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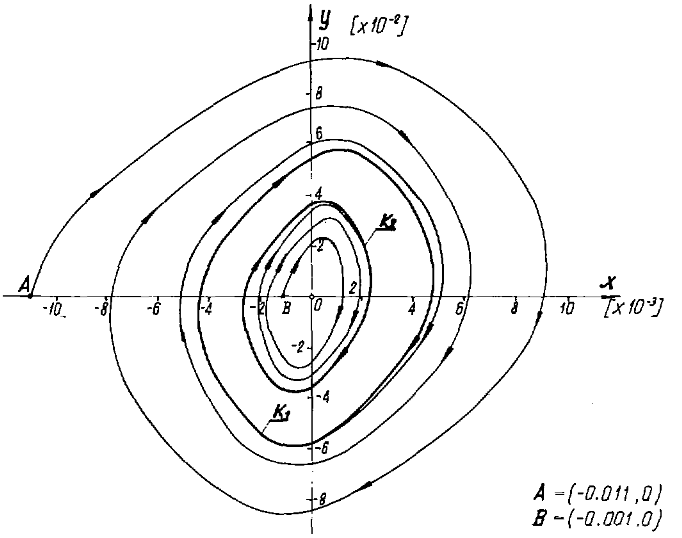

Fig. 3. Example of limit-cycles for $T = 0.1$ s.

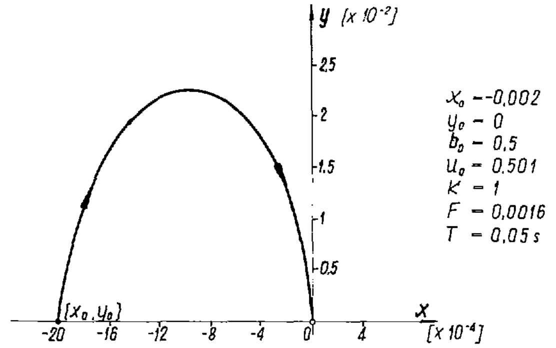

Fig. 4. Suboptimal sliding process.

V. Concluding Remarks and Suggestions for Applications

Difficulties in motion resistance performance identification make it often necessary to design the switching curve based on the assumed but probable motion resistance function $\hat{f}(y)$. In the technical aspect, in order to receive the suboptimal "chattering" process, it should be taken $|\hat{f}(y)| < |f_r(y)|$, which assures that the origin can be reached in finite time. The neglecting of constant T in the synthesis of the control system causes negative consequences, that is, the control structure generates limit-cycles.

The structure generating time-optimal control of the considered object is very complicated [4], so there is the tendency to apply suboptimal control systems, which slightly aggravates the control quality but highly simplifies the control structure [5].

Acceptance of $v(x, y)$ control as a simplification of time-optimal control $u^*(x, y)$ always leads to limit-cycle formation not allowed in real structures. The shortcomings of control discussed above suggest the conclusion that component u should be considered in the switching function. Fig. 4 shows the trajectory generated by control $v(x, y, u) = -\operatorname{sign}[\varphi(x, y) + Fu]$. Function φ has been calculated from (7),(24), where F is a real positive constant. The initial state and values of b, T, u_0 are the same as in the example of Fig. 2. On segment $S - O$ the "chattering" process occurs. Such a control process, being only suboptimal, is acceptable, however, from a technical viewpoint.

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The Disturbance Decoupling Problem for Nonlinear Control Systems

HENK NIJMEIJER AND ARJAN VAN DER SCHAFT

Abstract —**Necessary and sufficient conditions are derived for the solution of the disturbance decoupling problem for general nonlinear control systems. Some conceptual algorithms needed are discussed.**

I. Introduction

Consider the linear system

$$\begin{cases} \dot{x} = Ax + Bu + Eq \\ z = Hx \end{cases} \tag{1.1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, disturbance $q \in \mathbb{R}^r$, and the to-be-controlled variable $z \in \mathbb{R}^p$. A, B, E, and H are matrices of appropriate dimensions. The disturbance decoupling problem (DDP) consists of finding a state feedback $u = Fx + v$ which decouples the disturbance from the to-be-controlled variable z. Equivalently, after feedback the transfer function from q to z has to be zero. The solvability of DDP can be constructively checked in the following way (cf. [8]).

