Controllability of Diffusively-Coupled Multi-Agent Systems with General and Distance Regular Coupling Topologies

2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Controllability of Diffusively-Coupled Multi-Agent Systems with General and Distance Regular Coupling Topologies

Shuo Zhang, M. Kanat Camlibel, and Ming Cao

Abstract— **This paper studies the controllability of linearly diffusively coupled multi-agent systems when some agents, called *leaders*, are under the influence of external control inputs. We bound the system's controllable subspace using combinatorial characteristics of some partitions of the graph describing the neighbor relationships between the agents. In particular, when such graphs are distance regular, we provide a full characterization of the controllable subspace for single leader cases while for multi-leader cases, a necessary condition and a sufficient condition for controllability are given respectively. In the end, we discuss how to choose leaders among the agents to guarantee controllability when the graphs are cycles or complete graphs, which are special subclasses of distance regular graphs.**

I. INTRODUCTION

Recently significant work has been done to study distributed and cooperative control of multi-agent systems [1], [2]. It is of particular interest to study the case when the agents are linearly diffusively coupled since rich collective behaviors, such as synchronization [3]–[5] and clustering [6]–[8], may arise as a result of local interactions among agents without centralized coordination and control. People are especially interested in knowing how to influence the behavior of the overall system by just controlling some, usually a small fraction, of the agents [9], [10]. We call such agents that are under the forcing of external control inputs the *leaders* and correspondingly the rest of the agents *followers*. Hence, to study whether any desired collective behavior can be achieved in finite time by controlling the leaders is equivalent to study the controllability of the overall systems where the leaders and followers that are coupled together through linear diffusive couplings.

Graphs describing the neighbor relationships in multi-agent systems have been extensively used to study the cooperative control of multi-agent systems [11], [12]. The controllability problem of multi-agent systems has also been formulated and studied using tools from graph theory [13], [14]. In [14], the automorphism group and equitable partitions of graphs are utilized to identify necessary conditions for the controllability of the corresponding multi-agent systems. These results are further generalized in [15] to provide an upper bound of the system's controllable subspace using a more general notion of graph partitioning, called *almost equitable partitions* [16]. Similar graphical approaches have been taken to study the observability problem for multi-agent systems in [17].

In this paper, we first bound the controllable subspace for a given multi-agent system using the almost equitable partition of the associated neighbor relationship graph. When the graph is distance regular, it is shown that the controllable subspace can be fully characterized if there is a single leader and it is also shown that if there are multiple leaders, a necessary and a sufficient condition can be constructed respectively. In the end, we discuss how to choose leaders among the agents in order to guarantee controllability when the graphs are cycles or complete graphs, which are special subclasses of distance regular graphs.

The rest of this paper is organized as follows. The controllability problem of linearly diffusively coupled multi-agent systems is formulated in Section II. We then review and discuss several facts about graph partitions in Section III. In Sections IV and V, controllable subspaces for systems associated with general graphs and distance regular graphs are studied respectively using graph partitions. Then in Section VI we show how to choose leaders for systems associated with cycles and complete graphs to guarantee controllability.

II. DIFFUSIVELY COUPLED MULTI-AGENT SYSTEMS WITH EXTERNAL CONTROL INPUTS

A. Formulation of the controllability problem

We consider a multi-agent system consisting of $n > 0$ agents, labeled by $1, \ldots, n$, and use $\mathcal{V} = \{1, \ldots, n\}$ to denote the set of indices of all the agents. Let $x_i \in \mathbb{R}$, $i \in \mathcal{V}$, denote the state of agent i. For a pair of distinct agents i and j, $i, j \in \mathcal{V}$, we say agent j is a neighbor of agent i if $x_j(t)$, $t \geq 0$, is known by agent i. We assume that the neighbor relationships are fixed during the evolution of the system and are symmetric, namely j is a neighbor of i always implies that i is also a neighbor of j. We assign the roles of the leaders and followers to the agents and use $\mathcal{V}_L, \mathcal{V}_F \subseteq \mathcal{V}$ to denote the sets of indices of the leaders and followers respectively. For a finite set $\mathcal{S}$, let $\mathrm{card}(\mathcal{S})$ denote its cardinality. Then we assume there are altogether $0 < m = \mathrm{card}(\mathcal{V}_L) < n$ control inputs $u_i \in \mathbb{R}$, $1 \leq i \leq m$ and each leader is influenced by only one input. The followers' dynamics are governed by linear diffusive

S. Zhang and M. Cao are with the Research Institute of Industrial Technology and Management, University of Groningen, 9747AG, Groningen, the Netherlands ({shuo.zhang, m.cao}@rug.nl). M. Kanat Camlibel is with the Johann Bernoulli Institute for Mathematics and Computer Sciences, University of Groningen, 9747AG, Groningen, the Netherlands (m.k.camlibel@rug.nl).

