5 Is Recovery Definite after Physical Shocks? A Theoretical Elaboration

5.1 Introduction

In the previous chapter, we demonstrated that an economic system would converge to its steady state value in time, whatever the magnitude of a physical shock it will be exposed to, thus supporting the observation made by John Stuart Mill a long time ago cited in the introduction. Recall nonetheless that we also raised two critical questions in the introduction that asked (i) Are all shocks recoverable? (ii) Is there any indispensable role for the government during the period of recovery from a shock? This chapter focuses on that role of a social planner (i.e., the government) in restoring equilibrium after a physical shock (in our case, an earthquake). Our aim is not to give a complete answer to the above questions but to make a first effort to deal with them.
In this paper, we investigate the fundamental role that governments may play during restoring the long run equilibrium. The aim of this study is not the question of whether government involvement may accelerate restoring equilibrium but of whether governments do play an indispensable role during recovery. In that respect, an \( AK \) setup comes in very handy because (i) it generates endogenous growth (\( i.e., \) long run equilibrium) from the start, and hence, we refrain from transitional roles governments may undertake during recovery from a physical shock and focus directly on its fundamental role for restoring equilibrium; (ii) it lacks diminishing marginal productivity property. The latter property is critical because a shock lowering the stock of an input (say, physical capital) implies increasing productivity of the input, which brings off convergence to steady state equilibrium in the long run (a good example to this statement is the previous chapter, in which a shock on the housing stock is recovered due to the neoclassical properties of the model).

\( AK \) model has one more property that is worth to mention here. To our best knowledge, \( AK \) models and patterns that are reduced to \( AK \) models without generating transitional dynamics enforce a constant ratio among two or more quantities within the model \textit{from the start} that we call \textit{constancy conditions}. Hence, if the path of one variable is known, then, necessarily, the time-paths of the rest are also known in these setups, given parameter values. What makes these models interesting from the viewpoint of shocks is that the constant optimality ratios are \textit{not tolerant} to disturbances. In other words, the conditions need to be restored as quickly as possible (preferably immediately) if an unexpected shock (like an earthquake) causes a deviation from these conditions, because otherwise intertemporal maximization of the objective function cannot be
accomplished. Such an environment is a very special case, and the system is in need of a social planner’s intervention if the market dynamics do not contain a self-sufficient mechanism that ensures convergence to the normal path.

One may say that constancy conditions in growth models have been rarely paid attention to for two reasons. First, constancy conditions do arise only in a limited number of growth-modeling approaches, namely only in so-called AK models and in models that are reduced to AK form without generating transitional dynamics. Second, the issue of shocks itself has been rarely studied in deterministic growth modeling approaches (perhaps the job is left to (real) business cycle literature, which deals explicitly with shocks in dynamic general equilibrium frameworks). Hence, neither constancy conditions nor the question of how to restore constancy conditions after shocks have been studied sufficiently explicitly in deterministic growth models.

The aim of this chapter is to show how constancy conditions can be restored after a physical shock (we will explain below why we focus on physical shocks). We employ an augmented (two-sector) AK model to derive constancy conditions among quantities of the model. We assume that the model-economy accumulates two stock variables, namely physical capital and housing, by using unconsumed output. Hence, there is a trade-off, in terms of the use of the resources, between housing-accumulation and capital-accumulation. Our analysis shows that a constant ratio between capital and housing must be maintained from the start if welfare is to be maximized. Next, we assume that an earthquake hits the housing sector while leaving the capital sector unaffected. Since housing is an argument of
utility, the social planner cannot give up demanding for new housing, \emph{i.e.}, the constancy conditions have to be restored. Then, the problem of the social planner is how to do that. We will show that adjustments to constancy conditions after a shock require the limitation of growth of undisturbed quantities of the model. The simple mechanism we propose restores the constancy ratios under a temporary optimization problem.

The approach in this chapter can be seen as an extension (and revision) of Chapter 5 of Barro and Sala-i-Martin (1995). There, Barro and Sala-i-Martin (henceforth BSM) discuss, in an extended \textit{AK} model, how to restore the constancy condition between physical capital and human capital after a physical shock on capital (\emph{e.g.}, war) or on human capital (\emph{e.g.}, communicable disease). BSM argue that a temporary optimization policy that restricts the growth of the abundant quantity while letting the scarce variable grow after a shock is sufficient to restore constancy condition(s). We will follow the same idea in our work, but offer a more refined solution procedure. In addition, we will discuss the role of the social planner with regard to adjustment dynamics after a physical shock.