1) Construct the maximal controlled invariant subspace in the kernel of H: $\mathcal{V}^*_{\ker H}$.

2) Check if $\operatorname{Im} E \subset \mathcal{V}^*_{\ker H}$.

Recently, a similar theory has been developed for nonlinear systems where the inputs and the disturbances enter linearly in the equations (cf. [2], [3]).

$$\begin{cases} \dot{x} = A(x) + \sum_{i=1}^{m} B_i(x) u_i + \sum_{j=1}^{r} E_j(x) q_j \\ z = H(x). \end{cases} \tag{1.2}$$

The procedure is the same: construct the maximal controlled invariant distribution contained in $\ker dH$, and call this $D^*_{\ker dH}$. Then DDP is locally solvable if and only if $\operatorname{span}\{E_1, \cdots, E_r\} \subset D^*_{\ker dH}$. Applications of these results may be found in [1], [7].

In our previous paper [5], we treated controlled invariance for a general nonlinear system $\dot{x} = f(x, u)$. With the aid of this we can treat the DDP for the system

$$\begin{cases} \dot{x} = f(x, u) + \sum_{j=1}^{r} E_j(x) q_j \\ z = H(x). \end{cases} \tag{1.3}$$

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In fact, DDP is locally solvable if and only if there exists a controlled invariant distribution D [w.r.t. $\dot{x}=f(x,u)$] such that $\operatorname{span}\{E_1,\cdots,E_r\}\subset D$.

In this paper, we will treat the most general case where the disturbances also enter in a nonlinear way

$$\begin{cases} \dot{x}=f(x,u,q) \\ z=H(x). \end{cases} \tag{1.4}$$

To give a coordinate-free description of the disturbance decoupling problem in this case, we first have to generalize the definition of a control system, as in [5], to the definition of a control system with disturbances. Then the local solution will readily follow.

Furthermore, just as in the linear case, we will give some algorithms for checking solvability of DDP (see Section III).

II. Controlled Invariance for Nonlinear Control Systems with Disturbances

As in our previous paper [5], we use the following setting for a nonlinear control system. Let M be a smooth n-dimensional manifold, denoting the state space. Let $\pi\colon B\to M$ be a smooth fiber bundle, whose fibers represent the state-dependent input spaces. Then a *control system* $\Sigma(M,B,f)$ is defined by the commutative diagram

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "\pi"'] & & TM \arrow[dl, "\pi_M"] \\ & M & \end{tikzcd}$$

where TM denotes the tangent bundle of M, with natural projection π_M, and f is a smooth map.

In local coordinates x for M, (x,u) for B, this coordinate-free definition comes down to $\dot{x}=f(x,u)$.

We now want to formalize the situation in which our control system also contains disturbances (which also can be interpreted as unknown inputs). This leads to the following definition.

Definition 2.1: A control system with disturbances $\Sigma=\Sigma(M,B,\tilde{B},f)$ is given by the following. Let $\Sigma(M,\tilde{B},f)$ be a control system. Let $\tilde{\pi}\colon\tilde{B}\to B$ and $\pi\colon B\to M$ be fiber bundles, where the fibers of $\pi\colon B\to M$ represent the state-dependent input spaces and the fibers of $\tilde{\pi}\colon\tilde{B}\to B$ represent the state- and input-dependent disturbance spaces. If we let $\pi' := \pi\circ\tilde{\pi}$, then the fibers of the bundle $\pi'\colon\tilde{B}\to M$ represent the state-dependent input *and* disturbance spaces. So a control system with disturbances is given by the following commutative diagram:

$$\begin{tikzcd} \tilde{B} \arrow[rr, "f"] \arrow[d, "\tilde{\pi}"] \arrow[dd, bend right=60, "\pi'"'] & & TM \arrow[ddll, "\pi_M"] \\ B \arrow[d, "\pi"] & & \\ M & & \end{tikzcd}$$

In local coordinates x for M, (x,u) for B (u for the inputs), and (x,u,q) for $\tilde{B}$ (q for the disturbances), this simple definition comes down to $\dot{x}=f(x,u,q)$.