The work was supported in part, by grants from the Dutch Organization for Scientific Research (NWO) and the Dutch Technology Foundation (STW).

couplings

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i), \qquad i \in \mathcal{V}_F \tag{1}$$

where $\mathcal{N}_i$ is the set of indices of the neighbors of agent i. For a leader $i \in \mathcal{V}_L$, let $[i] \in \{1, \cdots, m\}$ denote the index of the control input acting on it. Then the dynamics of the leaders are determined by

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i) + u_{[i]}. \tag{2}$$

Note that the neighbor relationships can be conveniently described by graphs. We now define the graph $\mathbb{G}$ associated with the system (1) and (2) and use the matrices associated with $\mathbb{G}$ to write (1) and (2) into a more compact form. Consider the graph $\mathbb{G}$ with a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$ such that there is an edge in $\mathbb{G}$ from vertex j to i for any $i, j \in \mathcal{V}$ if and only if $j \in \mathcal{N}_i$. Since the neighbor relationships are fixed and symmetric, $\mathbb{G}$ is time invariant and undirected. The *adjacency matrix* A of $\mathbb{G}$ is an $n \times n$ symmetric matrix whose element A_{ij}, $1 \le i, j \le n$, is determined by

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

In $\mathbb{G}$, a vertex i's *degree* is defined to be the number of edges to i, denoted by $\deg(i)$. Then the *degree matrix* D of $\mathbb{G}$ is defined by

$$D = \operatorname{diag}(\deg(1), \deg(2), \ldots, \deg(n)),$$

where $\operatorname{diag}(a_1, a_2, \ldots, a_n)$, $a_1, \ldots, a_n \in \mathbb{R}$, is the diagonal matrix whose diagonal elements are $a_1, a_2, \ldots, a_n$. Then the *Laplacian matrix* of $\mathbb{G}$ is defined to be

$$L = D - A.$$

Define $x \triangleq \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}^T$ and $u \triangleq \begin{bmatrix} u_1 & \ldots & u_m \end{bmatrix}^T$, then (1) and (2) can be rewritten into a compact form

$$\dot{x} = -Lx + Mu \tag{3}$$

where for $1 \le i \le n$ and $1 \le j \le m$, the elements of M are defined by

$$M_{ij} = \begin{cases} 1 & \text{if } j = [i] \\ 0 & \text{otherwise.} \end{cases}$$

The goal of this paper is to study the controllability of the system (3). We denote its controllable subspace by $\mathcal{K}(L, M)$, which, when the context is clear, is also written as $\mathcal{K}$. It is obvious that $\mathcal{K}$ is the smallest L-invariant subspace that contains the subspace spanned by the columns of M, denoted by $\operatorname{im} M$.

We want to mention that in the literature, e.g. [11], [14], the controllability problem of multi-agent systems is sometimes studied in a different setting where it is assumed that the dynamics of the leaders are completely determined by the control input. In the next subsection, we show that the setting used in [11], [14] is in fact equivalent to the setting used in this paper.

B. Relationship to some different formulation of the controllability problem

In this subsection, we relabel the agents such that the first $n-m$ agents are followers and the last m agents are leaders. Then one can partition L into block submatrices

$$L = \begin{bmatrix} L_f & l_{fl} \\ l_{fl}^T & L_l \end{bmatrix}$$

where L_f and L_l are $(n-m) \times (n-m)$ and $m \times m$ dimensional matrices respectively. Let $x_l \in \mathbb{R}^m$ denote the state of the leaders and $x_f \in \mathbb{R}^{n-m}$ that of the followers. Then in [11], [14] the following model is studied

$$\dot{x}_f = -L_f x_f + l_{fl} x_l \tag{4}$$

where x_l is taken as the control input.

Now we compare the controllable subspaces $\mathcal{K}(L, M)$ of (3) and $\mathcal{K}(L_f, l_{fl})$ of (4). In fact, with the new agent labels we can rewrite (3) into

$$\begin{bmatrix} \dot{x}_f \\ \dot{x}_l \end{bmatrix} = -\begin{bmatrix} L_f & l_{fl} \\ l_{fl}^T & L_l \end{bmatrix} \begin{bmatrix} x_f \\ x_l \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u \tag{5}$$

where I is the $m \times m$ identity matrix. Taking the control input to be the state feedback $u = \begin{bmatrix} l_{fl}^T & L_l \end{bmatrix} \begin{bmatrix} x_f & x_l \end{bmatrix}^T + v$ with the new control input $v \in \mathbb{R}^m$, we have

$$\begin{bmatrix} \dot{x}_f \\ \dot{x}_l \end{bmatrix} = -\begin{bmatrix} L_f & l_{fl} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_f \\ x_l \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} v.$$

This is equivalent to

$$\dot{x}_f = -L_f x_f + l_{fl} x_l \text{ and } \dot{x}_l = v. \tag{6}$$

Since controllable subspaces are invariant under state feedback, it holds that

$$\mathcal{K}(L, M) = \mathcal{K}(L_f, l_{fl}) \times \mathbb{R}^m.$$

In the sequel, we will use graph partitions to bound $\mathcal{K}(L, M)$. Towards this end, in the next section we will first introduce some definitions and results related to partitioning graphs.

