac{\Gamma_0}{M_{i+1}}|\tilde{e}_{i+1}|$$
$$-\begin{bmatrix} \tilde{e}_{(i+1)} \\ \tilde{e}_{i+1} \end{bmatrix}^T \begin{bmatrix} k/2 & -\Gamma_1/2 \\ -\Gamma_1/2 & \mu_{i+1} \end{bmatrix} \begin{bmatrix} \tilde{e}_{(i+1)} \\ \tilde{e}_{i+1} \end{bmatrix}$$

Therefore, choosing

$$\mu_{i+1} \geq \frac{\Gamma_1^2(e_{(i+1)}, z_{(i+1)}^d)}{2k} \tag{46}$$

the thesis is proved with

$$\eta_{i+1}(z_{(i+1)}^d, k) = \frac{\Gamma_0(z_{(i+1)}^d)}{M_{i+1}} + \eta_i(z_{(i)}^d, k). \tag{47}$$

□

Since we have shown that the hypotheses of the claim are true for $i = 1$, applying $(\rho - 1)$-times the claim we can construct a function

$$e_{\rho+1}^* = e_{\rho+1}^*(e_1, \ldots, e_\rho, y_d, \ldots, y_d^{(\rho-1)}) \tag{48}$$

which determines the final control $v$ as

$$v = e_{\rho+1}^* \tag{49}$$

We also construct a change of coordinates

$$\tilde{e}_1 = e_1, \tilde{e}_2 = \frac{e_2 - e_2^*}{M_2}, \ldots, \tilde{e}_\rho = \frac{e_\rho - e_\rho^*}{M_\rho} \tag{50}$$

such that the function

$$V_\rho = \frac{1}{2}\sum_{j=1}^{\rho} \tilde{e}_j^2 \tag{51}$$

has time derivative satisfying the inequality (with $k$ suitably redefined)

$$\dot{V}_\rho \leq -k\left\|\begin{bmatrix} \tilde{e}_1 \\ \vdots \\ \tilde{e}_\rho \end{bmatrix}\right\|^2 + \left\|\begin{bmatrix} \tilde{e}_1 \\ \vdots \\ \tilde{e}_\rho \end{bmatrix}\right\| \eta_\rho(y_d, \ldots, y_d^{(\rho-1)}, k) \tag{52}$$

with

$$\lim_{k\to\infty} \frac{\eta_\rho(y_d, \ldots, y_d^{(\rho-1)}, k)}{k} = 0. \tag{53}$$

Therefore, $\tilde{e}_i$, $1 \leq i \leq \rho$, are bounded and consequently, since the reference signal $y_d(t)$ is bounded with its $\rho - 1$ time derivatives, $z_i(t)$, $1 \leq i \leq \rho$, are bounded. Since Assumption 2 holds $\|z_r(t)\|$ is also bounded. By virtue of (53), we have

$$|\eta_\rho| \leq \gamma > 0,$$

so that for every $|e_1| \geq \epsilon/2$

$$\frac{\dot{V}_\rho}{V_\rho} \leq -2k + 2\gamma/\epsilon,$$

which implies

$$e_1^2(t) \leq V_\rho(t) \leq V_\rho(0)e^{(-2k+2\gamma/\epsilon)t}.$$

For any $\epsilon > 0$, $T > 0$ there exists $k$ which solves the problem. Therefore, the problem is solved by the feedback controller given by (8) and (49) with a suitable choice of $k$. □

