On Nonlinear Observers

Schaft, A.J. van der

Published in:
IEEE Transactions on Automatic Control

the closed-loop system reaches the equilibrium position $\theta_i = \dot{\theta}_i = 0$ $i = 1, 2$. However, unlike [4], steady-state errors are not observed. Indeed, it can be shown that the origin is the unique equilibrium point of the closed-loop system. In Fig. 3, we plot the state trajectories after the decentralized control has been connected ($t = 40$ s).

VI. Conclusions

In this note, we proposed a new robustness condition for optimal regulators of nonlinear dynamic systems. We considered additive perturbations (state depending) on the control variables.

The stability condition does not depend on the solution of the optimal control problem under consideration. On the contrary, it depends only on the performance index and consequently can be used by the designer in order to simplify considerably the feedback gains determination. In this sense, functions $\psi(x)$ can be interpreted as coupling terms which can be neglected during the optimization process (Theorem 3).

In [3], many stability conditions were established for more general systems. However, we can say that for the class of nonlinear system considered here, the provided stability condition promotes important improvements as shown by a simple example (Section III).

References

[1] M. Athans, and P. L. Falb, *Optimal Control: An Introduction to the Theory and Its Applications.* New York: McGraw-Hill, 1966.

[2] J. C. Geromel, and J. Bernussou, "Optimal decentralized control of dynamic systems," *Automatica,* vol. 18, no. 5, 1982.

[3] S. T. Glad, "On the gain margin of nonlinear and optimal regulators," *IEEE Trans. Automat. Contr.,* vol. AC-29, 1984.

[4] S. Lefevre, S. Richter, and R. DeCarlo, "Decentralized variable structure control design for a two-pendulum system," *IEEE Trans. Automat. Contr.,* vol. AC-28, 1983.

[5] M. Safonov and M. Athans, "Gain and phase margin for multiloop LQG regulators," *IEEE Trans. Automat. Contr.,* vol. AC-22, pp. 173-179, 1977.

[6] P. K. Wong, and M. Athans, "Closed loop structural stability for linear quadratic optimal systems," *IEEE Trans. Automat. Contr.,* vol. AC-22, pp. 94-99, 1977.

On Nonlinear Observers

A. J. VAN DER SCHAFT

Abstract—**In this note we make some remarks about the general observer problem for nonlinear systems. We show how the existence of a partial observer without stability requirements is locally equivalent to the existence of a conditioned invariant distribution. We argue that by considering not only identity observers we can put the stability issue in a broader perspective.**

Introduction

Consider a smooth (i.e., C^∞ or C^k) vector field A on a smooth n-dimensional manifold M, in local coordinates $x = (x_1, \cdots, x_n)$ for M given by

$$\dot{x} = A(x) \tag{1a}$$

together with a smooth output mapping $C: M \to Y$, where Y is a smooth p-dimensional output manifold, in local coordinates x for M and $y = (y_1, \cdots, y_p)$ for Y given by

$$y_j = C_j(x), \qquad j = 1, \cdots, p. \tag{1b}$$

Manuscript received February 21, 1985; revised July 18, 1985.

The author is with the Department of Applied Mathematics, Twente University of Technology, Enschede, The Netherlands.

A Luenberger observer [8], or shortly *observer*, for the system (1) is another dynamical system

$$\dot{w} = K(w, y) \tag{2a}$$

evolving on a certain smooth k-dimensional manifold W, driven by the outputs y of system (1), which yields an estimate

$$\hat{x} = F(w, y) \tag{2b}$$

of the state x of the system (1) (where F is a smooth mapping). Such an observer has to satisfy the following two basic requirements.

Condition A: If for a certain t_o, $\hat{x}(t_o) = x(t_o)$, then $\hat{x}(t) = x(t)$ for all $t \geq t_o$.

Condition B: $\hat{x}(t)$ should converge to $x(t)$ for $t \to \infty$ sufficiently fast, irrespective of the initial conditions $x(t_o)$ and $w(t_o)$.

For the linear case $\dot{x} = Ax$, $y = Cx$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, this has been completely solved [8] resulting, for instance, in the identity observer (the estimate $\hat{x}$ equals the state of the observer)

$$\dot{w} = (A - GC)w + Gy, \qquad \hat{x} = w, \quad w \in \mathbb{R}^n \tag{3}$$

where the matrix G is such that $A - GC$ is asymptotically stable. (Such a G always exists if (C, A) is observable, or at least detectable.) In the nonlinear case the construction of an observer is much more delicate. There has been various attempts to construct identity-like observers (for instance [5], [11]). Recently, there has been considerable interest in finding conditions in order to transform a nonlinear system (1) by coordinate transformations on M and Y to the form (with A and C matrices)

$$\dot{x} = Ax + \gamma(y), \qquad y = Cx \tag{4}$$

(cf. [1], [6], [7]). An obvious (identity) observer for (4) is

$$\dot{w} = (A - GC)w + Gy + \gamma(y), \qquad \hat{x} = w \tag{5}$$

where G is chosen such that $A - GC$ is asymptotically stable.