III. Graph partitions

For the vertex set $\mathcal{V}$ of a graph $\mathbb{G}$ and a constant $1 \le k \le n$, we call its nonempty, disjoint subsets $\mathcal{C}_i$, $1 \le i \le k$, a *partition* of $\mathcal{V}$ if $\bigcup_i \mathcal{C}_i = \mathcal{V}$. We use $\pi = \{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$ to denote the partition. Accordingly, we call $\mathcal{C}_i$'s *cells* and k the *size* of the partition. The *characteristic matrix* $P(\pi) \in \mathbb{R}^{n \times k}$ of the partition π is defined by

$$P_{ij}(\pi) = \begin{cases} 1 & \text{if } i \in \mathcal{C}_j; \\ 0 & \text{otherwise,} \end{cases} \qquad 1 \le i \le n, 1 \le j \le k.$$

A. Binary relations over the set of graph partitions

Let Π denote the set of all the partitions of $\mathbb{G}$. We say that a partition π_1 is *finer* than another partition π_2 if each cell of π_1 is a subset of some cell of π_2 and we write $\pi_1 \leq \pi_2$. If $\pi_1 \leq \pi_2$, the following two relations are immediate

$$\mathrm{card}(\pi_1) \geq \mathrm{card}(\pi_2) \text{ and } \operatorname{im} P(\pi_1) \supseteq \operatorname{im} P(\pi_2). \tag{7}$$

With the ordering induced by the binary relation "$\leq$", Π becomes a partially ordered set. Furthermore, it is also a complete lattice [18], i.e. every subset of Π has the greatest lower bound, also called the *meet*, and the least upper bound, also called the *join*, within the set Π. For a subset Π' of Π, we use $\wedge\Pi'$ and $\vee\Pi'$ to denote its meet and join respectively. In particular, we use $\pi_1 \wedge \pi_2$ and $\pi_1 \vee \pi_2$ to denote $\wedge\{\pi_1, \pi_2\}$ and $\vee\{\pi_1, \pi_2\}$, respectively.

We prove the following result which will be used later.

Lemma 1: Let $\pi_1, \pi_2 \in \Pi$. Then,

$$\operatorname{im} P(\pi_1) \cap \operatorname{im} P(\pi_2) = \operatorname{im} P(\pi_1 \vee \pi_2).$$

Proof: For any $x \in \mathbb{R}^n$, let π_x be the partition such that i, j are within one cell if $x_i = x_j$ for any $1 \leq i, j \leq n$.

Consider a set $\kappa = \{\pi_x \mid x \in \operatorname{im} P(\pi_1) \cap \operatorname{im} P(\pi_2)\}$. Then for any $\pi_x \in \kappa$, it holds that $\operatorname{im} P(\pi_x) \subseteq \operatorname{im} P(\pi_i)$ and thus, $\pi_i \leq \pi_x$ for $i = 1, 2$. Let π^* denote the partition satisfying $\pi^* \leq \pi_x$ for any $\pi_x \in \kappa$. Then one has $\pi_i \leq \pi^*$ for $i = 1, 2$. Then $\pi_1 \vee \pi_2 \leq \pi^*$ and hence,

$$\operatorname{im} P(\pi^*) \subseteq \operatorname{im} P(\pi_1 \vee \pi_2). \tag{8}$$

Also, since $\pi_i \leq \pi_1 \vee \pi_2$ for $i = 1, 2$, it holds that

$$\operatorname{im} P(\pi_1 \vee \pi_2) \subseteq \operatorname{im} P(\pi_1) \cap \operatorname{im} P(\pi_2). \tag{9}$$

From (8) and (9), it is obvious that $\operatorname{im} P(\pi^*) \subseteq \operatorname{im} P(\pi_1) \cap \operatorname{im} P(\pi_2)$. On the other hand, for any $x \in \operatorname{im} P(\pi_1) \cap \operatorname{im} P(\pi_2)$, there exists some $\pi_x \in \kappa$ such that $x \in \operatorname{im} P(\pi_x)$. Then it follows from $\operatorname{im} P(\pi_x) \subseteq \operatorname{im} P(\pi^*)$ that $x \in \operatorname{im} P(\pi^*)$. Now it can be concluded that $\operatorname{im} P(\pi_1) \cap \operatorname{im} P(\pi_2) \subseteq \operatorname{im} P(\pi^*)$ and thus,

$$\operatorname{im} P(\pi_1) \cap \operatorname{im} P(\pi_2) = \operatorname{im} P(\pi^*). \tag{10}$$

Finally, the conclusion can be achieved by combining (8), (9) and (10). □

Clearly, one can generalize these relations to any number of partitions as follows.

Lemma 2: It holds that

$$\bigcap_{i=1}^{m} \operatorname{im} P(\pi_i) = \operatorname{im} P(\vee_{i=1}^{m} \pi_i). \tag{11}$$

In the next subsection, we look at some useful classes of graph partitions.

