Contraction Analysis of Monotone Systems via Separable Functions

Kawano, Yu; Besselink, Bart; Cao, Ming

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Yu Kawano, *Member,* Bart Besselink, *Member,* and Ming Cao, *Senior Member*

Abstract—In this paper, we study incremental stability of monotone nonlinear systems through contraction analysis. We provide sufficient conditions for incremental asymptotic stability in terms of the Lie derivatives of differential one-forms or Lie brackets of vector fields. These conditions can be viewed as sum- or max-separable conditions, respectively. For incremental exponential stability, we show that the existence of such separable functions is both necessary and sufficient under standard assumptions for the converse Lyapunov theorem of exponential stability. As a by-product, we also provide necessary and sufficient conditions for exponential stability of positive linear time-varying systems. The results are illustrated through examples.

Index Terms—Nonlinear systems; Contraction analysis; Monotone systems; Separable functions

I. INTRODUCTION

A dynamical system is called monotone if its trajectory preserves a partial order of the initial states [1], [2]. For instance, biological systems [3], chemical reaction systems [4], transportation networks [5], and social dynamics [6] are often modeled as monotone systems. Moreover, monotonicity properties naturally arise in the analysis of interconnected large-scale networks [7], [8]. In the linear time-invariant (LTI) case, positive systems provide a typical example of monotone systems, and there are rich theoretical developments for stability and gain analysis of positive LTI systems; see, e.g., [9], [10] for basic results. These works are mostly motivated by the fundamental observation that an asymptotically stable positive LTI system always admits linear (also called *separable*) Lyapunov functions [9], [10], which is essentially related to the Perron-Frobenius theorem on the dominant eigenvalue of a positive matrix [11] and its variant on Metzler matrices [2]. Since almost all practical applications of monotone systems mentioned above such as biological systems contain nonlinearity, stability analysis of monotone nonlinear systems using separable functions has also been studied [12]–[16].

In this paper, we are also interested in stability analysis but not in the standard Lyapunov framework. We pursue analysis in the *contraction* framework [17]–[22]. Contraction theory is a differential geometric framework for the analysis of a pair

Y. Kawano is with the Faculty of Engineering, Hiroshima University, Higashi-Hiroshima, Japan (email: ykawano@hiroshima-u.ac.jp).

B. Besselink is with the Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, Groningen, The Netherlands (email: b.besselink@rug.nl).

M. Cao is with the Faculty of Science and Engineering, University of Groningen, Groningen, The Netherlands (email: m.cao@rug.nl).

of trajectories of a given nonlinear system, which is applied to, for instance, the tracking problem, observer design, and synchronization. The main idea is to consider an infinitesimal metric instead of a feasible distance function, and thus the so-called variational system or prolonged system plays a key role. In fact, this prolonged system naturally arises in monotonicity analysis [1], [2]. A well known monotonicity condition, the Kamke condition [1], [2], is described by using the prolonged system, which motivates us to study monotone systems in this contraction framework rooted in differential geometry.

In classical nonlinear systems and control theory, differential geometry is exploited for controllability and observability analysis, and the Lie derivatives of differential one-forms and the Lie brackets of vector fields play essential roles [23], [24]. Recently, the papers [25]–[27] provide concepts of eigenvalues and eigenvectors for nonlinear systems via these Lie derivatives and Lie brackets, and the papers [27], [28] apply these nonlinear eigenvalues to contraction analysis. This suggests the possibility of applying nonlinear eigenvalues to contraction analysis of monotone systems. That is, the Perron-Frobenius theorem on the dominant eigenvalues might be generalized to monotone nonlinear systems as the Perron-Frobenius vector of a positive LTI system has been successively extended in the differential geometric framework [18], [29].

Partly motivated by the nonlinear eigenvalues and the Perron-Frobenius theorem about dominant eigenvalues in linear algebra, we investigate incremental stability conditions of monotone nonlinear systems in terms of the Lie derivatives and Lie brackets. First, for incremental asymptotic stability, we provide sufficient conditions, which can be viewed as separable conditions. Next, for incremental exponential stability, we prove necessary and sufficient conditions under the standard assumptions [30] for the converse Lyapunov stability of exponential stability. Our incremental exponential stability conditions can be viewed as a generalization of the well-known separable conditions for positive LTI systems. In [16], similar analysis can be found. However, our conditions are less conservative as illustrated by a simple example, and we study the converse for incremental exponential stability, which has not been done in the literature. As a by-product of our results, we also provide separable necessary and sufficient conditions for exponential stability of linear time-varying (LTV) systems based on the fact that variational systems can be viewed as LTV systems along trajectories of the original nonlinear systems.