Since the paper explores adjustment dynamics under the story of earthquakes, it may be useful to first look at the earthquake literature. Economic implications of earthquakes have been analyzed in few theoretical papers, such as Oulton (1993), Albala-Bertrand (1993b), Selcuk and Yeldan (2001), and the former chapter of this dissertation.\footnote{Albala-Bertrand (1993b) is in the Keynesian spirit, but the main message of his study underlines boldly that physical disturbances will be recovered in time by the system. His study proposes some government intervention in order to accelerate the recovery, but does not show any theoretical necessity for that.} The common characteristic of these studies is the (global) convergence property of

\footnote{It is becoming traditional to attribute this framework to Rebelo (1991) in the new growth literature.}
quantities after an earthquake. None of these models studies explicitly why government intervention may be a necessity in order to restore equilibrium.

The organization of the chapter is as follows. The second section is all about the elaboration of constancy conditions. We first labor the most basic \( AK \) model that generates constancy conditions as an introduction into the concept. Next, we discuss the solution procedure offered by the BSM (1995) model for restoring constancy conditions after an upset; our discussion on why the technical details of the solution procedure suggested by BSM need revision can be found in Annex E. The third section presents our model, which contributes to the literature in two ways. First, it demonstrates a refined solution procedure to return to constancy conditions. Second, it shows the possibility that the social planner must play a role in restoring constancy conditions. The last section is reserved for concluding remarks.

### 5.2 The Basics

Constancy conditions arise in \( AK \) type models or in models that ultimately reduce to \( AK \)-form without generating transitional dynamics. A natural starting point in familiarizing with constancy conditions is the basic \( AK \) model. The model goes as follows. Define the overall utility as:

\[
U(C) = \int_0^\infty e^{-\rho t} \frac{C^{1-\theta} - 1}{1-\theta} \, dt
\]  

(5.1)

where \( C \) is aggregate consumption, \( \rho \) is the subjective rate of discount factor, and \( \theta \) is the (absolute) value of elasticity of marginal utility. We
assume that $\rho > 0$ and $\theta > 0$, and that population is normalized to one and does not grow.

The production function is defined as

\[ Y = AK \]  \hspace{1cm} (5.2)

where $Y$ is aggregate output, $A$ is the exogenous technology parameter, and $K$ is the aggregate physical capital stock. The model is closed by the macroeconomic budget constraint:

\[ \dot{K} = AK - C - \delta K \]  \hspace{1cm} (5.3)

where $\dot{K}$ is the instantaneous rate of change in the capital stock and $\delta$ the rate of depreciation of capital (we assume that the economy is closed and there is no government in the model). The solution of this problem that we will not further elaborate here is part of textbooks (e.g., BSM (1995)). The system generates steady state growth without transitional dynamics:

\[ g = \dot{C} = \dot{K} = \frac{A - \rho - \delta}{\theta}. \]  \hspace{1cm} (5.4)

A steady state growth without transitional dynamics implies that variables of the system, namely consumption $C$ and physical capital $K$, hold a constant ratio between them from the start. In particular, it is straightforward to show that:
\[
\frac{C(t)}{K(t)} = g_A - g
\]  

(5.5)

where \( g_A = A - \delta \). So, there is also a constant ratio between capital and consumption at their initial values \( K(0) \) and \( C(0) \). In other words, consumption is not a free choice but some function of the initial capital stock, given parameter values. This is called the “closed-form policy function” (see BSM, 1995, p.143, footnote 3). Furthermore, the condition is ‘binding’ not only once-and-for-all but permanently, implying that the constant ratio between consumption and capital must be satisfied at all times. As an aside, it can be mentioned that constancy conditions are comparable to the saddle-path stability property of the Cass-Koopmans framework in that the value of the initial control variable is dependent on the initial value of state variable in the latter framework (the difference being that the latter property does not impose a constant ratio between quantities permanently). Finally, it is worth to note that changes (shocks) on the right hand side of the constancy condition do not violate the optimization rule but just change it in accordance with the shock. A violation arises if any of the quantities on the left-hand side would be upset. This is why we focus on physical shocks, because, given an AK framework, only physical shocks can generate imbalances between the right hand and left hand side of constancy conditions. Now, let us return to what BSM suggested (in chapter 5 of their 1995 book) with regard to restoring constancy conditions after a disturbance.\(^3\)