B. Equitable, almost equitable and distance partitions

A partition $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ of $\mathbb{G}$ is said to be an *equitable partition* if for every pair of $1 \leq i, j \leq k$ (not necessarily distinct) there exists a number b_{ij} such that any vertex $v \in \mathcal{C}_i$ has b_{ij} neighbors in $\mathcal{C}_j$ [19]. Further, it is said to be an *almost equitable partition* if for every pair of distinct $1 \leq i, j \leq k$ there exists a number b_{ij} such that any vertex $v \in \mathcal{C}_i$ has b_{ij} neighbors in $\mathcal{C}_j$ [16]. For a given $v \in \mathcal{V}$, π is said to be an *almost equitable partition relative to* v if it is almost equitable and $\{v\}$ is one of its cells. In general, for a given $\mathbb{G}$, the above three classes of partitions are not unique. Let Π_{EP}, Π_{AEP} and $\Pi_{\mathrm{AEP}}(v)$ denote the sets of all equitable, almost equitable and almost equitable partitions relative to v respectively. Obviously, it holds that $\Pi_{\mathrm{EP}} \subseteq \Pi_{\mathrm{AEP}}$ and $\Pi_{\mathrm{AEP}}(v) \subseteq \Pi_{\mathrm{AEP}}$.

Almost equitable partitions can be characterized in terms of the invariant subspaces of the Laplacian matrix L of $\mathbb{G}$ as follows.

Lemma 3: [16, Prop. 1] A partition π of $\mathbb{G}$ is almost equitable if and only if $\operatorname{im} P(\pi)$ is L-invariant.

In view of (11) and Lemma 3, it is true that the maximal partitions with respect to "$\leq$" for almost equitable partitions and almost equitable partitions relative to v are *unique*, which are denoted by π^*_{AEP} and $\pi^*_{\mathrm{AEP}}(v)$ respectively.

Another class of graph partitions depends on the *distance* of two vertices and the graph *diameter*. A *path* of length r in $\mathbb{G}$ is a sequence of $r+1$ distinct vertices $i_0, \ldots, i_r$ such that $(i_{k-1}, i_k) \in \mathcal{E}$ for all $k = 1, \ldots, r$. If there is a path between any pair of distinct vertices in $\mathbb{G}$, then $\mathbb{G}$ is said to be *connected*. In what follows, $\mathbb{G}$ is assumed to be connected. The *distance* between two vertices $i, j \in \mathcal{V}$ is the length of the shortest path from i to j and is denoted by $\mathrm{dist}(i, j)$. For convenience, we define $\mathrm{dist}(i, i) = 0$ for any $i \in \mathcal{V}$. The *diameter* of $\mathbb{G}$ is defined by

$$\mathrm{diam}(\mathbb{G}) = \max_{i,j \in \mathcal{V}} \mathrm{dist}(i, j).$$

Obviously, when $n > 1$, it holds that $0 < \mathrm{diam}(\mathbb{G}) \leq n - 1$.

A partition is called the *distance partition relative to* v if each of its cells is of the form $\{u \in \mathcal{V} | \mathrm{dist}(u, v) = i\}$ for some $0 \leq i \leq \mathrm{diam}(\mathbb{G})$. Hence, the distance partition relative to a vertex v of $\mathbb{G}$ is always unique and denoted by $\pi_{\mathrm{D}}(v)$. Let $\mathcal{C}_i$ denote the cell $\{u \in \mathcal{V} | \mathrm{dist}(u, v) = i\}$ in $\pi_{\mathrm{D}}(v)$. Then it is easy to prove the following result.

Lemma 4: For any $z \in \mathcal{C}_i$ and $w \in \mathcal{C}_j$, it always holds that $\mathrm{dist}(z, w) \geq |i - j|$.

From Lemma 4, it follows that if $|i - j| > 1$, then no vertex in $\mathcal{C}_i$ has a neighbor in the cell $\mathcal{C}_j$.

For any $v \in \mathcal{V}$ and $\pi \in \Pi_{\mathrm{AEP}}(v)$, one has

$$\pi \leq \pi_{\mathrm{D}}(v), \tag{12}$$

which implies the following result:

Lemma 5: For any $\pi \in \Pi_{\mathrm{AEP}}(v)$, it holds that

$$\mathrm{card}(\pi_{\mathrm{D}}(v)) \leq \mathrm{card}(\pi^*_{\mathrm{AEP}}(v)) \leq \mathrm{card}(\pi).$$

IV. Controllability of Systems with General Graphs

First, we give an upper bound for the controllable subspace of the system (3) in terms of almost equitable partitions. Such a bound for a multi-agent system with a single leader can be deduced from [15, Prop. 2] where almost equitable partitions are called relaxed equitable partitions. The following proposition provides an upper bound when the system has multiple leaders.

Proposition 1: Let $\pi \in \Pi_{\text{AEP}}$ be such that for each $v \in \mathcal{V}_L$, $\{v\}$ is a cell of π. Then

$$\mathcal{K} \subseteq \operatorname{im} P(\pi).$$

Proof: It follows from Lemma 3 that $\operatorname{im} P(\pi)$ is L-invariant. Since $\{v\}$ is a cell of π for each leader v, we have $\operatorname{im} M \subseteq \operatorname{im} P(\pi)$, i.e. $\operatorname{im} P(\pi)$ is L-invariant containing $\operatorname{im} M$. Then the claim follows from (7) and the fact that $\mathcal{K}$ is the smallest of such subspaces. □

Next we present a lower bound for the dimension of the controllable subspace when there is a single leader.