In this note we wish to make some remarks about the general case, where a transformation to (4) is not possible. We will mainly concentrate on Condition A. Instead of focusing on identity observers that automatically satisfy Condition A we will deal with general observers and general *partial* observers satisfying Condition A. (A partial observer yields an estimate of a *part* of the state.) We will show how the existence of a partial observer (without stability requirement B) is locally equivalent to the existence of a so-called *conditioned invariant* distribution. This generalizes some more or less well-known results in the linear case (see, for instance, [10], [12]). Finally, we discuss the (hard) stability problem related to Condition B.

(Partial) Observers and Conditioned Invariance

Recall the definition of a dynamical system with (everything is smooth) state-space manifold N and input manifold U. It is given by a mapping

$$N \times U \xrightarrow{F} TN \tag{6}$$

such that the diagram

$$\begin{array}{ccc} N \times U & \xrightarrow{F} & TN \\ & {\scriptstyle \pi_N} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & N & \end{array} \tag{7}$$

commutes (π_N projection on first factor, π usual projection from TN to N). If we write $N \times U \xrightarrow{F} TN$ we will throughout assume the commutativity of (7). Furthermore, we use the abuse of notation $F(n, u) = (n, F(n, u))$ with $n \in N$ and $F(n, u) \in T_nN$. In the same way we will denote a vector field A on a manifold M also as a section A: $M \to TM$ assuming that $\pi \circ A =$ identity, with $\pi: TM \to M$ the usual projection.

Definition 1: Let (1) be a nonlinear system with state-space manifold M. A nonlinear system with smooth k-dimensional state-space manifold

W and input manifold Y, defined by a smooth map $K: W \times Y \to TW$, is a *partial observer* for (1) if there exists a smooth mapping $\phi: M \to W$ such that the following diagram

$$\begin{array}{ccc} M & \xrightarrow{A} & TM \\ \phi \swarrow & \searrow C & \downarrow \phi_* \\ W \times Y & \xrightarrow[K]{} & TW \end{array} \tag{8}$$

commutes. (ϕ_* is the usual tangent bundle mapping, in local coordinates given by $(x, \dot{x}) \to (\phi(x)$, both cases with $\partial\phi(x)/\partial x$ the Jacobian.) If the mapping $(\phi, C): M \to W \times Y$ is a diffeomorphism from M to $W \times$ Image C, then it is a (full) observer.

Remark: This definition can be easily generalized from systems (1) to systems with inputs $\dot{x} = f(x, u)$, $y = h(x, u)$.

In local coordinates x for M and $w = (w_1, \cdots, w_k)$ for W a partial observer is given by $\dot{w} = K(w, y)$, and the commutativity of (8) means that

$$\sum_{j=1}^{n} \frac{\partial \phi_i(x)}{\partial x_j} A_j(x) = K_i(\phi(x), C(x)), \qquad i = 1, \cdots, k \tag{9}$$

where $A(x) = (A_1(x), \cdots, A_n(x))$, $\phi(x) = (\phi_1(x), \cdots, \phi_k(x))$ and $K(x) = (K_1(x), \cdots, K_k(x))$. Hence, if $x(t)$ is the solution of (1a) with $x(t_o) = x_o$, then $\bar{w}(t) = (\bar{w}_1(t), \cdots, \bar{w}_k(t)) := (\phi_1(x(t)), \cdots, \phi_k(x(t)) = \phi(x(t))$ satisfies

$$\frac{d}{dt} \bar{w}_i(t) = \frac{d}{dt} \phi_i(x(t)) = \sum_{j=1}^{n} \frac{\partial \phi_i(x(t))}{\partial x_j} A_j(x(t))$$
$$= K_i(\phi(x(t)), C(x(t))) = K_i(\bar{w}(t), y(t)) \tag{10}$$

with $y(t) = C(x(t))$. On the other hand the solution $w(t)$ of the observer for an arbitrary initial condition $w(t_o) = w_o$ and the same function $y(t)$, satisfies

$$\frac{d}{dt} w_i(t) = K_i(w(t), y(t)). \tag{11}$$

Hence, if $w_o = \phi(x_o)$, then $w(t) = \phi(x(t)) = \bar{w}(t)$, for all $t \geq t_o$ and so Condition A is satisfied.