\(^3\) We defer to Annex E for discussing the details of the BSM model and its solution procedure.
BSM (1995, pp. 172-9) use a two-sector growth model, which reduces into an $AK$ model; they (1995) assume a Cobb-Douglas production function that exhibits constant returns to physical and human capital, $K$ and $H$. Output can be used for consumption or (gross) investment in physical or human capital, which constitutes the economy’s resource constraint. It is assumed that gross investment net of depreciation $\delta$ is net investment for any stock (for simplicity, BSM assume that the stocks of physical and human capital depreciate at the same rate). The familiar first order conditions of the maximization problem yield that

$$K/H = \alpha/(1-\alpha).$$

(5.6)

In equation (5.6), there is a constant ratio between physical and human capital, where $\alpha$ is the production elasticity of capital. Next, BSM (1995) analyze what happens if the $K/H$ ratio deviates from the value $\alpha/(1-\alpha)$, e.g., due to a shock on one of the quantities. BSM (1995) state that the constancy condition dictates adjustments, preferably instantaneously, in the two stocks in order to attain the value $\alpha/(1-\alpha)$. Because, as they argue, instantaneous adjustment (“reversible investment”) is not viable as “it depends on the possibility of an infinite positive rate of investment in one form of capital and an infinite negative rate of investment in the other form” (BSM, 1995, p.175), they propose the more realistic assumption to limit the growth of the abundant stock variable while allowing the scarce stock variable to grow. To illustrate, suppose that a war destroys part of the capital stock and upsets the constancy condition between physical and human capital. Then, human capital is abundant compared to physical capital. BSM now propose to limit the growth of the abundant stock, which
effectively means keeping the investment in human capital at zero: \( H(t) = H(0)e^{-\delta t} \). Their interpretation is that the social planner realizes that the economy has too much \( H \) in relation to \( K \), but since to have negative gross investment in \( H \) is infeasible, they allow \( H \) to depreciate at the exogenously given rate \( \delta \). BSM argue that the \( K/H \) ratio will then rise and reach the value \( \alpha/(1-\alpha) \) in finite time; thereafter the system returns to the original position where all quantities grow at the rate \( g \).

We argue in Annex E that the BSM (1995) solution procedure has two important caveats that would need to be resolved. In the next section, we therefore offer a revised version of the BSM solution procedure, and demonstrate how to restore constancy conditions.

### 5.3 The Model

Below, we develop a two-sector \( AK \) model, where one sector produces physical capital and output and the other generates housing stock by using capital as input.\(^4\) The overall utility is defined as

\[
U(C,H) = e^{-\rho (\frac{(C \cdot H^\gamma)^{1-\theta}}{1-\theta} - 1)}
\]  

(5.7)

where \( C \) is aggregate consumption, \( H \) is the stock of accommodation units (houses), \( \rho \) is the subjective rate of discount factor, \( \gamma \) is the housing characterization parameter, and \( \theta \) is the (absolute) value of elasticity of

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\(^4\) See Smith, Rosen, and Fallis (1988, p.33), and Nielsen and Sørensen (1994) on how to introduce a housing sector into a dynamic general equilibrium model.
marginal utility. We assume that \( \rho > 0 \), \( 0 < \gamma < 1 \), \( \theta > 0 \), and that the population size is normalized to one and does not grow. We also assume \( \gamma(1-\theta) < 1 \) to get diminishing marginal utility with respect to housing. The overall utility function has the following properties. First, the elasticity of substitution between consumption and the housing stock is one. Second, elasticities of marginal utility with respect to consumption and housing are constant (\( \theta \) and \( 1-\gamma(1-\theta) \), respectively). Thus, \( \theta > 1-\gamma(1-\theta) \) \( (\theta < 1-\gamma(1-\theta)) \) is guaranteed under \( \theta > 1 \) (\( \theta < 1 \)), implying that, *ceteris paribus*, the intertemporal elasticity of substitution of consumption (housing) is greater than the intertemporal elasticity of substitution of housing (consumption). Hence, the more rapid is the proportionate decline in marginal utility of consumption (housing) in response to increases in \( C(\gamma) \), and hence the households are less willing to accept deviations from a uniform pattern of \( C(\gamma) \) over time. It is clear that \( \gamma \) identifies housing in the analysis.

