# Relations between (H∞) optimal control of a nonlinear system and its linearization

Schaft, A.J. van der

*Published in:*
Proceedings of the 30th IEEE Conference on Decision and Control

Proceedings of the 30th Conference
on Decision and Control
Brighton, England • December 1991



# Relations between ($\mathcal{H}_\infty$) optimal control of a nonlinear system and its linearization

A.J. van der Schaft

Department of Applied Mathematics, University of Twente
P.O. Box 217, 7500 AE Enschede, The Netherlands
E-mail: twarjan@math.utwente.nl

February 25, 1991; revised version August 16, 1991.

**Abstract.** In a previous paper we showed some basic connections between $\mathcal{H}_\infty$ control of a nonlinear control system and $\mathcal{H}_\infty$ control of its linearization. A key argument was that the existence and parametrization, at least locally, of the stable invariant manifold of a certain Hamiltonian vector field is determined by the Hamiltonian matrix corresponding to the linearized problem. Using the same methodology we are able to give a quick proof of the fact that a nonlinear optimal control problem is locally solvable if the associated LQ problem is solvable. This was proved before by Lukes under much stronger conditions.

Consider a smooth nonlinear control system, affected by (unknown) disturbances $d$, which in local coordinates $x = (x_1, \ldots, x_n)$ for a state space manifold $M$ is given as

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + k(x)d, \quad u \in \mathbf{R}^m, d \in \mathbf{R}^q, \\ y &= h(x), \qquad\qquad\qquad y \in \mathbf{R}^p, \end{aligned} \tag{1}$$

with $g(x)$ and $k(x)$ denoting an $n \times m$, respectively $n \times q$, matrix with entries depending smoothly on $x$. We will assume throughout the existence of an equilibrium $x_0 \in M$, i.e., $f(x_0) = 0$, and without loss of generality we assume that $h(x_0) = 0$. Also we consider the linearization of (1) around $x_0$, denoted as

$$\begin{aligned} \dot{\bar{x}} &= F\bar{x} + G\bar{u} + K\bar{d}, \quad \bar{u} \in \mathbf{R}^m, \bar{d} \in \mathbf{R}^q, \bar{x} \in \mathbf{R}, \\ \bar{y} &= H\bar{x}, \qquad \bar{y} \in \mathbf{R}^p, \end{aligned} \tag{2}$$

where

$$F = \frac{\partial f}{\partial x}(x_0), G = g(x_0), K = k(x_0), H = \frac{\partial h}{\partial x}(x_0). \tag{3}$$

The main theorem obtained in [7], see [9] for further information, reads as follows

**Theorem 1** *Assume that $(F, G)$ is stabilizable, and that $(H, F)$ is detectable. Let $\gamma > 0$. Suppose there exists a symmetric solution $P \geq 0$ of the Riccati equation*

$$F^T P + PF - P(GG^T - \frac{1}{\gamma^2}KK^T)P + H^T H = 0, \tag{4}$$

*satisfying*

$$\sigma(F - GG^T P + \frac{1}{\gamma^2}KK^T P) \subset C^-. \tag{5}$$

*Then there exists a neighborhood $W$ of $x_0$, and a nonlinear feedback $u = l(x)$ such that*

$$\dot{x} = f(x) + g(x)l(x) \text{ is asymptotically stable on } W \tag{6}$$

$$||y||^2_{L_2} + ||u||^2_{L_2} < \gamma^2 ||d||^2_{L_2} \qquad x(0) = x_0, \tag{7}$$

*for all disturbance functions $d \in L_2$ such that the state space trajectories starting from $x(0) = x_0$ remain in $W$.* □