If $(\phi, C): M \to W \times$ Image C is a diffeomorphism, then we denote the inverse mapping by $F(w, y)$, and we have obtained an observer (2) satisfying Condition A. If $(\phi, C): M \to W \times$ Image C is not a diffeomorphism, we have obtained a partial observer, giving an estimate of a *part* of the state. In fact, locally this part is given by M modulo kernel (ϕ, C). This will now be made clear using the notion of conditioned invariance or "(h, f)-invariance" [4].

Remark: In case (1) is a *linear* system $\dot{x} = Ax$, $y = Cx$, $x \in \mathbb{R}^n$, ϕ is a *linear* mapping $\phi(x) = Tx$, and (2) is a *linear* observer $\dot{w} = Kw + Ly$, the condition of commutativity (9) reduces to $TA = KT + LC$. This is exactly the condition for a linear observer as stated in [8]. Furthermore, in this case (ϕ, C) is a diffeomorphism from M to $W \times$ Image C if and only if the matrix $[T \vdots C]$ has rank n, which following [8] is the condition for a reduced order linear observer.

Define for a mapping $C: M \to Y$ the codistribution dC by

$$dC(x) = \text{span}_{\mathbb{R}} \{dC_1(x), \cdots, dC_p(x)\} (= C^*(T^*Y)(x)). \tag{12}$$

Definition 2: Let (1) be a nonlinear system on M. Let D be an involutive distribution on M. Then D is conditioned invariant if

$$[A, D \cap \ker dC] \subset D \tag{13}$$

or equivalently [4]

$$L_A P \subset P + dC \tag{14}$$

where P is a codistribution such that Ker P (all vector fields X on M such that $\theta(X) = 0$ for all one-forms in P) equals D. ($L_A P$ means the codistribution containing all Lie-derivatives along A of all one-forms in P.)

Proposition 3: Let $K: W \times Y \to TW$ be a partial observer for (1), with $\phi: M \to W$ of constant rank. Then the distribution Ker ϕ_* on M is conditioned invariant. Conversely, suppose D is an involutive conditioned invariant distribution of constant dimension. Then M can be locally factored out by the leaves of the foliation induced by D to a manifold W, i.e., there exists locally a mapping $\phi: M \to W$ of constant dimension such that Ker $\phi_* = D$. Assume furthermore that the codistribution $P + dC$, with Ker $P = D$, has constant dimension. Then there exists locally a mapping $K: W \times Y \to TW$ such that (8) commutes, i.e., a partial observer.

Proof: Let $K: W \times Y \to TW$ be a partial observer. Write $\phi = (\phi_1, \cdots, \phi_k)$ in local coordinates w for W. Then by (9)

$$L_A(\phi_i)(x) = \sum_{j=1}^{n} \frac{\partial \phi_i(x)}{\partial x_j} A_j(x) = K_i(\phi(x), C(x)), \qquad i = 1, \cdots, k \tag{15}$$

and hence $L_A(d\phi_i) = d(L_A\phi_i) = dF_i(\phi(x), C(x)) \subset d\phi + dC$, with the codistribution $d\phi$ defined by $d\phi(x) = \text{span}_{\mathbb{R}} \{d\varphi_1(x), \cdots, d\varphi_k(x)\} (= \phi^*(T^*W))$. Therefore, $L_A(d\phi) \subset d\phi + dC$, and so Ker $d\phi =$ Ker ϕ_* is conditioned invariant.

Conversely assume D to be conditioned invariant. Since D is involutive and of constant dimension there exist locally k independent functions $\phi_1, \cdots, \phi_k$ such that $D =$ Ker $d\phi :=$ Ker $\text{span}_{\mathbb{R}} \{d\phi_1, \cdots, d\phi_k\}$ with $\phi = (\phi_1, \cdots, \phi_k): M \to W := \mathbb{R}^k$. Because D is conditioned invariant

$$d(L_A\phi_i) = L_A(d\phi_i) \subset d\phi + dC \qquad i = 1, \cdots, k. \tag{16}$$

Since $d\phi + dC$ is of constant dimension (and involutive) there exist by Frobenius' theorem local coordinates $x = (x_1, \cdots, x_n)$ for M such that $d\phi + dC = \text{span}\{dx_1, \cdots, dx_r\} (r \leq n)$. Now write

$$L_A\phi_i(x) = \tilde{K}_i(x_1, \cdots, x_n) \qquad i = 1, \cdots, k. \tag{17}$$