Proposition 2: If $\mathcal{V}_L = \{v\}$, then

$$\operatorname{card}(\pi_{\text{D}}(v)) \leq \dim(\mathcal{K}).$$

Proof: Without loss of generality, we take $v = 1$ and $\pi_{\text{D}}(1) = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_r\}$ where $0 < r \leq \operatorname{diam} \mathbb{G}$, $\mathcal{C}_0 = \{1\}$, and $\mathcal{C}_q = \{i_q+1, i_q+2, \ldots, i_{q+1}\}$ with $1 = i_1 < i_2 < \ldots < i_{r+1} = n$. Then in view of Lemma 4, the matrix L becomes a tri-diagonal matrix

$$L = \begin{bmatrix} \deg(1) & \mathbf{1}^T & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mathbf{1} & L_{11} & L_{12} & 0 & \cdots & 0 & 0 & 0 \\ 0 & L_{21} & L_{22} & L_{23} & \cdots & 0 & 0 & 0 \\ 0 & 0 & L_{32} & L_{33} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & L_{r-2,r-2} & L_{r-2,r-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & L_{r-1,r-2} & L_{r-1,r-1} & L_{r-1,r} \\ 0 & 0 & 0 & 0 & \cdots & 0 & L_{r,r-1} & L_{r,r} \end{bmatrix} \quad (13)$$

where $\mathbf{1}$ is the column vector of 1's with the dimension of $\operatorname{card}(\mathcal{C}_1)$ and L_{kl} is a $\operatorname{card}(\mathcal{C}_k) \times \operatorname{card}(\mathcal{C}_l)$ matrix for all $1 \leq k, l \leq r$; the matrix M becomes e_1 which is the first standard basis vector in the form of $\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ for $\mathbb{R}^n$.

Let $E = \begin{bmatrix} e_1 & Le_1 & \cdots & L^r e_1 \end{bmatrix}$, then it can be written as

$$E = \begin{bmatrix} 1 & \deg(1) & * & * & \cdots & * & * \\ 0 & \mathbf{1} & * & * & \cdots & * & * \\ 0 & 0 & L_{21}\mathbf{1} & * & \cdots & * & * \\ 0 & 0 & 0 & L_{32}L_{21}\mathbf{1} & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & L_{r-1,r-2} \ldots L_{21}\mathbf{1} & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & L_{r,r-1} \ldots L_{21}\mathbf{1} \end{bmatrix}$$

where '*' denotes the corresponding elements of less interest. Since the graph $\mathbb{G}$ is connected, each diagonal block must be nonzero. Then $\operatorname{rank} E = \operatorname{card}(\pi_{\text{D}}(1)) = r + 1$. Since $r \leq \operatorname{diam}(\mathbb{G}) \leq n - 1$, we have

$$\operatorname{rank} E \leq \operatorname{rank} \begin{bmatrix} e_1 & Le_1 & \cdots & L^{n-1} e_1 \end{bmatrix} = \dim(\mathcal{K}).$$

□

Combining Propositions 1 and 2, we obtain the following bounds for systems with a single leader.

Proposition 3: If $\mathcal{V}_L = \{v\}$, then

$$\operatorname{card}(\pi_{\text{D}}(v)) \leq \dim(\mathcal{K}) \leq \dim(\operatorname{im} P(\pi^*_{\text{AEP}}(v))). \quad (14)$$

For a general graph, the bounds presented in Proposition 3 are tight and cannot be further improved. This can be illustrated by the examples shown in Figure 1. The first agent is chosen as the leader in each example. For the first example shown in Figure 1 left, the lower bound is achieved which is strictly less than the upper bound. For the second example shown in Figure 1 middle, the upper bound is achieved which is strictly greater than the lower bound. An interesting question to ask is whether the controllable subspace can be expressed as the image of the characteristic matrix of a certain partition. In general, the answer is not affirmative as illustrated by the third example shown in Figure 1 right. One can check that neither of the two bounds in Proposition 3 is achieved for this example [20]. Moreover, there is *no* partition π of the corresponding graph $\mathbb{G}$ such that $\mathcal{K} = \operatorname{im} P(\pi)$.

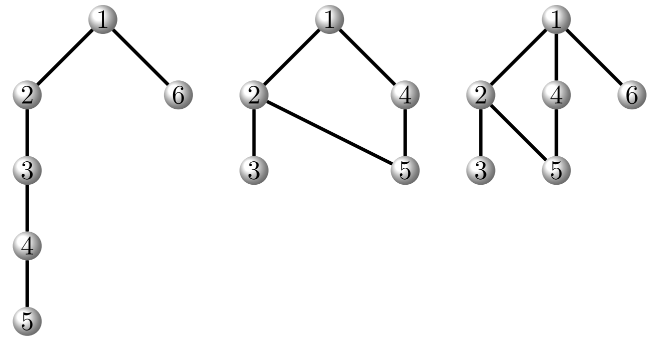

Fig. 1. Examples of Proposition 3

In the next section, we will focus on distance regular graphs for which the lower and upper bounds discussed in this section are identical. This will enable us to investigate the minimum number of leaders needed to render a multi-agent system controllable.

V. Controllability of Systems with Distance Regular Graphs

We begin with a brief review of the properties of distance regular graphs. For more details, readers can refer to [21].