It is well-known from the state space approach to $\mathcal{H}_\infty$-control for linear systems ([4,5]) that the existence of a solution $P \geq 0$ to (4,5) is equivalent to the existence of a linear feedback $\bar{u} = L\bar{x}$ such that the linearized system (2) satisfies

$$F + GL \text{ is asymptotically stable,} \tag{8}$$

$$||\bar{y}||^2_{L_2} + |\bar{u}|^2_{L_2} < \gamma^2 ||\bar{d}||^2_{L_2}, \qquad \bar{x}(0) = 0, \tag{9}$$

for all disturbance functions $\bar{d} \in L_2$; i.e., the $\mathcal{H}_\infty$ norm from disturbances $\bar{d}$ to the vector of inputs $\bar{u}$ and outputs $\bar{y}$ can be made smaller than $\gamma$ by state feedback. (In fact one can take $\bar{u} = -G^T P\bar{x}$.) Thus Theorem 1 can be rephrased by saying that the "$\mathcal{H}_\infty$-norm" (or better, $L_2$-induced norm) of the nonlinear system (1) can be made smaller than a given constant $\gamma$ *if* the $\mathcal{H}_\infty$ norm of the linearized system (2) can be made smaller than $\gamma$ by linear feedback. (However if the neighborhood $W$ is strictly contained in $M$ then we have to restrict to disturbances $d$ for (1) which are sufficiently small.) For a preliminary analysis of the *size* of the neighborhood $W$ we refer to [8]. The key argument in the proof of Theorem 1 is the fact that the existence of a solution $P$ to (4,5) is *equivalent* to the local existence (around $x_0$) of a function $V : M \to \mathbf{R}^+$ satisfying the Hamilton-Jacobi equation

$$\begin{aligned} &\tfrac{\partial V}{\partial x}(x)f(x) + \tfrac{1}{2}h^T(x)h(x) \\ &-\tfrac{1}{2}\tfrac{\partial V}{\partial x}(x)\left[g(x)g^T(x) - \tfrac{1}{\gamma^2}k(x)k^T(x)\right]\left[\tfrac{\partial V}{\partial x}(x)\right]^T = 0, \end{aligned} \tag{10}$$

with boundary conditions

$$V(x_0) = 0, \; \frac{\partial V}{\partial x}(x_0) = 0, \; \frac{\partial^2 V}{\partial x^2}(x_0) = P. \tag{11}$$

Notice that, as in the linear case [4], the Hamilton-Jacobi equation (10) tends for $\gamma \to \infty$ to the Hamilton-Jacobi-Bellman equation corresponding to the optimal control problem

$$\begin{aligned} &\min_u \textstyle\int_0^\infty \left(\frac{1}{2}||y||^2 + \frac{1}{2}||u||^2\right) dt, \\ &\dot{x} = f(x) + g(x)u, \; y = h(x). \end{aligned} \tag{12}$$

We now wish to elaborate more generally on the connections with nonlinear optimal control. In particular we will show how our methods provide a much quicker, and in our opinion more transparent, proof of a result by Lukes [6] (see also [2], [1]) relating the existence of a *local* solution of the general nonlinear optimal control problem to the existence of a solution of a particular LQ problem obtained by *linearizing* the nonlinear problem. Also we will be able to weaken the conditions imposed in [6] considerably. Consider the infinite horizon optimal control problem

$$\begin{aligned} &\min_u \textstyle\int_0^\infty L(x(t), u(t))\, dt \\ &\dot{x} = f(x, u). \end{aligned} \tag{13}$$

Here, as before, $x = (x_1, \ldots, x_n)$ are local coordinates for a state space manifold $M$, $u \in \mathbf{R}^m$, and $f$ and $L$ are smooth functions of their arguments. We assume throughout the existence of an equilibrium $(x_0, u_0)$ such that

$$\begin{aligned} &(a) \qquad f(x_0, u_0) = 0 \\ &(b) \qquad L(x_0, u_0) = 0 \\ &(c) \qquad dL(x_0, u_0) = 0 \end{aligned} \tag{14}$$