In this framework, *state feedback* is given by the following procedure. Let α be a fiber-preserving diffeomorphism on B such that the diagram

$$\begin{tikzcd} B \arrow[rr, "\alpha"] \arrow[dr, "\pi"'] & & B \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

commutes. Consider an *arbitrary* fiber-preserving diffeomorphism $\tilde{\alpha}$ on $\tilde{B}$, such that we also have that the diagram

$$\begin{tikzcd} \tilde{B} \arrow[r, "\tilde{\alpha}"] \arrow[d, "\tilde{\pi}"'] & \tilde{B} \arrow[d, "\tilde{\pi}"] \\ B \arrow[r, phantom, "\alpha"] & B \end{tikzcd}$$

commutes. Then the system $\Sigma(M,B,\tilde{B},f)$ after *state feedback* $\tilde{\alpha}$ is given by $\Sigma(M,B,\tilde{B},\hat{f})$ with $\hat{f}=f\circ\alpha$ (compare to [6]).

Remark: In local coordinates, this means that the system $\dot{x}=f(x,u,q)$ is modified by the state feedback $(u,q)=\tilde{\alpha}(x,v,q')=(\alpha(x,v),\tilde{q}(x,v,q'))$ to the system $\dot{x}=f(x,\alpha(x,v),\tilde{q}(x,v,q'))$, where $\tilde{q}(x,v,\cdot)\colon\tilde{\pi}^{-1}(x,v)\to\tilde{\pi}^{-1}(x,v)$ is an arbitrary diffeomorphism (induced by $\tilde{\alpha}$).

From the above discussion of this notion of state feedback and from [5], the next definition should be clear.

Definition 2.2: An involutive distribution D, of fixed dimension, on M, is a *locally controlled invariant for the control system with disturbances* $\Sigma(M,B,\tilde{B},f)$, if locally around each point $x_0\in M$ there exist fiber respecting coordinates (x,u) for B, such that for all fiber respecting coordinates (x,u,q) for $\tilde{B}$ we have that for every fixed u and q $[f(\cdot,u,q),D]\subset D$.

Remark: This implies that for every time function $\bar{u}(\cdot)$ and $\bar{q}(\cdot)$, also $[f(\cdot,\bar{u},\bar{q}),D]\subset D$.

What are the conditions such that a distribution D is locally controlled invariant for the control system with disturbances? The next theorem, which is a combination of the results of [5] and [6] yields the exact solution.

Theorem 2.3: Let $\Sigma=\Sigma(M,B,\tilde{B},f)$ *be a control system with disturbances. Let* $Q:=\ker\tilde{\pi}_*$ *and* $R:=\ker\pi'_*$. *Then an involutive distribution* D *of fixed dimension is locally controlled invariant for the control system with disturbances, if and only if the following three conditions hold:*

1) $f_*(\pi'^{-1}_*(\dot{D}))\subset D+f_*(R)$
2) $f_*(Q)\subset D$
3) $\dot{D}+f_*(R)$ and $f_*(Q)$ have fixed dimension.

Remark: For the definition of $\dot{D}$ we refer to [5] or [6].

Proof of Theorem 2.3: The proof resembles that of Theorem 3.1 of [6]. We note that from 1), it follows that we can locally construct a state feedback for the system $\Sigma(M,\tilde{B},f)$ (cf. [5]). But in principle, this feedback depends upon the input space of the bundle $\pi'\colon\tilde{B}\to M$; i.e., the state feedback can also depend upon the disturbances. Following [5], we know that the condition 1) is also equivalent to the existence of a distribution $\tilde{D}_{\text{lift}}$ on $\tilde{B}$ generated by an integrable connection on the bundle $\pi'\colon\tilde{B}\to M$. In local coordinates, this distribution $\tilde{D}_{\text{lift}}$ is generated by the vector fields