A. Distance regular graphs and their properties

A graph $\mathbb{G}$ is said to be *regular* if $\deg(i) = \deg(j)$ for all $i, j \in \mathcal{V}$. It is called *distance-regular* if it is regular and for any pair of vertices $u, v \in \mathcal{V}$ with $\operatorname{dist}(u, v) = i$, $0 < i < \operatorname{diam}(\mathbb{G})$, there exist numbers c_i and b_i such that there are c_i neighbors of v that are of distance $i-1$ from u and b_i neighbors of v that are of distance $i+1$ from u [21].

It is easy to see that if $\mathbb{G}$ is regular, then $\Pi_{\text{EP}} = \Pi_{\text{AEP}}$.

Consider a distance regular graph $\mathbb{G}$. Let $d = \operatorname{diam}(\mathbb{G})$. The sequence

$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$$

is called the *intersection array* of $\mathbb{G}$. For a pair of vertices $u, v \in \mathcal{V}$ with $\operatorname{dist}(u, v) = h$, we define the numbers $p^h_{ij} \triangleq \operatorname{card}(\{w \in \mathcal{V} | \operatorname{dist}(u, w) = i \text{ and } \operatorname{dist}(v, w) = j\})$ for all $0 \leq i, j, h \leq d$. Then we have

Lemma 6: [21] For a distance regular graph, it holds that

$$b_{i-1} > 0, c_i > 0 \quad \text{for all } 1 \leq i \leq d$$

$$p^{i-h}_{ih} = \frac{b_{i-1} \cdots b_{i-h}}{c_1 \cdots c_h} \quad \text{for all } 0 \leq h \leq i \leq d.$$

Define the i distance regular graph of $\mathbb{G}$ for $i = 0, 1, \ldots, d$, denoted by $\mathbb{G}_i = (\mathcal{V}, \mathcal{E}_i)$, such that for $u, v \in \mathcal{V}$, $(u, v) \in \mathcal{E}_i$

if and only if $\mathrm{dist}(u, v) = i$ in $\mathbb{G}$. Denote the adjacency matrix of $\mathbb{G}_i$ by A_i. Note that $A_0 = I$ and $A_1 = A$. These matrices satisfy

$$I + A_1 + \cdots + A_d = J \tag{15}$$

where J is the matrix of all 1's. Further, there exist ith degree matrix polynomials v_i such that

$$A_i = v_i(A). \tag{16}$$

Moreover, the matrices $\{I, A_1, \ldots, A_d\}$ are linearly independent.

B. Single-leader cases

Consider system (3) with a distance regular graph $\mathbb{G}$ and a single leader. When $\mathcal{V}_L = \{v\}$, we denote its controllable subspace by $\mathcal{K}(v)$. Then $\mathcal{K}(v)$ can be fully characterized by the following result.

Proposition 4: For any $v \in \mathcal{V}$,

$$\mathcal{K}(v) = \mathrm{im}\, P(\pi_{\mathrm{D}}(v)) = \mathrm{im}\, P(\pi^*_{\mathrm{AEP}}(v)).$$

Proof: It follows from the definition of distance regularity that the distance partition $\pi_{\mathrm{D}}(v)$ is an almost equitable partition relative to v. Hence, we have $\pi_{\mathrm{D}}(v) \leq \pi^*_{\mathrm{AEP}}(v)$. Further, (12) implies that $\pi_{\mathrm{D}}(v) = \pi^*_{\mathrm{AEP}}(v)$. Then the claim follows from Propositions 1 and 2. ☐

It turns out that the dimension of $\mathcal{K}(v)$ does not depend on the choice of the leader due to the distance regularity.

Proposition 5: For any $v \in \mathcal{V}$,

$$\dim(\mathcal{K}(v)) = d + 1.$$

Proof: It follows from Lemma 6 that $p^0_{dd} > 0$. This implies that for each vertex v, there exists at least one other vertex u such that $\mathrm{dist}(v, u) = d$. Since d is the diameter of the graph, we conclude that $\mathrm{card}(\pi_{\mathrm{D}}(v)) = d+1$. Then the claim follows from Proposition 4. ☐

Proposition 5 implies that system (3) with a distance regular graph cannot be controllable with a single leader unless $n = d + 1$. This condition is satisfied if and only if the graph consists of two vertices and one edge.

In the next subsection, we will turn our attention to the multi-leader case.

C. Multi-leader cases

Consider system (3) with a distance regular graph $\mathbb{G}$ and multiple leaders. Let

$$N = \begin{bmatrix} A_d M & A_{d-1} M & \cdots A_1 M & A_0 M \end{bmatrix}. \tag{17}$$

where A_ℓ's ($0 \leq \ell \leq d$) are defined in (15). Then we have the following result.

Proposition 6: It holds that

$$\mathrm{im}\, N = \mathcal{K}. \tag{18}$$

Proof: For a subspace $\mathcal{W} \subseteq \mathbb{R}^n$, we denote its orthogonal complement by $\mathcal{W}^\perp$. Let $z \in \mathbb{R}^n$.

$$\begin{aligned} z^T \in \mathcal{K}^\perp &\iff z^T L^k M = 0 \text{ for all } k = 0, 1, \ldots, n-1 \\ &\iff z^T A^k M = 0 \text{ for all } k = 0, 1, \ldots, n-1 \\ &\overset{(15)}{\iff} z^T A_\ell M = 0 \text{ for all } \ell = 0, 1, \ldots, d \\ &\overset{(16)}{\iff} z \in \ker N^T. \end{aligned}$$

The second relation follows from the regularity of the graph, i.e. $L = aI - A$ where $a = \deg(i)$ for an $i \in \mathcal{V}$. ☐

Theorem 1: The system (3) with a distance regular graph $\mathbb{G}$ is controllable only if the number of inputs m satisfies

$$dm \geq n - 1 \tag{19}$$

where $d = \mathrm{diam}(\mathbb{G})$.