Then the linearized version of the nonlinear optimal control problem (13) is given by the following LQ problem (see [6])

$$\min_{\bar{u}} \int_0^\infty \left(\tfrac{1}{2}\bar{x}^T Q\bar{x} + \bar{x}^T N\bar{u} + \tfrac{1}{2}\bar{u}^T R\bar{u}\right) dt$$
$$\dot{\bar{x}} = A\bar{x} + B\bar{u}, \tag{15}$$

where (compare with (3))

$$A = \tfrac{\partial f}{\partial x}(x_0,u_0), B = \tfrac{\partial f}{\partial u}(x_0,u_0)$$
$$Q = \tfrac{\partial^2 L}{\partial x^2}(x_0,u_0), R = \tfrac{\partial^2 L}{\partial u^2}(x_0,u_0), N = \tfrac{\partial^2 L}{\partial x \partial u}(x_0,x_0) \tag{16}$$

It is well-known from linear optimal control theory that the LQ problem (15) admits a solution if the following standard assumptions are satisfied.

### Assumptions

(1) $R > 0$, (2) $\begin{pmatrix} Q & N \\ N^T & R \end{pmatrix} \geq 0$

(3) $(A - BR^{-1}N^T,\ BR^{-1}B^T)$ is stabilizable
$(A - BR^{-1}N^T,\ Q - NR^{-1}N^T)$ is detectable

**Theorem 2** *Suppose Assumptions* 1, 2, 3 *are satisfied (i.e., the* LQ *problem* (15) *is solvable). Then there exists a neighborhood $W$ of $x_0$ such that the nonlinear optimal control problem is solvable for $x \in W$. The optimal control is given in feedback form by a smooth nonlinear feedback $u = l(x)$, which is such that $\bar{u} = L\bar{x}$ with*

$$L = \frac{\partial l}{\partial x}(x_0) \tag{17}$$

*is the solution of the* LQ *problem (15).*

**Proof** Define the pseudo-Hamiltonian on $T^*M$

$$H(x,p,u) = p^T f(x,u) + L(x,u) \tag{18}$$

Then $H$ satisfies because of (14) and Assumption 1

$$(a)\ \ H(x_0,0,u_0) = 0, \qquad (b)\ dH(x_0,0,u_0) = 0,$$
$$(c)\ \ \tfrac{\partial^2 H}{\partial u^2}(x_0,0,u_0) > 0 \tag{19}$$

By (c) the equation

$$\frac{\partial H}{\partial u}(x,p,u) = 0 \tag{20}$$

has a unique solution $u = u^*(x,p)$ satisfying $u^*(x_0,0) = u_0$ for all $(x,p)$ near $(x_0,0)$. Furthermore by continuity

$$\frac{\partial^2 H}{\partial u^2}(x,p,u^*(x,p)) > 0 \tag{21}$$

for $(x,p)$ near $(x_0,0)$. Hence for $(x,p)$ near $(x_0,0)$

$$H(x,p,u) > H(x,p,u^*(x,p)) \qquad \text{for all } u \text{ near } u^*(x,p) \tag{22}$$

Consider now the Hamilton-Jacobi equation

$$\frac{\partial V}{\partial x}(x)f(x,u^*(x,\frac{\partial V}{\partial x}(x))) + L(x,u^*(x,\frac{\partial V}{\partial x}(x))) = 0 \tag{23}$$

corresponding to the optimal Hamiltonian $H^*(x,p) = p^T f(x,u^*(x,p)) + L(x,u^*(x,p))$. The Jacobian in $(x_0,0)$ of the Hamiltonian vector field $X_{H^*}$ on $T^*M$ with Hamiltonian $H^*$ is given by the Hamiltonian matrix

$$\begin{bmatrix} A - BR^{-1}N^T & BR^{-1}B^T \\ Q - NR^{-1}N^T & -(A - BR^{-1}N^T)^T \end{bmatrix} \tag{24}$$