By (16) $d\tilde{K}_i$ is contained in span $\{dx_1, \cdots, dx_r\}$. This implies that $\tilde{K}_i$ only depends on $x_1, \cdots, x_r$. Hence there exists locally functions K_i, $i = 1, \cdots, k$, such that

$$L_A\phi_i = K_i(\phi(x), C(x)) \tag{18}$$

and so by (15) $K = (K_1, \cdots, K_k): \mathbb{R}^k \times Y \to \mathbb{R}^k$ is a partial observer. □

Motivated by the above proposition one could give an alternative and more intrinsic definition of conditioned invariance, which is locally equivalent to the usual Definition 2, in terms of partial observers. (In the linear case this is explicitly done in [12].) Notice that the relation between conditioned invariance (Definition 2) and a partial observer (Definition 1) is similar to the relation between *controlled invariance* and a *quotient system* as treated in [9].

Finally, we note that if $D \cap$ Ker $dC = o$, then D is automatically conditioned invariant. Let $D =$ Ker $d\phi$, with $\phi: M \to \mathbb{R}^k$ and $K: \mathbb{R}^k \times Y \to \mathbb{R}^k$ the corresponding (locally defined) partial observer, then it follows from Ker $d\phi \cap$ Ker $dC = o$ that the rank of the map $(\phi, C): M \to \mathbb{R}^k \times Y$ is moreover equal to dim M and hence by the implicit function theorem the full state $x \in M$ can be locally recovered from the state w of the observer and the output map y. In the linear case this results in the reduced order observer as proposed by Luenberger [8].

CONVERGENCE

As we remarked in the Introduction one of the basic requirements for an observer or partial observer is Condition B: The estimate $\hat{x}$ of the state of the system, or in the case of a partial observer the estimate of a part of the state, should converge to the real value of (a part of) the state-vector. In terms of Definition 1 this means that the solution $w(t)$ of (11) should converge to the solution $\bar{w}(t)$ of (10) for any initial data $w(t_o) = w_o$ and $\bar{w}(t_o) = \phi(x_o)$. Or said otherwise, the dynamical system with inputs

$$\frac{d}{dt} w(t) = K(w(t), y(t)) \tag{19}$$

should have a *unique steady-state* solution for "any" function $y(t)$.

In general it is very hard to give conditions on K in order that (19) has this property (see, e.g., [3]). For instance, if (19) is reachable from the origin and the operator from $y(\cdot)$ to $w(\cdot)$ defined by (19) has fading memory [2], then (19) has this unique steady-state property [2]. Only in case (19) has the almost linear form

$$\frac{d}{dt} w(t) = Kw(t) + L(y(t)), \qquad w \in \mathbb{R}^k \tag{20}$$

with K a constant matrix and L a smooth mapping, it is easy to give conditions: (20) has the unique steady-state property if and only if K is an *asymptotically stable* matrix. Of course this brings us back to the aforementioned research of finding conditions on A and C in (1) in order to transform (1) into (4). We notice however that this line of research may be considerably broadened by using Definition 1. Indeed in the terminology of Definition 1 the current research only deals with finding a (local) diffeomorphism ϕ (a transformation on the state space M) and coordinates on the output manifold Y in order that there exists an *identity* observer of the form (20), with K asymptotically stable, namely (5). It may be worthwhile to look for conditioned invariant distributions D for (1) such that at least the resulting *partial* observer (Proposition 3) can be made linear in w. Especially if D satisfies $D \cap \text{Ker } dC = 0$ we may in this way obtain stable *reduced order* observers.