The production function is defined as:

\[
Y = AK
\]

(5.8)

where \( Y \) is aggregate output, \( A \) is the exogenous technology parameter, and \( K \) is the aggregate physical capital stock.

In this model economy, part of the resources is used for producing new houses. We conjecture that the net housing investment is captured by:

\[
\dot{H} = I_H - \delta_H H
\]

(5.9)
In Equation (5.9), $\dot{H}$ is the instantaneous change in the housing stock, $I_{H}$ is the gross investment for producing housing goods, and $\delta_{H}$ is the depreciation rate of houses. Note that for a matter of focus we completely ignore housing quality.

The model is closed by the macroeconomic budget equation. The constraint is:

$$\dot{K} = AK - C - \delta_{K} K - I_{H}$$

(5.10)

where $\dot{K}$ is the instantaneous rate of change in the capital stock, and $\delta_{K}$ the rate of depreciation of capital. Thus, we described the full properties of the model.

The full solution of the model is represented in Annex F. In particular, the steady state growth rate found is:

$$g = \dot{C} = \dot{H} = \dot{K} = \frac{A - \delta_{K} - \rho}{\theta - \gamma(1 - \theta)}.$$  

(5.11)

As Annex F explains, the two-sector AK model imposes constancy conditions on $C(t)/H(t)$ and $K(t)/H(t)$ for all $t$. In particular, we find these conditions as

$$\frac{C(t)}{H(t)} = \frac{A - \delta_{K} + \delta_{H}}{\gamma}$$  

(5.12)

$$\frac{K(t)}{H(t)} = \frac{K(0)}{H(0)}$$  

(5.13)
The conditions express that all variables are interdependent, and that we can trace the time paths of all variables from an initial condition of one variable, say housing.

What is the importance of the ‘restrictions’ imposed by the constancy conditions from the viewpoint of physical shocks in our model? Suppose that a shock hits the housing stock at time $T_e$ and destroys it to a significant degree. Then, constancy conditions are upset, and we need to restore them in the model. The setup requires (discrete or gradual) adjustments in such a way that all these conditions must be restored within a temporary setting. Discrete adjustment, which would include infinite resource transfers between variables, is not appealing practically. Therefore, we need a scheme for gradual adjustments to constancy conditions. The next subsection discusses such a scheme.

### 5.3.1 Restoring Constancy conditions

We assume that an earthquake destroys a significant amount of the housing stock. In particular, we assume that due to an earthquake at time $T_e$ the housing stock has declined by some, say $\chi$, percentage:

$$H(T_e^+) = (1 - \chi)H(T_e^-)$$

(5.14)
where $T_e^-$ denotes time $T_e$ just before an earthquake, and $T_e^+$ the time immediately after an earthquake. How to restore constancy conditions?

**Simple response Policy and others**

We conjecture that there are *infinite* ways of restoring constancy conditions. For example, first the constancy condition between housing and consumption and next the constancy condition between housing and physical capital can be restored. Evidently, any combination of the abovementioned program such as restoring halfway the constancy condition between housing and physical capital following a halfway in restoring the constancy condition between housing and consumption after completing the first halfway in restoring the constancy condition between housing and physical capital and so on can also be setup. Recall however that the motivation of the social planner in this temporary optimization program is to restore the original programme as soon as possible as off-equilibrium path implies welfare losses. In that respect, we *speculate* that the *immediate* restriction of growth of *all undisturbed* variables in the model seems to be the best policy in order to restore constancy conditions, given that discrete adjustment is not possible. We do not have any theoretical proof to this argument but intuition dictates that a program with multiple stages should prolong the duration for restoring the conditions. This policy can be called as ‘simple response policy’ in the sense that the policy-maker follows a very simple scheme in order to restore optimality conditions (note that this is also the policy suggested by BSM(1995)). Figure 5.1 below illustrates the simple response policy:

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5 We will use $T_e$ and $T_e^-$ interchangeable whenever it is clear that time $t$ refers to just-before-earthquake.
In figure 5.1, at time $T_e$, an earthquake hits the economy (the housing sector). Thus the optimality between $K$ and $H$, and $C$ and $H$ are disturbed. Since there is more than one condition, it is not possible to restore them without constraining the growth of all undisturbed variables. As figure 5.1 shows, in order to do that one should restrict the growth of $K$ and $C$ immediately after the earthquake, and release them to grow at the point that $H$ reaches the pre-earthquake level. This is the requirement that an algebraic formulation of the problem should solve. Let us return to our example for the technical representation of this scheme.
Suppose that the social planner agrees to restrict undisturbed variables immediately after the disturbance. Then the planner will have required setting up a temporary optimization problem, where the only unknowns are housing and ‘restoration’ time. We are now diverging from BSM (1995) in terms of the way in which we define the temporary problem (the solution procedure), because we argue that the constancy condition is not only between $H$ and $K$, but also between $H$ and $C$. In that respect, the only way to restore constancy conditions is to restrict the growth of all undisturbed variables. The temporary problem starts at $T^*$ and ends at time $T$, in which the social planner maximizes:

$$\text{Max} \quad \int_{T^*}^{T} e^{-\gamma t} \left( \frac{\overline{C} \ast H}{1 - \theta} \right) dt$$

subject to:

$$\dot{H} = I_H - \delta_H H$$

$$0 = (A - \delta_H) \overline{K} - \overline{C} - I_H$$

where the initial value is $H(T^*) = (1 - \chi)H(0)e^{\gamma T^*}$, the terminal value is $H(T) = H(0)e^{\gamma T}$, the terminal time $T$ is unknown, and $\overline{C} = C(0)e^{\gamma T}$ and $\overline{K} = K(0)e^{\gamma T}$. The problem specified in equation (5.15) is a calculus of variations problem. Furthermore, there is no need to impose a state-space constraint on the problem because, given that there is a single unknown in the model, the terminal time of the temporary problem is effectively the constraint on the variable. Hence, by construction, we do not allow the disturbed variable to exceed the value whereby it satisfies the optimality ratios with other variables.
We can rewrite the maximization problem after eliminating \( H \) as follows:

\[
\begin{align*}
\text{Max} & \quad \int_{T_0}^{T} F(t, \dot{H}) dt \\
& = \int_{T_0}^{T} \frac{(\overline{Z} - \dot{H})^{\gamma(1-\theta)}}{\delta_H} - 1 \\
\end{align*}
\]

(5.16)

where \( F = e^{-\gamma s} \) and \( \overline{Z} = (A - \delta_K)K - C \). The special form of \( F() \) implies that the Euler equation is

\[
F_{\cdot t} \dot{H} + F_{\cdot tt} = 0 .
\]

(5.17)

Together with the fixed endpoint transversality condition \([F - F_{\cdot t} \dot{H}]_{t=T}\) and initial and terminal values, the Euler equation identifies the path of \( H \). \(^6\)

Details of the solution are as follows. \(^7\) First, applying the Euler equation formulation, we end up with a second order differential equation:

\[
\ddot{H} + \frac{\rho}{1 - \gamma(1-\theta)} \dot{H} = \frac{\rho \overline{Z}}{1 - \gamma(1-\theta)}
\]

(5.18)

Solving (5.18) yields that

\[
H(s) = \overline{Z} \cdot (s - T_c^+) + c_1 e^{\frac{-\rho}{\gamma(1-\theta)}(s - T_c^+)} + c_2
\]

(5.19)