Proof: Since the system is controllable, we have $\dim(\mathcal{K}) = n$. Then, we get

$$\begin{aligned} n &= \dim(\mathcal{K}) \\ &\overset{(18)}{=} \mathrm{rank}\, N \\ &\overset{(15)}{=} \mathrm{rank} \begin{bmatrix} JM & A_{d-1}M & \cdots & A_1 M & M \end{bmatrix} \\ &= \mathrm{rank} \begin{bmatrix} \mathbf{1} & A_{d-1}M & \cdots & A_1 M & M \end{bmatrix} \end{aligned}$$

where $\mathbf{1}$ is the column vector of 1's with the dimension of n. It follows from linear independence of $\{I, A_1, \ldots, A_d\}$ and (15) that

$$\mathrm{rank} \begin{bmatrix} A_{d-1}M & \cdots & A_1 M & M \end{bmatrix} \geq n - 1.$$

Note that the matrix on the left has n rows and dm columns. Hence, we get $dm \geq n - 1$. ☐

D. Leader selection

We begin with a procedure of choosing leaders and later we will show that this procedure guarantees controllability. Consider $w \in \mathcal{V}$. Denote the distance partition relative to w by $\pi_{\mathrm{D}}(w) = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_d\}$. For each $1 \leq \ell \leq d$, choose $w_\ell \in \mathcal{C}_\ell$. Define $\mathcal{V}^*_{\mathrm{F}}$ to be the set of all such w_ℓ. Let $\mathcal{V}' = \mathcal{V} \setminus \mathcal{V}^*_{\mathrm{F}}$. Then $w \in \mathcal{V}'$ and $\mathrm{card}(\mathcal{V}') = n - d$.

The main result of the paper is stated as follows.

Theorem 2: When $\mathbb{G}$ is distance regular, system (3) is controllable if $\mathcal{V}_{\mathrm{L}} = \mathcal{V}'$.

Proof: Let $z \in \mathcal{K}^\perp$. Then we have $z \in \ker N^T$ due to Proposition 6. This means that

$$z^T A_\ell M = 0$$

for all $\ell = 0, 1, \ldots, d$. In particular, we have

$$z^T M = 0.$$

If $v \in \mathcal{V}_{\mathrm{L}}$ then the vth standard basis vector must be a column of M. This leads to

$$z_v = 0 \tag{20}$$

where z_v is the vth element of z and $v \in \mathcal{V}_{\mathrm{L}}$. We claim that the other components of z are zero too. To see this, we consider the relation

$$z^T A_\ell M = 0$$

with $\ell \in \{1, 2, \ldots, d\}$. Let q be the wth column of $A_\ell M$. Note that the jth element of q, $1 \leq j \leq n$, is

$$q_j = \begin{cases} 1 & \text{if } j \in \mathcal{C}_\ell \\ 0 & \text{otherwise.} \end{cases}$$

Since all elements of $\mathcal{C}_\ell$ except w_ℓ belong to $\mathcal{V}_{\mathrm{L}}$, it follows from (20) that

$$0 = z^T q = z_{w_\ell}.$$

This means that $z = 0$ and hence $\mathcal{K} = \mathbb{R}^n$. $\square$

We have the following result as a direct consequence of Theorem 2.

Corollary 1: Every distance regular graph can be rendered controllable with $n - d$ leaders.

In the next section, we will discuss how to choose leaders for two special classes of distance regular graphs: cycles and complete graphs.

VI. Choosing leaders for systems with cycles or complete graphs

Cycle and complete graphs are two classes of distance regular graphs. A graph $\mathbb{G}$ is a *cycle* if $\deg(i) = 2$ for all $i \in \mathcal{V}$ and is *complete* if $\deg(i) = n - 1$ for all i. In the following, $\mathbb{C}_n$ and $\mathbb{K}_n$ are used to denote a cycle and a complete graph with n vertices respectively. Note that $n \geq 3$ for any $\mathbb{C}_n$.

Consider system (3) with $\mathbb{G}$ being $\mathbb{C}_n$. Let $d = \mathrm{diam}(\mathbb{C}_n)$. From Proposition 5 and the fact that $n = 2d$ or $n = 2d + 1$, we know that such a system can never be controllable by a single leader. Moreover, from Theorem 2, we know that it is controllable by d leaders when n is even and $d + 1$ leaders when n is odd. In addition, we have the following result.

Theorem 3: System (3) with $\mathbb{C}_n$ and two leaders is controllable if the two vertices corresponding to the leaders are adjacent.

Proof: The two adjacent leaders are denoted by v_1 and v_2 and their controllable subspaces $\mathcal{K}(v_1)$ and $\mathcal{K}(v_2)$ respectively. We use $\mathcal{K}(v_1, v_2)$ to denote the joint controllable subspace of v_1 and v_2.

