A Behavioral Approach to Passivity and Bounded Reality Preserving Balanced Truncation with Error Bounds

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Abstract—In this paper we revisit the problems of passivity and bounded reality preserving model reduction by balanced truncation. In the behavioral framework, these problems can be considered as special cases of balanced truncation of strictly half line dissipative system behaviors, where the number of input variables of the behavior is equal to the positive signature of the supply rate. Instead of input-state-output representations, the balancing algorithm uses normalized driving variable representations of the behavior. We show that the diagonal elements of the minimal solution of the balanced algebraic Riccati equation are the singular values of the map that assigns to each past trajectory the optimal storage extracting future continuation. Since the future behavior is only an indefinite inner product space, the term singular values should be interpreted here in a generalized sense. We establish some new error bounds for this model reduction method.

Keywords: Model reduction, strictly half line dissipative behaviors, balanced truncation, bounded real balancing, positive real balancing, normalized driving-variable representations.

I. INTRODUCTION

The method of balanced truncation is the most prominent method of model reduction for linear dynamical systems. The method is straightforward and simple, has a nice and convincing physical interpretation, preserves stability, and, last but not least, comes with simple and effective $\mathcal{H}_\infty$ error bounds.

Starting with the seminal paper [3] by Desai and Pal on stochastic model reduction, there has been also an interest in balanced truncation methods that preserve typical structural properties of the original system. The paper by Desai and Pal introduces a balanced truncation method to approximate a given positive real transfer matrix by a reduced order positive real transfer matrix. In [5] this problem was revisited, and it was shown that also stability and minimality are preserved under this balanced truncation method. In [14], and later in [2], $\mathcal{H}_\infty$ error bounds for balanced reduction of strictly positive real transfer matrices were found. The related problem of balanced truncation of bounded real transfer matrices, including $\mathcal{H}_\infty$ error bounds, was studied extensively in [8]. For a nice overview, we refer to [1].

Research on balanced reduction methods using ideas from stochastic model reduction can also be found in the work of Weiland [15]. In [15], the problem of model reduction by balancing is put into a more general, behavioral, framework. It is shown that the system invariants that appear as diagonal elements in the solutions of the algebraic Riccati equations after balancing are, in fact, the nonzero singular values of a given map from past to future behavior. This map assigns to any past trajectory its optimal continuation, in an appropriate sense.

In the present paper we revisit the problems of reduction by balancing of passive real and bounded real systems from a behavioral point of view. Positive real (or passive) and bounded real (or contractive) systems are special cases of systems with a given input-output partition that are dissipative on the negative half line, and whose number of input components is equal to the positive signature of the supply rate. In this paper we study the general problem of model reduction by balancing for such strictly half line dissipative systems. In particular, the property of strict half line dissipativity should be preserved, and the input-output partition of the original system should be respected.

We find a number of frequency domain inequalities involving the error transfer matrix, i.e. the difference between the original and reduced order transfer matrix from driving variable to manifest variable. We study in what sense these inequalities can be interpreted as error bounds. In particular, for the special case of strictly bounded real systems we find a new error bound for bounded real balanced truncation.

Most of the proofs of the results in this paper are omitted. For these, we refer to a future, full version of the paper.

Notation and background material. $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ denotes the space of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^d$. $\mathcal{D}(\mathbb{R}, \mathbb{R}^d)$ denotes its subspace of functions with compact support. For this space we use the shorthand notation $\mathcal{D}$. We denote by $\mathcal{L}_{loc}^\infty(\mathbb{R}, \mathbb{R}^d)$ the space of all measurable functions $w$ from $\mathbb{R}$ to $\mathbb{R}^d$ such that $\int_{\mathbb{R}}^b |w|^2 dt < \infty$ for all $a, b \in \mathbb{R}$. $\mathcal{L}_2(\mathbb{R}, \mathbb{R}^d)$ denotes the ambient space of all measurable functions $w$ from $\mathbb{R}$ to $\mathbb{R}^d$ such that $\int_{\mathbb{R}}^\infty |w|^2 dt < \infty$. The $L_2$-norm of $w$ is $\|w\|_2 := (\int_{\mathbb{R}}^\infty |w|^2 dt)^{1/2}$. We denote by $\mathcal{R}_+$ the set of negative real numbers, and by $\mathcal{R}_+$ the complementary set of nonnegative real numbers. $\mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^d)$ denotes the space of all measurable functions $w$ from $\mathbb{R}_+$ to $\mathbb{R}^d$ such that $\int_{\mathbb{R}_+}^\infty |w|^2 dt < \infty$.

II. REDUCTION OF DISSIPATIVE LINEAR DIFFERENTIAL BEHAVIORS

This paper deals with model reduction of dissipative linear differential systems. A subspace $\mathcal{B} \subset \mathcal{L}_{loc}^\infty(\mathbb{R}, \mathbb{R}^d)$ is called a linear differential system (or a linear differential behavior) if it is equal to the space of (weak) solutions $w : \mathbb{R} \rightarrow \mathbb{R}^d$ of a system of linear, constant coefficient, higher order differential equations, i.e., there exists a polynomial matrix $R \in \mathbb{R}^{n \times d}[s]$ such that $\mathcal{B} = \{ w \in \mathcal{L}_{loc}^\infty(\mathbb{R}, \mathbb{R}^d) \mid R \frac{d^k}{ds^k} w = 0 \}$ (see [9]). The variable $w$ is called the manifest variable of the system $\mathcal{B}$. The set of all linear differential systems with $w$ variables is denoted by $\mathcal{L}_w$.