\(^6\) See equation (2.19) for the Euler equation, and (3.11) for the transversality condition in Chiang (1992).
where \( s \in [T_e^+, T] \), and \( c_1 \) and \( c_2 \) are constants. Equation (5.19) indicates that the housing stock increases as new investments are made. The three unknowns of (5.19) are \( c_1 \), \( c_2 \), and \( T \). We also have three equations:

\[
H(T_e^+) = c_1 + c_2 \equiv (1 - \chi)H(0)e^{gT_e} 
\]  
(5.20)

\[
H(T) = Z \cdot (T - T_e^+) + c_1 e^{-\rho(1-\theta)(T - T_e^+)} + c_2 \equiv H(0)e^{gT_e} 
\]  
(5.21)

\[
\left( e^{-\alpha T} \frac{C}{1-\theta} \right)^{\gamma(1-\theta)-1} \left[ \frac{Z - \dot{H}}{\delta_H} \right]_{t=T}^* \left[ \frac{Z - \dot{H}}{\delta_H} + (\gamma(1-\theta))\dot{H} - \frac{C}{1-\theta} \left( \frac{Z - \dot{H}}{\delta_H} \right)^{1-\gamma(1-\theta)} \right]_{t=T} = 0 
\]  
(5.22)

Equation (5.20) and (5.21) are derived from the initial and terminal time conditions, respectively, and equation (5.22) is obtained from the transversality condition. Note from equation (5.21) that:

\[
\dot{H} \bigg|_{t=T} = Z + \frac{c_1(-\rho)}{1-\gamma(1-\theta)} e^{-\rho(1-\theta)(T - T_e^+)} . 
\]  
(5.23)

\footnote{An alternative solution procedure is possible. Note that equation (5.17) implies that \( d/dt(F_H) = 0 \). Hence, \( F_H = c_1 \), where \( c_1 \) is a constant. Starting from this observation, we can easily determine the housing path. Refer to Chiang (1992) for details.}
Then, given that the first component in square brackets of equation (5.22) cannot be zero, the term in the second square brackets of that equation must be zero. Hence, we derive from (5.22) that

\[
\frac{c_1 \rho}{\delta_H} e^{\frac{-\rho}{\gamma(t)}(T-T')} + \frac{\gamma(1-\theta)Z}{\delta_H} \left( \frac{c_1 \rho}{1-\gamma(1-\theta)} \right)^{1-\gamma(1-\theta)} e^{-\rho(T-T')}.
\]

(5.24)

Unfortunately, it is not possible to find explicit values of \(c_1\), \(c_2\), and \(T\) from (5.20), (5.21) and (5.24). We run a small experiment for a set of hypothetical parameter values just to check what these equations imply.\(^8\) Our numerical experiment shows that it takes 2.334 ‘years’ to recover a 50 percent reduction in housing stock after an earthquake when \(\theta = 0.8\) and 12.974 years when \(\theta = 1.8\), where the earthquake hits the model economy at ‘year’ 80. Hence, we deduce from our solution procedure that the housing stock will increase in time while other quantities in the model are kept constant. Naturally, the temporary problem imposed by the social planner will be lifted at the time that the constancy conditions are restored.

Finally, we would like to discuss briefly induced shocks. It is a theoretical possibility that the social planner (government) may use policy shocks to accelerate the pace of restoring constancy conditions. Examples of such policy shocks are consumption, income, capital, and lump-sum taxation, each of which in our context will aim at transferring additional resources to housing accumulation. Since these shocks would not change the

\(^8\) The hypothetical parameter values are as follows: \(\theta = 0.8\) or \(\theta = 1.8\), \(A = 0.07\), \(\gamma = 0.8\), \(\rho = 0.02\), \(H(0) = 0.2\), \(\chi = 0.5\), \(Te = 80\), \(\delta_K = 0.04\), and \(\delta_H = 0.05\).
essential characteristics of our adjustment analysis, we skipped them in this paper, however.

5.4 Conclusion

This chapter analyzed how to restore constant optimality ratios between variables after a shock hits an AK economy. We first showed that AK models (and frames that reduce to an AK model without generating transitional dynamics) have the very special property that constant ratios are imposed among quantities from the start. This property implies that, for a given initial value of one of the stock variables, the time paths of other variables are known to the system. This characteristic becomes crucial when (physical) shocks hit any of these quantities, because in these models it is then not possible to restore constancy conditions without the active involvement of a social planner.

Second, we discussed the solution procedure offered by BSM (1995) on how to restore constancy conditions, and conjectured that although the intuition of the procedure seemed right, it nevertheless includes serious flaws (shown in Appendix E).

Third, we offered a simple alternative solution procedure. The main premise of that procedure was that all undisturbed quantities have to be kept constant until the shocked variable returns to its pre-shock value.