The remainder of this paper is organized as follows. Section II introduces monotone systems and incremental stability. Section III provides sufficient conditions for incremental

asymptotic stability in terms of the Lie derivative of differential one-forms or the Lie bracket of vector fields. From their structures, the proposed conditions are called sum- or max-separable conditions, respectively. In Section IV, we study incremental exponential stability by sum- and max-separable functions. Based on our sum- and max-separable results, we also provide an alternative proof for the existence of a diagonal contraction Riemannian metric [15]. Next, the application of our results to positive LTV systems yields separable exponential stability conditions. In Section V, as applications of our sufficient conditions, we mention extensions of the stabilizing controller design for systems that are not necessarily monotone [31], [32] to monotone systems; we also perform partial stability or so-called partial contraction analysis [33], [34]. The proof of each theorem is presented in the Appendix.

Notation: Let $\mathcal{I}_n := \{1, \ldots, n\}$. The sets of real numbers and non-negative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. The n-dimensional vector of which the i-th element is 1 and the others are 0 is denoted by e_i. The n-dimensional vector of which all elements are 1 is denoted by $\mathbb{1}_n$. For $x, y \in \mathbb{R}^n$, a partial order $\preceq$ is defined by writing $x \preceq y$ if and only if $x_i \leq y_i$ for all $i \in \mathcal{I}_n$, where x_i denotes the i-th component of x. Similarly, $x \prec y$ if and only if $x_i < y_i$. For $x \in \mathbb{R}^n$, $|x| \in \mathbb{R}_+^n$ is the vector consisting of the absolute values of elements of x, i.e., $|x| = [|x_1|, \ldots, |x_n|]^{\mathrm{T}}$. Also, vector norms are denoted as $|x|_1 := \sum_{i \in \mathcal{I}_n} |x_i|$, $|x|_2 := (\sum_{i \in \mathcal{I}_n} x_i^2)^{1/2}$, and $|x|_\infty := \max_{i \in \mathcal{I}_n} |x_i|$. A continuous function $\alpha : [0, a) \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed $r > 0$, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

II. PRELIMINARIES

A. Nonlinear Monotone Systems

Consider the nonlinear system

$$\Sigma : \dot{x}(t) = f(x(t)), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^2. The unique solution $x(t)$ to the system Σ at time $t \in \mathbb{R}_+$ starting from $x(0) = x_0$ is denoted by $\phi(t, x_0)$ if it exists. Let $\mathcal{X} \subset \mathbb{R}^n$ be a convex subset, e.g., $\mathcal{X} = \mathbb{R}_+^n$, and suppose that there exists a connected and forward invariant subset $\mathcal{S} \subset \mathcal{X}$, i.e., $\phi(t, x_0) \in \mathcal{S}$ for any $x_0 \in \mathcal{S}$ and $t \in \mathbb{R}_+$.

In contraction analysis, we use the so called prolongation of the system Σ, which consists of the system Σ and its variational system along the trajectory $\phi(t, x_0)$ given as

$$\delta \dot{x}(t) := \frac{d \delta x(t)}{dt} = \frac{\partial f(\phi(t, x_0))}{\partial x} \delta x(t). \tag{2}$$

If $\mathcal{S}$ is forward invariant and $x_0 \in \mathcal{S}$, the variational system (2) has a unique solution $\delta x(t)$ for any $\delta x(0) = \delta x_0 \in \mathbb{R}^n$, which reads

$$\delta x(t) = \Phi(t, x_0) \delta x_0, \quad \Phi(t, x) := \frac{\partial \phi(t, x)}{\partial x}. \tag{3}$$

The variational system can be viewed as a linear time-varying (LTV) system along the trajectory $\phi(t, x_0)$ of the original system Σ with the transition matrix $\Phi(t, x_0)$. Indeed, one can confirm that $\Phi(0, x_0) = I_n$ and

$$
\begin{aligned}
\Phi(t_2, x_0) &= \frac{\partial \phi(t_2, x_0)}{\partial x_0} \\
&= \left. \frac{\partial \phi(t_2 - t_1, x)}{\partial x} \right|_{x = \phi(t_1, x_0)} \frac{\partial \phi(t_1, x_0)}{\partial x_0} \\
&= \Phi(t_2 - t_1, \phi(t_1, x_0)) \Phi(t_1, x_0), \tag{4}
\end{aligned}
$$

for any $t_2 \geq t_1$.