It is well known, see e.g., [9], [15], that any $\mathcal{B} \in \mathcal{L}_w$ admits state space representations. In this paper we will use mainly one type of state space representation, namely driving variable representations (DV-representations). Consider the equations

$$\dot{x} = Ax + Bv, \quad w = Cx + Dv, \quad (1)$$

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with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{w \times n}$ and $D \in \mathbb{R}^{w \times m}$. These equations represent the full behavior

$$BDV(A, B, C, D) := \{(w, x, v) : (w, x, v) = (col(w_1), col(w_2)), \quad \exists \sigma \in \mathbb{R}, \quad \exists \xi \in \mathbb{R}^n \}$$

The variable $w$ is a state variable, taking its values in $\mathbb{R}^w$, the state space, and $v$ is called the driving variable, taking its values in $\mathbb{R}^w$. The external behavior corresponding to this full behavior is defined as

$$BDV(A, B, C, D)_{ext} = \{w \in \mathbb{R}^{2w} : \exists \sigma \in \mathbb{R}, \quad \exists \xi \in \mathbb{R}^n\}$$

If $B = BDV(A, B, C, D)_{ext}$, then we call $BDV(A, B, C, D)$ a driving variable representation of $B$.

We now review the notions of input and output. For a given system $B$, a partition of the manifest variable $w$ into $w = col(w_1, w_2)$ is called an input-output partition if $w_1$ is maximally free, meaning that it is free (i.e. for any $w_1 \in L_2^{loc}(\mathbb{R}^w)$ there exists $w_2$ such that $col(w_1, w_2) \in B$), and one can not enlarge $w_1$ to a new variable $w_1'$ by adding components of $w_2$ such that the new variable $w_1'$ is free. If $w = col(w_1, w_2)$ is an input-output partition then $w_1$ is called input and $w_2$ is called output of $B$. For details we refer to [9]. The number of input components in any input-output partition of $B \in L_2^{loc}$ is an integer invariant of $B$, and is called the input cardinality of $B$, denoted by $\omega(B)$.

Now assume $B \in L_2^{loc}$ and partition $w = col(w_1, w_2)$. Let $BDV(A, B, C, D)$ be a driving variable representation of $B$. Of course, the partition of $w$ into $col(w_1, w_2)$ induces a partition of the matrices $C$ and $D$ into

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}.$$

A natural question is now: under what conditions on the driving variable representation is the partition $w = col(w_1, w_2)$ an input-output partition? The answer is given in the next lemma:

**Lemma 2.1:** Under the above assumptions, $w = col(w_1, w_2)$ is an input-output partition for $B$ if and only if the rational matrix $G_1(\xi) := C_1(\xi I - A)^{-1} B + D_1$ is square and nonsingular. In that case, the transfer matrix from $w_1$ to $w_2$ is equal to $G_2(\xi)G_1^{-1}(\xi)$, with $G_2(\xi) := C_2(\xi I - A)^{-1} B + D_2$.

Another important integer invariant of a given behavior $B$ is the minimal dimension of the state space over all its state space representations. This integer is called the McMillan degree of $B$, denoted with $\sigma(B)$.

A driving variable representation $BDV(A, B, C, D)$ of $B$, with state space dimension $n$ and driving variable dimension $n$ is called minimal if $n$ and $m$ are minimal over all such driving variable representations. The minimal $n$ is equal to the McMillan degree $n(B)$ and the minimal $m$ is equal to the input cardinality $\omega(B)$. A given DV-representation $BDV(A, B, C, D)$ of $B$ is a minimal DV-representation if and only if $A, B, C, D$ is strongly observable (meaning that the pair $(C + DF, A + BF)$ is observable for every $F$) and $D$ has full column rank (see [15]).

We restrict ourselves to controllable behaviors in this paper. A behavior $B \in L_2^{loc}$ is called controllable if for all $w_1, w_2 \in B$ there exists $T \geq 0$ and $w \in B$ such that $w(t) = w_1(t)$ for $t < 0$, and $w(t) = w_2(t - T)$ for $t \geq 0$. Properties of controllable behaviors are discussed in [9]. $L_2^{cont}$ (a subset of $L_2^{loc}$) will denote the set of all controllable behaviors.

If $BDV(A, B, C, D)$ is a minimal DV-representation of $B$, then $B$ is controllable if and only if the pair $(A, B)$ is controllable, see [15].

We will now review the basic material on dissipative behaviors. For an extensive treatment we refer to [12], [13], [17], [18]. Let $B \in L_2^{cont}$ and let $\Sigma = \Sigma^T = R^{w \times w}$ be nonsingular. The quadratic form $\Sigma w^T \Sigma w$ is called a supply rate. $B$ is said to be $\Sigma$-dissipative if $\int_{-\infty}^{+\infty} w^T \Sigma w dt \geq 0$ for all $w \in B \cap \Sigma$. $B$ is said to be strictly $\Sigma$-dissipative if $\int_{-\infty}^{0} Q(w) dt \geq 0$ for all $w \in B \cap \Sigma$. We will also call such behaviors half line dissipative. It is easily seen that if $B$ is $\Sigma$-dissipative on $\mathbb{R}_-$, then it is $\Sigma$-dissipative. A controllable behavior $B$ is said to be strictly $\Sigma$-dissipative if there exists an $\epsilon > 0$ such that $B$ is $(\Sigma - \epsilon I)$-dissipative. We have the obvious definition for strict dissipativity on $\mathbb{R}_-$. If $B$ is strictly $\Sigma$-dissipative on $\mathbb{R}_-$, then it is strictly $\Sigma$-dissipative.