By a standard result in the theory of Riccati equations Assumptions 1,2,3 imply that this matrix has no eigenvalues on the imaginary axis and that its generalized stable eigenspace is of the form span $\begin{bmatrix} I \\ P \end{bmatrix}$ for some $P > 0$ (resulting in the solvability of the LQ-problem (15)). However by the same reasoning as in [7] this also implies that the vector field $X_{H^*}$ possesses an $n$-dimensional stable invariant manifold through $(x_0,0)$ which around $x_0$ is parametrized by $x$, and that there exists a neighborhood $W$ of $x_0$ on which the Hamilton-Jacobi equation (17) has a solution $V$ with $V(x_0) = 0$ such that this stable invariant manifold consists of all points $(x, p = \frac{\partial V}{\partial x}(x))$, $x$ around $x_0$. It immediately follows from (23) that

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}(x)f(x,u) = \Big[\frac{\partial V}{\partial x}(x)f(x,u) + L(x,u)$$
$$-\frac{\partial V}{\partial x}(x)f(x,u^*(x,\frac{\partial V}{\partial x}(x)) - L(x,u^*(x,\frac{\partial V}{\partial x}(x)))\Big] \tag{25}$$
$$-L(x,u)$$

where the expression between brackets on the right-hand side is always nonnegative by (22). Integration yields (for $x(0) \in W$)

$$\int_0^\infty L(x(t),u(t))\,dt =$$
$$\int_0^\infty [*]\,dt - V(x(\infty)) + V(x(0)) \tag{26}$$

where the expression between brackets is the same as in (25). Since $x(\infty) = x_0$ it follows that the optimal control is given as

$$u^*(t) = l(x(t)) := u^*(x(t), \frac{\partial V}{\partial x}(x(t))) \tag{27}$$

while $\min_u \int_0^\infty L(x(t),u(t))\,dt = V(x(0))$. The rest of the proof uses the same arguments as the proof of Theorem 8 in [7]. □

**Remark 1** In [6] more or less the same statement was obtained under the much stronger assumption (instead of Assumptions 1,2,3) that $\begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} > 0$.

**Remark 2** After submitting this paper I found out that a statement similar to Theorem 2 for the restricted class of nonlinear systems $\dot{x} = f(x) + g(x)u$ with restricted cost criterion $L(x,u) = \frac{1}{2}h^T(x)h(x) + \frac{1}{2}u^T u$ has been obtained in the interesting paper [3].

### References


[1] E.G. Albrecht, On the optimal stabilization of nonlinear systems, J. Appl. Math. Mech. 25 (1962), 1254-1266.

[2] P. Brunovsky, On optimal stabilization of nonlinear systems, Mathematical Theory of Control (eds. A.V. Balakrishnan, L.W. Neustadt), Academic Press, New York (1963).

[3] C.I. Byrnes, New methods for nonlinear optimal control, Proc. 1st ECC, Grenoble, July 2-5, 1991, Hermes, Paris, 2043-2047.

[4] J.C. Doyle, K. Glover, P.P. Khargonekar, B.A. Francis, State space solutions to standard $\mathcal{H}_\in$ and $\mathcal{H}_\infty$ control problems, IEEE Trans. Automat. Control 34 (1990), 831-846.

[5] P.P. Khargonekar, I.R. Petersen, M.A. Rotea, $\mathcal{H}^\infty$ optimal control with state feedback, IEEE Trans. Automat. Control 33 (1988), 786-788.

[6] D.L. Lukes, Optimal regulation of nonlinear dynamical systems, SIAM J. Control 7 (1969), 75-101.

[7] A.J. van der Schaft, On a state space approach to nonlinear $H_\infty$ control, Systems & Control Letters 16 (January 1991), 1-8.

[8] A.J. van der Schaft, On the Hamilton-Jacobi equation of nonlinear $\mathcal{H}_\infty$ optimal control, in Proc. 1st ECC, Grenoble, July 2-5, 1991, Hermes, Paris, 649-654.

[9] A.J. van der Schaft, $L_2$-gain analysis of nonlinear systems and nonlinear $\mathcal{H}_\infty$ control, April 1991, Memorandum No. 969, University of Twente, submitted to IEEE Trans. Automat. Control.