$$\frac{\partial}{\partial x_i}+h_i(x,u,q)\frac{\partial}{\partial u}+g_i(x,u,q)\frac{\partial}{\partial q},\qquad i=1,\cdots,k \tag{2.1}$$

whereas D is generated by the vector fields (Frobenius)

$$\frac{\partial}{\partial x_i},\qquad i=1,\cdots,k \tag{2.2}$$

and the coefficients h_i and g_i in (2.1) satisfy certain integrability conditions [5, eq. (4.30)]. Now the second condition 2), in fact implies that we are able to choose the coefficients h_i in (2.1) such that h_i does not depend upon q. Namely, as in [6], we have that

$$\tilde{D}_{\text{lift}}+Q=\operatorname{span}\left\{\frac{\partial}{\partial x_i}+h_i(x,u,q)\frac{\partial}{\partial u},\frac{\partial}{\partial q};\,i=1,\cdots,k\right\} \tag{2.3}$$

and then from 2) it follows that

$$\left[\tilde{D}_{\text{lift}},Q\right]\subset\tilde{D}_{\text{lift}}+Q. \tag{2.4}$$

Now, (2.4)—which is equivalent to the fact that $\tilde{\pi}_*(\tilde{D}_{\text{lift}})$ is a well-defined distribution on B—implies that after an easy computation $h_i(x,u,q)$ is independent from q'. Knowing this, we can locally construct a state feedback independent of q for $\Sigma(M,B,\tilde{B},f)$, similarly as in [6]. □

III. Algorithms

In this section, we will prove that every involutive distribution on the state space contains a maximal locally controlled invariant distribution. Furthermore, we will give (conceptual) algorithms to compute this maximal locally controlled invariant distribution, and apply these to the general disturbance decoupling problem. First, we will start with the affine system given locally by

$$\dot{x}=A(x)+\sum_{i=1}^{m}u_iB_i(x), \qquad x\in M \quad \text{(a manifold)}. \tag{3.1}$$

Following [4], we define

$$\Delta(x):=A(x)+\operatorname{span}\{B_1(x),\cdots,B_m(x)\}$$
$$\Delta_0(x):=\operatorname{span}\{B_1(x),\cdots,B_m(x)\}$$

and $\Delta^{-1}(\Delta_0+D):=\{X$ a smooth vector field on M such that $[\Delta,X]\subset \Delta_0+D\}$. Then we can state the following theorem.

Theorem 3.1:

a) *Let D_1 and D_2 be controlled invariant distributions on M for the affine system (3.1). Then $\overline{D_1+D_2}$ (the involutive closure of D_1+D_2) is again controlled invariant.*

b) *Let K be an involutive distribution on M of dimension k. Then K contains a maximal controlled invariant distribution. Moreover, define*

$$D^0=K$$
$$D^{m+1}=D^m\cap\Delta^{-1}(D^m+\Delta_0), \qquad m=0,1,\cdots.$$

Then $\lim_{m\to\infty}D^m=D^k$, and when we assume that $\overline{D}^k$ has fixed dimension, $\overline{D}^k$ is the maximal controlled invariant distribution in K.

Proof:

a) The essential part in the proof of a) is the Jacobi identity $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ (see [2], [3]).

b) From a), it follows that K contains a maximal locally controlled invariant distribution (see [2], [3]). The algorithm above is given in [4] (for a related algorithm see [3]). □

Next, we will consider the situation for a general nonlinear system $\Sigma(M,B,f)$.

We define the extended system (see [5] for references) of $\Sigma(M,B,f)$ as the affine system on B given by

$$\Delta^e(x,u):=\{X\in T_{(x,u)}B\,|\,\pi_*X=f(x,u)\}$$
$$\Delta^e_0(x,u):=\{X\in T_{(x,u)}B\,|\,\pi_*X=0\},$$

i.e., in local coordinates simple

$$\begin{cases}\dot{x}=f(x,u)\\ \dot{u}=v\end{cases}$$

(v is the new input).