In this paper we deal with linear differential behaviors $B \in L_2^{cont}$ that are strictly $\Sigma$-dissipative on $\mathbb{R}_-$. In addition, we assume that $m(B) = \sigma_+(\Sigma)$, i.e. the input cardinality of $B$ is equal to the positive signature of $\Sigma$. Two important special cases of such systems are

1. strictly bounded real input-output systems, where the manifest variable is partitioned as $w = col(u, y)$, with $u$ input and $y$ output, and where $\Sigma = diag(I_1, -I_1)$, and
2. strictly passive input-output systems, where $w = col(u, y)$, with $u$ input and $y$ output (having the same dimension $n$, and where $\Sigma = \frac{1}{2} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$).

As announced in the introduction, this paper deals with approximation of strictly passive input-output systems by a strictly passive input-output systems with a given lower McMillan degree, and approximation of strictly bounded real input-output systems by a strictly bounded real input-output systems with a given lower McMillan degree. By the above remarks, these problems can be put into one single framework, namely the following: given a strictly half line $\Sigma$-dissipative behavior $B$ with input cardinality $\omega(B)$ equal to the number of $\sigma_+(\Sigma)$ of positive eigenvalues of $\Sigma$, and a given input-output partition of its manifest variable, approximate it by a strictly half line $\Sigma$-dissipative behavior of a given lower McMillan degree, with the same input-output partition. More precise, the problem can be formulated as follows:

**Main Problem.** Let $B \in L_2^{cont}$, with system variable $w = col(w_1, w_2)$, where $w_1$ is input and $w_2$ is output. Let $\Sigma = \Sigma^T \in R^{w \times w}$ be nonsingular and assume that $m(B) (= \dim(w_1)) = \sigma_+(\Sigma)$. Assume that $B$ is strictly $\Sigma$-dissipative on $\mathbb{R}_-$ and let $k$ be an integer such that $0 < k < \omega(B)$. Find $\hat{B} \in L_2^{cont}$ such that

1. $\hat{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}_-$,
2. in $\hat{B}$ $w_1$ is input and $w_2$ is output,
3. $\hat{n}(\hat{B}) \leq k$,
4. $\hat{B}$ is an approximation of $B$.

We will show that for the two special cases of strictly passive and strictly bounded real systems, i.e. $\Sigma = \frac{1}{2} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ and $\Sigma = diag(I_n, -I_n)$, respectively, reduction by balancing using normalized driving variable representations leads to a behavior $B$ satisfying properties 1,2, and 3. Finally, the question whether our balanced truncation method leads to a reasonable approximation is studied afterwards, and amounts to finding reasonable error bounds.

III. $\Sigma$-CHARACTERISTIC VALUES OF SYSTEM BEHAVIORS

In this section we introduce the notion of $\Sigma$-characteristic values of behaviors that are strictly $\Sigma$-dissipative on $\mathbb{R}_-$ and that have the property $m(\hat{B}) = \sigma_+(\Sigma)$.

We will use the property that $B$ is strictly dissipative on $\mathbb{R}_-$ to endow the past behavior with an inner product, with the inner product given by the integral of the supply rate. In the same way, the supply rate will only yield an indefinite inner product on the future behavior. We will formulate a theorem that states that certain operators between past and future behavior allow singular value decompositions. This terminology should however be interpreted carefully, since the future behavior is not an inner product space. The "singular values" will form a set of invariants of the strictly $\Sigma$-dissipative behavior, and will be called the $\Sigma$-characteristic values of $B$. 

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Let $B \in L^\infty_{\text{cont}}$ and let a supply rate be given by the nonsingular symmetric matrix $\Sigma = \Sigma^\top \in R^{w \times w}$. Assume $B$ is strictly $\Sigma$-dissipative. Introduce the following notation:

$$B_+ := \{ w |_{r_w} \mid w \in B \}, \quad B_- := \{ w |_{r_w} \mid w \in B \}.$$ 

Furthermore, for a given past trajectory $w_\ast \in B_-$ define the set of all future trajectories $w_i$ whose concatenation at time zero with past trajectory $w_\ast$ is in $B$ by

$$B_+(w_\ast) := \{ w_+ \in B_+ \mid \text{there exists } w \in B \text{ such that } w |_{r_w} = w_\ast \text{ and } w |_{r_+} = w_+ \},$$

and for a given future trajectory $w_+ \in B_+$ define the set of all past trajectories $w_\ast$ whose concatenation at time zero with future trajectory $w_+ \ast$ is in $B$ by

$$B_-(w_+) := \{ w_- \in B_- \mid \text{there exists } w \in B \text{ such that } w |_{r_w} = w_- \text{ and } w |_{r_+} = w_+ \}.$$

For a given past trajectory $w_\ast \in B_- \cap L^2_{\text{R}}(R_-)$ we define the associated available storage by

$$V_{av}(w_\ast) := \sup \{ -\int_0^\infty w_+^T \Sigma w_+ dt \mid w_+ \in B_+(w_\ast) \cap L^2_{\text{R}}(R_+) \},$$

and for a given future trajectory $w_+ \in B_+ \cap L^2_{\text{R}}(R_+)$ we define the associated required supply by

$$V_{req}(w_+) := \inf \{ \int_0^\infty w_-^T \Sigma w_- dt \mid w_- \in B_-(w_+) \cap L^2_{\text{R}}(R_-) \}.$$

The available storage associated with past trajectory $w_\ast$ is the maximal amount of supply that can be extracted from the system over all future trajectories $w_+ \in B_+(w_\ast) \cap L^2_{\text{R}}(R_+)$. The required supply associated with future trajectory $w_+$ is the minimal amount of supply that has to be delivered to the system over all past trajectories $w_- \in B_-(w_+) \cap L^2_{\text{R}}(R_-)$. 