**Remark.** Theorem 2.1 generalizes the main result in [6] in several ways. The disturbances are only allowed to enter linearly in [6] and are required to have bounded time derivatives while in this paper they may enter nonlinearly and no requirement is made on their time derivatives. While condition (ii) is common to both theorems the most important difference lies in condition (i) which considerably weakens the corresponding condition in two ways. The result in [6] requires for $\nu - 1 \leq i \leq \rho - 1$,

$$d(L_q L_f^i h) \in \text{span}\{dh, \ldots, d(L_f^{\nu-1}h)\}, \tag{54}$$

while condition (i) in Theorem 2.1 only requires

$$d(L_q L_f^i h) \in \text{span}\{dh, \ldots, d(L_f^i h)\}, \quad \nu-1 \leq i \leq \rho-1, \tag{55}$$

or even the weaker condition on some bounding functions

$$|L_q L_f^i h| \leq \alpha_i, \quad \nu - 1, \leq i \leq \rho - 1,$$

with

$$d\alpha_i \in \text{span}\{dh, \ldots, d(L_f^i h)\}.$$

For instance condition (55) applies to system (3) while the stronger condition (54) does not.

## 3 Example

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 + \theta_1(t) \\ \dot{x}_2 &= x_2^3\theta_2(t) + u \\ y &= x_1\end{aligned} \tag{56}$$

with $|\theta_1(t)| \leq 1$ and $|\theta_2(t)| \leq 1$, where $\theta(t) = [\theta_1(t), \theta_2(t)]^T$ is a disturbance signal. It is easily seen that system (31) satisfies the conditions of Theorem 2.1 with

$$\begin{aligned} \alpha_0 &= 1 \geq L_q h = \theta_1(t) \\ \alpha_1 &= x_2^3 \geq L_q L_f h = x_2^3 \theta_2(t). \end{aligned} \tag{57}$$

Suppose that $y_d(t)$ is the desired output reference to be tracked. Define

$$e_1 = x_1 - y_d, \qquad e_2 = x_2 - \dot{y}_d. \tag{58}$$

From (31) and (32), we have

$$\begin{aligned} \dot{e}_1 &= e_2 + \theta_1(t) \\ \dot{e}_2 &= x_2^3 \theta_2(t) + u - \ddot{y}_d \stackrel{\Delta}{=} x_2^3 \theta_2(t) + v. \end{aligned}$$

Define as in (10)

$$e_2^* = -ke_1 - e_1 \mu_1(t)$$

Since by (16)

$$\mu_1 \geq \bar{\alpha}_0(e_1, y_d) = 0$$

we choose $\mu_1 = 0$ so that

$$e_2^* = -ke_1.$$

Define as in (21) and (33)

$$\begin{aligned} \tilde{e}_2 &= \frac{e_2 - e_2^*}{M_2} \\ v &= -M_2(k\tilde{e}_2 + \tilde{e}_2 \mu_2(t) + M_2 e_1) - ke_2 \end{aligned}$$

According to (44), we have

$$M_2 = k$$

and according to (42)

$$\begin{aligned} \Gamma_0 &= \dot{y}_d^3 + k \\ \Gamma_1 &= \frac{1}{k} 2k \left| \frac{x_2^3 - \dot{y}_d^3}{e_2} \right| = 2|e_2^2 + 3e_2\dot{y}_d + 3\dot{y}_d^2| \end{aligned}$$

which imply

$$\mu_2 = \frac{2}{k}(e_2^2 + 3e_2\dot{y}_d + 3\dot{y}_d^2)^2.$$

The final control $u$ is given by

$$u = \ddot{y}_d - 2k^2(\tilde{e}_2 + e_1) - 2\tilde{e}_2(e_2^2 + 3e_2\dot{y}_d + 3\dot{y}_d^2)^2. \tag{59}$$

If we considered as in [6] the problem of stabilizing the linear approximation of system (56) we would obtain, using the same design technique, the following control law

$$u = -k(k\tilde{x}_2 + kx_1) - kx_2 = -2k(x_2 + kx_1)$$

which is very similar to the one obtained in [6]

$$u = -\frac{1}{\epsilon^2}(\epsilon x_2 + x_1) \tag{60}$$

once we define $\epsilon = \frac{1}{k}$. In [6] it was shown that the control algorithm (60) does not guarantee almost disturbance decoupling for any bounded disturbance. Therefore, the nonlinear part of the control law (59) is crucial in order to obtain disturbance attenuation.

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