REFERENCES

[1] D. Bestle and M. Zeitz, "Canonical form observer design for non-linear time-variable systems," *Int. J. Contr.*, vol. 38, pp. 419–431, 1983.
[2] S. Boyd and L. O. Chua, "Fading memory and the problem of approximating nonlinear operators with Volterra series," University of California, Berkeley, Memo UCB/ERL M84/96, 1984.
[3] L. O. Chua and D. N. Green, "A qualitative analysis of the behavior of dynamic nonlinear networks: Steady-state solutions of non-autonomous networks," *IEEE Trans. Circuits Syst.*, vol. CAS-23, pp. 530–550, 1976.
[4] A. Isidori, A. J. Krener, C. Gori-Giorgi, and S. Monaco, "Nonlinear decoupling via feedback: A differential geometric approach, *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 331–345, 1981.
[5] S. R. Kou, D. L. Elliot, and T. J. Tarn, "Exponential observers for nonlinear dynamic systems," *Inform. Contr.*, vol. 29, pp. 204–216, 1975.
[6] A. J. Krener and A. Isidori, "Linearization by output injection and nonlinear observers," *Syst. Contr. Lett.*, vol. 3, pp. 47–52, 1983.
[7] A. J. Krener and W. Respondek, "Nonlinear observers with linearizable error dynamics," *SIAM J. Contr. Optimiz.*, vol. 23, pp. 197–216, 1985.
[8] D. G. Luenberger, "An introduction to observers," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 596–602, 1971.
[9] H. Nijmeijer and A. J. van der Schaft, "Controlled invariance for nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 904–914, 1982.
[10] J. M. Schumacher, "Dynamic feedback in finite and infinite-dimensional linear systems," Amsterdam, Math. Centre Tract 143, 1981.
[11] F. E. Thau, "Observing the state of non-linear dynamic systems," *Int. J. Contr.*, vol. 17, pp. 471–479, 1973.
[12] J. C. Willems, Almost invariant subspaces: An approach to high gain feedback design, Part II: Almost conditionally invariant subspaces." *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 1071–1085, 1982.

Comments on "Quadratic-Type Lyapunov Functions for Singularly Perturbed Systems"

LJUBOMIR T. GRUJIĆ

Abstract—**A reference information is given to reflect better the existing results on the topic treated in the paper.[1]**

In the Introduction of their interesting paper[1] presenting an effective new approach, Saberi and Khalil stated the following.

"Previous work, that is relevant to ours, is due to Habets [2], Grujić [3], and Chow [4]. Habets and Grujić employed composite Lyapunov methods (cf. [6]–[8]) to derive asymptotic stability conditions. They were concerned with establishing the existence of a composite Lyapunov function but did not use that Lyapunov function to investigate the stability properties of the system, like estimating the domain of attraction."

However, in the paper [1] there were defined stability-, attraction-, and asymptotic stability-domain of an invariant set (that can be a singleton containing only an equilibrium state) for a nonlinear singularly perturbed system. In the framework of large-scale singularly perturbed systems there was also established, by using the composite Lyapunov function, a criterion for the asymptotic stability domain estimation, hence for the attraction domain estimation. The paper did not present any example or application. New more effective results can be obtained at least for special classes of nonlinearities, which was well shown in the paper[1] and effectively illustrated by its Example 2.

Manuscript received November 20, 1984.
The author is with the Faculty of Mechanical Engineering, University of Belgrade, Belgrade, Yugoslavia.
[1] A. Saberi and H. Khalil, *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 542–550, June 1984.

REFERENCES

[1] Lj. T. Grujić, "Sets and singularly perturbed systems," *Syst. Sci.* (Poland), vol. 5, no. 4, 327–338, 1979.

Authors' Reply[2]

A. SABERI AND H. KHALIL

The authors thank Dr. Grujić for pointing out that the quoted statement from their paper[1] can be misinterpreted to mean that the results of Habets and Grujić do not contain estimates of the domain of attraction. Of course they do, because, whenever there is a function $V(x)$ that satisfies all the requirements of a Lyapunov function in a neighborhood N of an equilibrium point $x = 0$, then the set $\{V(x) \leq C\} \subset N$ is included in the domain of attraction. This is the estimate given in [1]. The paper[1] further exploits the freedom in forming a composite Lyapunov function to obtain the largest possible estimate of the domain of attraction, while keeping an acceptable upper bound on the singular perturbation parameter.

[2] Manuscript received April 22, 1985.
A Saberi is with the Department of Electrical and Computer Engineering, Washington State University, Pullman, WA 99613.
H. Khalil is with the Department of Electrical Engineering and Systems Science, Michigan State University, East Lansing, MI 48824.

Comments on "Controllability and Observability Criteria for Multivariable Linear Second-Order Models"

E. G. COLLINS, JR.

In [1] the modal controllability and observability tests for matrix-second-order systems

$$\mathcal{M}\ddot{q} + \mathcal{D}\dot{q} + \mathcal{K}q = \mathcal{B}u, \; y = \mathcal{P}q + \mathcal{R}\dot{q} \qquad q \in R^n \tag{1}$$

$$\mathcal{M}^T = \mathcal{M} > 0 \;\; \mathcal{K} = \mathcal{K}^T \geq 0 \qquad u \in R^{n_u}$$

are reduced to their simplest form without relying on the standard state-space techniques. The state-space techniques are inefficient because they do not take advantage of the special matrix properties of the system (1).

Manuscript received August 30, 1984; revised April 9, 1985.
The author is with Purdue University, West Lafayette, IN 47907.

0018-9286/85/1200-1256$01.00 © 1985 IEEE