Due to $\Sigma$-dissipativity of $B$, the supremum and infimum above are finite for all $w_\ast$ and $w_+$, respectively (see [12], [17], [18]).

Also, by strict $\Sigma$-dissipativity, both the supremum and infimum are attained for all $w_\ast$ and $w_+$. In particular, for given $w_\ast \in B_- \cap L^2_{\text{R}}(R_-)$ there is a unique $w_+^\ast \in B_+(w_\ast) \cap L^2_{\text{R}}(R_+)$ such that

$$V_{av}(w_\ast) = -\int_0^\infty w_+^\ast T \Sigma w_+^\ast dt$$

and for given $w_+ \in B_+ \cap L^2_{\text{R}}(R_+)$ there is a unique $w_-^\ast \in B_-(w_+) \cap L^2_{\text{R}}(R_-)$ such that

$$V_{req}(w_+) = \int_0^\infty w_-^\ast T \Sigma w_-^\ast dt.$$

By associating with any past trajectory $w_\ast \in B_- \cap L^2_{\text{R}}(R_-)$ the unique optimal future trajectory $w_+^\ast \in B_+(w_\ast) \cap L^2_{\text{R}}(R_+)$ we obtain a map

$$\Gamma_- : B_- \cap L^2_{\text{R}}(R_-) \to B_+ \cap L^2_{\text{R}}(R_+), \quad \Gamma_-(w_\ast) = w_+^\ast,$$

and by associating with any future trajectory $w_+ \in B_+ \cap L^2_{\text{R}}(R_+)$ the unique optimal past trajectory $w_-^\ast \in B_-(w_+) \cap L^2_{\text{R}}(R_-)$ we obtain a map

$$\Gamma_+ : B_+ \cap L^2_{\text{R}}(R_+) \to B_- \cap L^2_{\text{R}}(R_-), \quad \Gamma_+(w_+) = w_-^\ast.$$

In the remainder of this section, assume that $B$ is strictly $\Sigma$-dissipative on $R_-$. This implies that there exists $\epsilon > 0$ such that

$$\int_0^\infty w^T \Sigma w dt \geq \epsilon \int_0^\infty w^T w dt$$

for all $w \in L^2_{\text{R}}(R_-)$. Consequently, the bilinear form

$$\langle w_1, w_2 \rangle_\Sigma := \int_0^\infty w_1^T \Sigma w_2 dt$$

defines an inner product on $B_- \cap L^2_{\text{R}}(R_-)$. On $B_+ \cap L^2_{\text{R}}(R_+)$ consider the bilinear form

$$\langle w_1, w_2 \rangle_{+, \Sigma} := -\int_0^\infty w_1^T \Sigma w_2 dt.$$

Since $\Sigma$ is a nonsymmetric matrix, this defines an indefinite inner product on $B_+ \cap L^2_{\text{R}}(R_+)$. (Note that we have assumed strict $\Sigma$-dissipativity on $R_-$, implying only positive definiteness of the bilinear form over the past, and not necessarily over the future!). Now, in the sequel it will be shown that the maps $\Gamma_-$ and $\Gamma_+$ are linear. We will denote by $\Gamma_- : B_- \cap L^2_{\text{R}}(R_-) \to B_+ \cap L^2_{\text{R}}(R_+)$ the adjoint of $\Gamma_-$, i.e. (the unique) linear map $\Gamma_-^* : B_+ \cap L^2_{\text{R}}(R_+) \to B_- \cap L^2_{\text{R}}(R_-)$ that satisfies

$$\langle w_1, \Gamma_-^*(w_2) \rangle_{+, \Sigma} = \langle \Gamma_-^*(w_1), w_2 \rangle_{-, \Sigma}$$

for all $w_1 \in B_+ \cap L^2_{\text{R}}(R_+)$ and $w_2 \in B_- \cap L^2_{\text{R}}(R_-)$. The existence and uniqueness of this adjoint can be easily proven, see e.g. [4], chapter 4. Likewise, $\Gamma_+^* : B_- \cap L^2_{\text{R}}(R_-) \to B_+ \cap L^2_{\text{R}}(R_+)$ will denote the adjoint of $\Gamma_+$, i.e. the unique linear map that satisfies

$$\langle w_1, \Gamma_+^*(w_2) \rangle_{+, \Sigma} = \langle \Gamma_+^*(w_1), w_2 \rangle_{-, \Sigma}$$

for all $w_1 \in B_- \cap L^2_{\text{R}}(R_-)$ and $w_2 \in B_+ \cap L^2_{\text{R}}(R_+)$.