Theorem 3.2: Let D^1 and D^2 be locally controlled invariant distribution on M for the system $\Sigma(M,B,f)$ (see Section II). Then $\overline{D^1+D^2}$ is also locally controlled invariant. Therefore, given an involutive distribution K on M, there exists a maximal locally controlled invariant distribution contained in K.

Proof: From [5], we know that there exist involutive distributions D^1_{lift} and D^2_{lift} on B such that

$$\pi_*D^i_{\text{lift}}=D^i \qquad i=1,2. \tag{3.2}$$
$$[\Delta^e,D^i_{\text{lift}}]\subset D^i_{\text{lift}}+\Delta^e_0 \qquad i=1,2. \tag{3.3}$$

When we define $D:=\overline{D^1_{\text{lift}}+D^2_{\text{lift}}}$, it is clear from (3.2) that $\pi_*D=\overline{D^1+D^2}$. From (3.3) it follows that, by using the Jacobi identity, $[\Delta^e,D]\subset D+\Delta^e_0$. Therefore, $\overline{D_1+D_2}$ is locally controlled invariant (the connection above $\overline{D_1+D_2}$ is determined by D). □

The algorithmic side becomes very simple by reducing it to the extended system.

Algorithm 3.3: Let K be an involutive distribution on M. Consider the extended system (Δ^e,Δ^e_0) of $\Sigma(M,B,f)$, and define the following distributions on B:

$$D^0=K+\Delta^e_0$$
$$D^{m+1}=D^m\cap\Delta e^{-1}(D^m-\Delta^e_0), \qquad m=0,1,\cdots.$$

Then, $\lim_{m\to\infty}D^m=D^k$ (k is the dimension of D^0), and when we assume that $\overline{D}^k$ has fixed dimension, $\overline{D}^k$ is the maximal locally controlled invariant distribution in D^0 for the extended system. Furthermore because $[\Delta^e_0,\overline{D}^k]\subset\Delta^e_0+\overline{D}^k$, $\pi_*\overline{D}^k$ is a well-defined distribution on M. In fact, $\pi_*\overline{D}^k$ is the maximal locally controlled invariant distribution for $\Sigma(M,B,f)$ contained in K.

Proof: The algorithm is just the algorithm of Theorem 3.1 for the (affine) extended system. That $\pi_*\overline{D}^k$ is the maximal controlled invariant distribution contained in K follows from the one-to-one correspondence between locally controlled invariant distributions of $\Delta(M,B,f)$ and its extended system (see [5]). □

Remarks:

1) (Compare to [8, exercise 4.6].) Notice that while the Algorithm 3.3 applies at first instance to the case where we have an output function H: $M\to Z$ and $K=\ker dH$, it can also be applied to the case that H: $B\to Z$. For instance, we can consider the disturbance decoupling problem where the to-be-controlled variable z equals $H(x,u)$. In this case, we only have to change in the algorithm $D^0=K+\Delta^e_0$ to $D^0=\ker dH$, with H: $B\to Z$.

2) Note that Theorem 3.2 is not valid for measured controlled invariance (i.e., controlled invariance by static output feedback, see [6]). In fact, in general a maximal measured controlled invariant distribution does not exist.

Corollary 3.4: Consider a control system with disturbances $\Sigma(M,B,\tilde{B},f)$ (see Section II). Let H: $M\to Z$ be a smooth function, with $z=H(x)$ the to-be-controlled variable. Then apply Algorithm 3.3 to construct the maximal locally controlled invariant distribution contained in $K:=\ker dH$ for the control system $\Sigma(M,\tilde{B},f)$ (i.e., we compute controlled invariance with respect to the whole input space R). Call this distribution D. Then the disturbance decoupling problem is solvable if and only if $f_*Q\subset\tilde{D}$, or equivalently, if and only if D is locally controlled invariant for the system with disturbances $\Sigma(M,B,\tilde{B},f)$.

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Descriptor Systems: Expanded Descriptor Equation and Markov Parameters

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Abstract—**Generalized concepts of solvability and conditionability are presented for descriptor systems, and tests are given for these generalized properties. An expression is derived for descriptor system Markov parameters. The notion of "expanded descriptor equation" is presented.**

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