From proposition 5, $\dim \mathcal{K}(v_1) = \dim \mathcal{K}(v_2) = d + 1$. Note that when $\mathrm{dist}(v_1, v_2) = 1$ in $\mathbb{C}_n$, we have

$$\dim(\mathcal{K}(v_1) \cap \mathcal{K}(v_2)) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

The conclusion follows from the fact that $\dim \mathcal{K}(v_1, v_2) = \dim \mathcal{K}(v_1) + \dim \mathcal{K}(v_2) - \dim(\mathcal{K}(v_1) \cap \mathcal{K}(v_2))$. $\square$

As Theorem 3 suggests, the controllability of the system with two leaders associated with $\mathbb{C}_n$ depends on the distance between the two leaders. We illustrate this point by an example of $\mathbb{C}_6$. We label the vertices of the graph clockwise by 1 to 6 and choose vertices 1 and 4 to be the two leaders. Then $\dim(\mathcal{K}(u_1) + \mathcal{K}(u_2)) = 4 < 6$, which implies that the system is uncontrollable when the distance between the two leaders is 3.

When $\mathbb{G}$ is $\mathbb{K}_n$, we have the following result.

Theorem 4: System (3) with $\mathbb{K}_n$ is controllable if and only if at least $n - 1$ agents are leaders.

Proof: Since $\mathrm{diam}(\mathbb{K}_n) = 1$, the necessity and the sufficiency directly follow from Theorems 1 and 2 respectively. $\square$

VII. Concluding remarks

In this paper, graph partitions have been employed to characterize controllable subspaces of multi-agent systems associated with general graphs and distance regular graphs. In particular, if the graphs are distance regular, this characterization is complete when there is a single leader, and a necessary condition and a sufficient condition have been provided when multiple leaders are present. For systems associated with cycles and complete graphs, we have shown how to choose leaders to guarantee their controllability. For the future work, we are interested in systematic ways of choosing leaders for other classes of distance regular graphs. It is also of great interest to deal with the challenging case when $\mathbb{G}$ is time-varying.

References

[1] V. Kumar, N. E. Leonard, and A. S. Morse. *Cooperative Control*. Springer-Verlag New York, Inc., 2005.

[2] F. Bullo, J. Cortes, and S. Martinez. *Distributed Control of Robotic Networks*. Princeton University Press, Princeton, 2009.

[3] S. H. Strogatz. *SYNC: The emerging science of spontaneous order*. Hyperion, New York, 2003.

[4] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor ruls. *IEEE Transactions on Automatic Control*, 48:985–1001, 2003.

[5] R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in multi-agent networked systems. *Proceedings of the IEEE*, 45:215–233, 2007.

[6] W. Lu, B. Liu, and T. Chen. Cluster synchronization in networks of coupled nonidentical dynamical systems. *Chaos*, 20:013120, 2010.

[7] W. Xia and M. Cao. Cluster synchronization algorithms. In *Proc. of the 2010 American Control Conference (ACC)*, pages 6513–6518, 2010.

[8] W. Xia and M. Cao. Clustering in diffusively coupled networks. *Automatica*, 47(11), 2011.

[9] F. Sorrentino, M. di Bernardo, F. Garofalo, and G. Chen. Controllability of complex networks via pinning. *Physical Review E*, 75:046103, 2007.

[10] W. Wu, W. Zhou, and T. Chen. Cluster synchronization of linearly coupled complex networks under pinning control. *IEEE Transactions on Circuits and Systems I*, 56:829–839, 2009.

[11] M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[12] M. Cao, A. S. Morse, and B. D. O. Anderson. Reaching a consensus in a dynamically changing environment: A graphical approach. *SIAM Journal on Control and Optimization*, 47:575–600, 2008.

[13] M. Mesbahi. On state-dependent dynamic graphs and their controllability properties. *IEEE Transactions on Automatic Control*, 50:387–392, 2005.

[14] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt. Controllability of multi-agent systems from a graph-theoretic perspective. *SIAM Journal on Control and Optimization*, 48:162–186, 2009.

[15] S. Martini, M. Egerstedt, and A. Bicchi. Controllability analysis of multi-agent systems using relaxed equitable partitions. *International Journal of Systems, Control and Communications*, 2:100–121, 2010.

[16] D. M. Cardoso, C. Delorme, and P. Rama. Laplacian eigenvectors and eigenvalues and almost equitable partitions. *European Journal of Combinatorics*, 28:665–673, 2007.

[17] G. Parlangeli and G. Notarstefano. On the observability of path and cycle graphs. In *Proceedings of the 49th IEEE Conference on Decision and Control*, pages 1492–1497, 2010.

[18] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra*. Springer-Verlag, New York, 1981.

[19] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer-Verlag, New York, 2001.

[20] M. K. Camlibel, S. Zhang, and M. Cao. Comments on 'Controllability analysis of multi-agent systems using relaxed equitable partitions'. *International Journal of Systems, Control and Communication*, To appear.

[21] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-Regular Graphs*, pages 126–134. Springer-Verlag, Berlin, Heidelberg, 1989.