We now formulate a theorem stating that if $B$ is strictly $\Sigma$-dissipative on $R_-$ and $m(B) = \sigma_+(\Sigma)$, then the maps $\Gamma_-$ and $\Gamma_+$ allow singular value decompositions that, in a certain sense, are compatible. It should however be understood that, strictly speaking, the terminology singular value decomposition is not appropriate in the present context, since our maps do not act between genuine inner product spaces; only the past behavior is an inner product space, on the future behavior we have an indefinite inner product. The notion singular value should therefore be interpreted in a generalized sense:

**Theorem 3.1:** Assume that $B$ is strictly $\Sigma$-dissipative on $R_-$ and $m(B) = \sigma_+(\Sigma)$. The maps $\Gamma_-$ and $\Gamma_+$ are linear. The map $\Gamma_-^* \Gamma_- : B_- \cap L^2_{\text{R}}(R_-) \to B_- \cap L^2_{\text{R}}(R_-)$ has a finite-dimensional image, and it is Hermitian and nonnegative. There exist positive real numbers $\sigma_1 \geq \sigma_2 \geq \ldots > \sigma_n > 0$, where $n = n(B)$, the McMillan degree of $B$, such that $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \geq \sigma_n^2 > 0$ are the nonzero eigenvalues of $\Gamma_-^* \Gamma_-$. There exists an orthonormal set $\{ w_{1,1}, w_{1,2}, \ldots, w_{1,n} \} \subset B_- \cap L^2_{\text{R}}(R_-)$, and an orthonormal set $\{ w_{2,1}, w_{2,2}, \ldots, w_{2,n} \} \subset B_+ \cap L^2_{\text{R}}(R_+)$ such that

$$\Gamma_- = \sum_{i=1}^n \sigma_i <, w_{1,i} >_{-, \Sigma} w_{1,i}^+,$$

$$\Gamma_+ = \sum_{i=1}^n \frac{1}{\sigma_i} <, w_{2,i}^+ >_{+, \Sigma} w_{2,i}^-.$$
bounded real characteristic values, respectively. This will follow immediately from the characterization of the \( \Sigma \)-characteristic values in terms of solutions of the algebraic Riccati equation in the next section.

IV. STATE SPACE CHARACTERIZATIONS AND REPRESENTATIONS

In this section, we will review the characterizations of (strict) \( \Sigma \)-dissipativity in terms of the algebraic Riccati equation associated with a minimal DV-representation of the given behavior \( \mathcal{B} \). We implicitly compute representations of the linear maps (and their adjoints) that assign to each past (future) trajectory the unique state at time zero, and we characterize the extremal solutions of the Riccati equation in terms of these maps. We also compute the maps \( \Gamma_- \) and \( \Gamma_+ \) in terms of compositions of these maps. It will turn out that the \( \Sigma \)-characteristic values are the eigenvalues of the product of the inverse of the maximal solution and the minimal solution of the algebraic Riccati equation, and that \( \mathcal{B} \) admits a \( \Sigma \)-balanced minimal DV-representation. Much of the material in this section is an extension of results in [15] to the case that the future behavior is an indefinite inner product space.

**Proposition 4.1:** Let \( \mathcal{B} \in \mathcal{S}_{\text{cont}} \) with minimal DV-representation \( \mathcal{B}_{\text{DV}}(A, B, C, D) \) and let \( \Sigma = \Sigma^T \in \mathbb{R}^{n \times n} \). Assume \( D^T \Sigma D > 0 \).

1) If \( \mathcal{B} \) is strictly dissipative, then there exists a real symmetric solution \( P \geq 0 \) of the algebraic Riccati equation (ARE)

\[
A^T P + P A - C^T \Sigma C + (PB - C^T \Sigma D)(D^T \Sigma D)^{-1}(B^T P - D^T \Sigma C) = 0.
\]

2) \( \mathcal{B} \) is dissipative on \( \mathbb{R}_- \) and only if there exists a minimal solution \( P_- \) of the ARE (6) with minimal DV-representation

\[
A_- := A + B(D^T \Sigma D)^{-1}(B^T P_+ - D^T \Sigma C),
\]

\[
A_+ := A + B(D^T \Sigma D)^{-1}(B^T P_- - D^T \Sigma C),
\]

Finally, the following statements are equivalent:

1) \( \mathcal{B} \) is strictly dissipative on \( \mathbb{R}_- \),

2) \( D^T \Sigma D > 0 \) and the maximal solution \( P_+ \) of the ARE (6) is positive definite and anti-stabilizing, i.e., \( \sigma(A_-) \subset \mathbb{C}^- \) and \( \sigma(A_+) \subset \mathbb{C}^+ \).

We will now study the maps \( \Gamma_- \) and \( \Gamma_+ \) in terms of DV-representations of the given behavior \( \mathcal{B} \). Let \( \mathcal{B} \in \mathcal{S}_{\text{cont}} \) with minimal DV-representation \( \mathcal{B}_{\text{DV}}(A, B, C, D) \). By minimality, for every \( \mathcal{B} \in \mathcal{S}_{\text{cont}} \) there is a unique state trajectory \( x \). For any given \( x \in \mathbb{R}_n \), let \( \mathcal{B}(x) \) denote the set of \( w \in \mathcal{B} \) such that the corresponding state trajectory \( x(t) = x_0 \). Thus, for every \( w \in \mathcal{B} \), there is a unique \( x_0 \in \mathbb{R}_n \) such that \( w = \mathcal{B}(x_0) \). Moreover (see [15]), there exists linear surjective maps \( R_- : \mathcal{B}_- \cap L_2(\mathbb{R}_-) \to \mathbb{R}_n \) and \( R_+ : \mathcal{B}_+ \cap L_2(\mathbb{R}_+) \to \mathbb{R}_n \) such that for all \( x \in \mathbb{R}_n \) we have

\[
\{ \mathcal{B}(x) \} = \{ R_-(w_-) = x_0 + R_+(w_+) = x_0 \},
\]

where \( w_- := \mathcal{B}_-(x) \) and \( w_+ := \mathcal{B}_+(x) \). In the sequel, we will explicitly compute representations of the maps \( R_- \) and \( R_+ \), and their adjoints \( R_-^* \) and \( R_+^* \) in terms of the systems matrices \( A, B, C \) and \( D \). On \( \mathbb{R}_n \) we take the standard Euclidean inner product. Note that \( R_-^* \) denotes the generalized adjoint with respect to the indefinite inner product on \( \mathcal{B}_- \cap L_2(\mathbb{R}_-) \).

It is well known (see [13]) that the extremal solutions of the Riccati equation (6) are associated with the available storage and required supply as reviewed in the previous section:

**Proposition 4.3:** Let \( \mathcal{B} \in \mathcal{S}_{\text{cont}} \) with minimal DV-representation \( \mathcal{B}_{\text{DV}}(A, B, C, D) \). Assume that \( D^T \Sigma D > 0 \). Let \( \mathcal{B} \) be dissipative and let \( P_- \) and \( P_+ \) be the minimal and maximal real symmetric solution of the ARE (6). Then for any \( w \in \mathcal{B}_- \cap L_2(\mathbb{R}_-) \), we have

\[
V_{\mathcal{B}_-}(w_-) = x_0^T P_- x_0, \quad x_0 := R_-(w_-).
\]

Also, for any \( w \in \mathcal{B}_+ \cap L_2(\mathbb{R}_+) \), we have

\[
V_{\mathcal{B}_+}(w_+) = x_0^T P_+ x_0, \quad x_0 := R_+(w_+).
\]

If \( \mathcal{B} \) is strictly dissipative, then \( P_- \) and \( P_+ \) satisfy \( \sigma(A_-) \subset \mathbb{C}^- \) and \( \sigma(A_+) \subset \mathbb{C}^+ \) (see Theorem 4.1). Introduce the following notation:

\[
C_- = C + D(D^T \Sigma D)^{-1}(B^T P_- - D^T \Sigma C),
\]

\[
C_+ = C + D(D^T \Sigma D)^{-1}(B^T P_+ - D^T \Sigma C).
\]

The following is also well-known (see also [16]):

**Proposition 4.4:** Let \( \mathcal{B} \in \mathcal{S}_{\text{cont}} \) with minimal DV-representation \( \mathcal{B}_{\text{DV}}(A, B, C, D) \). Assume that \( \mathcal{B} \) is strictly dissipative. Then for any \( w \in \mathcal{B}_- \cap L_2(\mathbb{R}_-) \) and \( w \in \mathcal{B}_+ \cap L_2(\mathbb{R}_+) \), we have

\[
R_-(w_-) = \int_0^\infty e^{-(A_- - P_-^1 C_+^T \Sigma C_-) s} P_-^1 C_+^T \Sigma w_-(s) ds,
\]

\[
R_+(w_+) = \int_0^\infty e^{-(A_+ - P_+^1 C_-^T \Sigma C_+) s} P_+^1 C_-^T \Sigma w_+(s) ds.
\]

Furthermore, for any \( x \in \mathbb{R}_n \), we have

\[
R_-^*(x_0) = C_- P_-^1 e^{-(A_- - P_-^1 C_+^T \Sigma C_-)^T} x_0,
\]

\[
R_+^*(x_0) = C_+ P_+^1 e^{-(A_+ - P_+^1 C_-^T \Sigma C_+)^T} x_0.
\]

Finally, \( P_- = (R_- R_-^*)^{-1} \), \( P_+ = (R_+ R_+^*)^{-1} \).

**Remark 4.6:** In the case that both the past and the future behavior are inner product spaces a result analogous to \( P_- = (R_- R_-^*)^{-1} \) and \( P_+ = (R_+ R_+^*)^{-1} \) was proven in [15] using a general least squares argument, without computing explicit representations of \( R_- R_-^* \) and \( R_+ R_+^* \).

**Corollary 4.7:** Let \( \mathcal{B} \in \mathcal{S}_{\text{cont}} \) with minimal DV-representation \( \mathcal{B}_{\text{DV}}(A, B, C, D) \). Assume that \( \mathcal{B} \) is strictly dissipative on \( \mathbb{R}_- \) and that \( \mathcal{B}(\mathbb{R}_-) = \mathbb{R}_- \). Then we have \( \Gamma_- = R_- R_-^* \Gamma_- \) and \( \Gamma_+ = R_+ R_+^* \Gamma_+ \).

The eigenvalues \( \sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_n^2 > 0 \) of \( \Gamma_- \) are in fact the eigenvalues of \( P_-^{-1} P_+ \), with \( 0 < P_- < P_+ \) the extremal solutions of the ARE (6), for any minimal DV-representation of \( \mathcal{B} \).
Theorem 4.8: Assume that $\mathcal{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}^-$ and $m(\mathcal{B}) = \sigma_+(\Sigma)$. Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$ be the $\Sigma$-characteristic values of $\mathcal{B}$. Let $\mathcal{B}_{DV}(A, B, C, D)$ be a minimal DV-representation of $\mathcal{B}$ with $0 < P_0 < P_\delta$ the extremal solutions of the ARE (6). Then $\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2\} = \sigma(P_+^1 P_-)$. Furthermore $0 < \sigma_i < 1$ for all $i$.

After a suitable coordinate transformation $\hat{x} = Tx$ in the state space of the DV-representation $\mathcal{B}_{DV}(A, B, C, D)$, the maps $R_-$ and $R_+$ transform to $TR_-R_-^T$ and $TR_+R_+^T$ to $TR_R_-R_-^T$ and $P_4$ to $P_1 - T^1 P_4 T^1$. It is well known, see e.g. [20], that there exists a coordinate transformation $T$ such that $T^1 P_4 T^1$ and $T^1 P_4 T^1$ are equal and diagonal. Since the set of eigenvalues of $P_+^1 P_-$ is $\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2\}$ this diagonal matrix must be equal to the diagonal matrix $\Pi := \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$. Thus we obtain:

Corollary 4.9: Assume that $\mathcal{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}^-$ and $m(\mathcal{B}) = \sigma_+(\Sigma)$. Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$ be the $\Sigma$-characteristic values of $\mathcal{B}$. Then there exists a minimal driving variable representation $T\mathcal{B}$ such that $\Pi$ is elaborated in the following lemma.

Theorem 4.8: Assume that $\mathcal{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}^-$ and $m(\mathcal{B}) = \sigma_+(\Sigma)$. Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$ be the $\Sigma$-characteristic values of $\mathcal{B}$. Let $\mathcal{B}_{DV}(A, B, C, D)$ be a minimal DV-representation of $\mathcal{B}$, $\mathcal{B}$ is strictly $\Sigma$-dissipative on $\mathbb{R}^-$, $n(\mathcal{B}) \leq k$, and in $\mathcal{B}$ $w_1$ is input and $w_2$ is output.

VI. REDUCTION BY BALANCED TRUNCATION

Let $\mathcal{B} \in \mathcal{L}_{\Sigma}\text{cont}$ be strictly $\Sigma$-dissipative on $\mathbb{R}^-$, and partition $w = (w_1, w_2)$ with $w_1$ input and $w_2$ output. Assume that $m(\mathcal{B}) = \dim(w_1) = \sigma_+(\Sigma)$. Let $\mathcal{B}_{DV}(A, B, C, D)$ be a minimal $\Sigma$-normalized and $\Sigma$-balanced DV-representation of $\mathcal{B}$. Define $G(\xi) = C(\xi I - A)^{-1} B + D$. We have $P_+^1 = P_+ = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$.

Pick $k < n$ such that $\sigma_k > \sigma_{k+1}$. We will compute a reduced order approximation $\hat{\mathcal{B}} \in \mathcal{L}_{\Sigma}\text{cont}$ of $\mathcal{B}$ that inherits the input-output partition of $\mathcal{B}$, i.e., also in $\hat{\mathcal{B}} w_1$ is input and $w_2$ is output, such that its McMillan degree $n(\hat{\mathcal{B}}) \leq k$, and $\hat{\mathcal{B}}$ is strictly $\Sigma$-dissipative on $\mathbb{R}^-$. This computation consists of the following steps:

1. Partition $A$, $B$ and $C$:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

(16)

with $A_{11} k \times k$, $B_1 k \times w_1$ and $C_1 p \times k$, and define the truncated system $\mathcal{B}_{\text{trunc}} := B_{DV}(A_{11}, B_1, C_1, D)$.

2. Define the reduced order approximation as the controllable part of $\mathcal{B}_{\text{trunc}}$:

$$\hat{\mathcal{B}} := (\mathcal{B}_{\text{trunc}})_{\text{cont}}.$$  

A DV-representation of $\hat{\mathcal{B}}$ can be obtained by performing a Kalman controllability decomposition (see [7], proposition 22):

$$T^{-1} A_{11} T = \begin{bmatrix} \hat{A} & * \\ 0 & * \end{bmatrix}, T^{-1} B_1 = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix},$$

$$C_1 T = \begin{bmatrix} \hat{C} & * \end{bmatrix}, D = \hat{D}.$$  

(17)

We then have $\hat{\mathcal{B}} = \mathcal{B}_{DV}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$. Our main result now states that the for the special cases of strictly passive and strictly bounded real systems, the reduced order behavior $\hat{\mathcal{B}}$ obtained in this way satisfies the required properties.

Theorem 6.1: Assume

$$\Sigma = \frac{1}{2} \begin{bmatrix} 0 & I_{w_1} \\ I_{w_1} & 0 \end{bmatrix}$$

or $\Sigma = \text{diag}(I_{w_1}, -I_{w_2})$.

Let $\hat{\mathcal{B}}$ be defined by (16). Then

1. $\hat{\mathcal{B}}$ is controllable,
2. $\sigma(A) \subset \mathbb{C}^-$,
3. $\mathcal{B}_{DV}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is a $\Sigma$-normalized DV-representation of $\hat{\mathcal{B}}$,
4. $\hat{\mathcal{B}}$ is strictly $\Sigma$-dissipative on $\mathbb{R}^-$, $n(\hat{\mathcal{B}}) \leq k$, and in $\hat{\mathcal{B}} w_1$ is input and $w_2$ is output.
VII. Error bounds

A. A one-step frequency domain inequality

Starting with $\mathcal{B}$, strictly $\Sigma$-dissipative on $\mathbb{R}_-$, with $\sigma(\mathcal{B}) = \sigma_1(\Sigma)$, let $\mathcal{B}_{DV}(A, B, C, D)$ be a $\Sigma$-balanced, $\Sigma$-normalized minimal $DV$-representation. Let $G(\xi) = D + C(\xi I - A)^{-1} B$. Assume that the distinct $\Sigma$-characteristic values of $\mathcal{B}$ are $\sigma_1 > \sigma_2 > \ldots > \sigma_N$, where $\sigma_i$ appears $n_i$ times. Then $\Pi = \operatorname{diag}(\sigma_1 I_{n_1}, \sigma_2 I_{n_2}, \ldots, \sigma_N I_{n_N})$, with $I_n$ the $n \times n$ identity matrix.

Suppose now that we do a one-step reduction by eliminating the state components corresponding to the last singular value $\sigma_N$; partition $\Pi = \operatorname{blockdiag}(\Pi_{n_1}, \Pi_{n_2}, \ldots)$. Let $\mathcal{B}_{\text{trunc}} = B_{\text{DV}}(A_{n_1}, B_1, C_1, D_1)$ be the truncated behavior as defined in step 2 of our algorithm. Let $G_{\text{trunc}}(\xi) = D + C_1(\xi I - A_{n_1})^{-1} B_1$. Let $\mathcal{B} = \mathcal{B}_{DV}(A, B, C, D)$ be the reduced order behavior, and $\breve{G}(\xi) = \breve{D} + \breve{C}(\xi I - A)^{-1} \breve{B}$. Obviously, $G_{\text{trunc}} = \breve{G}$. We will now derive some properties of the error transfer matrix $E := G - \breve{G}$.

Theorem 7.1: The rational matrix $E$ is stable. For all $\omega \in \mathbb{R}$ we have

$$0 \leq -E^T(-i\omega)\Sigma E(i\omega) \leq \frac{4\sigma_N^2}{1 - \sigma_N^2} I_N. \quad (18)$$

Of course, the question arises in which sense the inequality (18) can be interpreted as an error bound. Note that the supply rate $\Sigma$ is still an arbitrary nonsingular symmetric matrix with the property that $\sigma_i(\Sigma) = \sigma_i(\mathcal{B})$. In the following, denote $G^*(\xi) := G(\xi)$. Since $G^* \Sigma G = \breve{I}$ and $G \Sigma G = I$, we have

$$I = (G - E)^T \Sigma (G - E) = I - G^* \Sigma G - G^* E \Sigma E^* + E^* \Sigma E,$$

which implies $E^* \Sigma E = E^* \Sigma G + G^* \Sigma E$. Thus (18) is equivalent with: for all $\omega \in \mathbb{R}$

$$0 \leq -[G^T(-i\omega)\Sigma E(i\omega) + E^T(-i\omega)\Sigma G(i\omega)] \leq \frac{4\sigma_N^2}{1 - \sigma_N^2} I_N.$$

B. Error bounds for the strictly bounded real case

We will now study the case that our system $\mathcal{B}$ comes with an input-output partition $w = \operatorname{col}(w_1, w_2)$, and that it is strictly bounded real, i.e. is $\Sigma$-dissipative on $\mathbb{R}_-$ given by $\operatorname{diag}(I_{n_1}, I_{n_2})$. We already know that our balanced truncation method retains the given input-output partition. In this section we will denote the input variable $w_1$ simply by $u$ and the output variable $w_2$ by $y$.

Let the original system $\mathcal{B}$ be represented by the minimal normalized $DV$-representation

$$\begin{align*}
\dot{x} &= Ax + Bu, \\
(\begin{array}{c}
u \\
y
\end{array}) &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} x + \begin{pmatrix} D_1 & D_2 \end{pmatrix} v.
\end{align*}$$

Apply then the algorithm outlined in section VI to obtain a reduced order behavior $\mathcal{B}$ given in normalized $DV$ representation by

$$\begin{align*}
\dot{z} &= \breve{A} z + \breve{B} u, \\
(\begin{array}{c}
\breve{u} \\
\breve{y}
\end{array}) &= \begin{pmatrix} \breve{C}_1 & \breve{C}_2 \end{pmatrix} z + \begin{pmatrix} \breve{D}_1 & \breve{D}_2 \end{pmatrix} v.
\end{align*}$$

Next, we investigate the frequency domain inequality (18) for this special case. Let

$$G(\xi) = \begin{pmatrix} G_1(\xi) \\ G_2(\xi) \end{pmatrix} = \begin{pmatrix} C_1(\xi I - A)^{-1} B + D_1 \\ C_2(\xi I - A)^{-1} B + D_2 \end{pmatrix}$$

compatibly with the partition $w = (u, y)$ (i.e., $u = G_1 v$ and $y = G_2 v$). Likewise, define $\breve{G}_1$ and $\breve{G}_2$. Denote $E_1 := G_1 - \breve{G}_1, E_2 := G_2 - \breve{G}_2$. Assume now that we truncate one step, as explained in subsection VII-A. According to Theorem 7.1, for all $\omega \in \mathbb{R}$ we have

$$\begin{align*}
E_1(-i\omega)^T E_1(-i\omega) &\leq E_2(-i\omega)^T E_2(-i\omega) \\
&\leq \frac{4\sigma_N^2}{1 - \sigma_N^2} I + E_1(-i\omega)^T E_1(-i\omega)
\end{align*}$$

This implies that if we 'drive' both $\mathcal{B}$ and $\breve{\mathcal{B}}$ with the same driving variable trajectory $v \in \mathcal{L}_2(\mathbb{R})$ with $\|v\|_2 = 1$, and denote the corresponding (unique) $\mathcal{L}_2(\mathbb{R})$ input trajectories and output trajectories for $\mathcal{B}$ and $\breve{\mathcal{B}}$ by $u, \breve{u}$ and $y, \breve{y}$, respectively, then we have

$$\|u - \breve{u}\|_2 \leq \|y - \breve{y}\|_2 \leq \frac{4\sigma_N^2}{1 - \sigma_N^2} + \|u - \breve{u}\|_2,$$

so

$$\|u - \breve{u}\|_2 \leq \|y - \breve{y}\|_2 \leq \frac{2\sigma_N}{\sqrt{1 - \sigma_N^2}} + \|u - \breve{u}\|_2.$$