In this paper, we assume that the nonlinear system Σ is *monotone* [1] on $\mathcal{X}$, i.e., the implication

$$x^1 \preceq x^2 \implies \phi(t, x^1) \preceq \phi(t, x^2), \ \forall t \geq 0,$$

holds for any $x^1, x^2 \in \mathcal{X}$. The monotonicity property of Σ is directly related to the properties of the variational system (2), as stated next.

Proposition 2.1 (Kamke condition, [1], [2]): The system Σ in (1) is monotone on a convex set $\mathcal{X} \subset \mathbb{R}^n$ if and only if

$$\frac{\partial f_i(x)}{\partial x_j} \geq 0, \ \forall i \neq j \tag{5}$$

for any $x \in \mathcal{X}$. ◁

Remark 2.2: Condition (5) implies that the variational system (2) is a positive system when regarded as an LTV system along $\phi(t, x_0) \in \mathcal{S}$, i.e.,

$$\Phi_{i,j}(t, x) = \frac{\partial \phi_i(t, x)}{\partial x_j} \geq 0, \ \forall i, j \in \mathcal{I}_n, \forall t \in \mathbb{R}_+, \tag{6}$$

and for all $x \in \mathcal{S}$ with being $\mathcal{S} \subset \mathcal{X}$ forward invariant. Therefore, $\mathcal{S} \times \mathbb{R}_+^n$ is a forward invariant set of the prolongation of Σ. ◁

The objective of this paper is to derive incremental stability conditions for monotone systems. In particular, by exploiting the variational system (2), we aim to extend the following conditions for positive (i.e., monotone) linear time-invariant (LTI) systems to monotone nonlinear systems.

Proposition 2.3 ([9]): Let $\dot{x} = Ax$ with $x \in \mathbb{R}^n$ be a positive system, i.e., A is Metzler ($A_{i,j} \geq 0$ for all $i \neq j$). Then, the following statements are equivalent:

1) A is Hurwitz;
2) there exists $v \in \mathbb{R}_+^n$ such that $v \succ 0$ and $v^{\mathrm{T}} A \prec 0$;
3) there exists $w \in \mathbb{R}_+^n$ such that $w \succ 0$ and $Aw \prec 0$;
4) there exists a diagonal positive definite matrix $P \in \mathbb{R}_+^{n \times n}$ such that $PA + A^{\mathrm{T}} P$ is negative definite. ◁

Conditions 2), 3), and 4) correspond to Lyapunov functions of the forms $\sum_{i \in \mathcal{I}_n} v_i |x_i|$, $\max_{i \in \mathcal{I}_n} |x_i|/w_i$, and $\sum_{i \in \mathcal{I}_n} p_i x_i^2$, respectively [9]. That is, stability is evaluated with respect to different distances induced by 1-, ∞-, and 2-norms. From their structures, the first two Lyapunov functions are referred to as sum- and max-separable functions.

It is worth mentioning that even for linear time-varying (LTV) systems, Conditions 2) and 3) have not been generalized. As a byproduct of our results, similar connections as in Proposition 2.3 are established for exponential stability of positive LTV systems.

Remark 2.4: Conditions 2) and 3) have strong connections with the Perron-Frobenius theorem [2], [11] in linear algebra. Suppose that the Metzler matrix A has a negative real eigenvalue $\lambda < 0$ whose corresponding left eigenvector v has strictly positive components, i.e., $v \succ 0$ then $v^{\mathrm{T}} A = \lambda v^{\mathrm{T}} \prec 0$. Then, Condition 2) holds. A similar conclusion holds for the right eigenvector. Then, the question is when such a pair of eigenvalue and eigenvector exists. The Perron-Frobenius theorem gives a partial answer. Namely, this theorem shows that a real square matrix with positive entries has a unique largest eigenvalue, called the dominant eigenvalue, and that the corresponding eigenvector can be chosen to have strictly positive components. There is a variant of this theorem for a Metzler matrix under the condition of irreducibility [2]. ◁

B. Incremental Stability and Induced Distances

In this paper, we study incremental stability with respect to a distance[1] $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$, which leads to the following definitions of incremental stability studied in this paper.

Definition 2.5 ([17], [35]): The system Σ in (1) is said to be

1) *incrementally stable (IS)* on $\mathcal{S}$ (with respect to distance d) if there exists a class $\mathcal{K}$ function α defined on the range of d such that

$$d(\phi(t, x^1), \phi(t, x^2)) \leq \alpha(d(x^1, x^2)), \ \forall t \in \mathbb{R}_+$$

for any $(x^1, x^2) \in \mathcal{S} \times \mathcal{S}$;

2) *incremental asymptotically stable (IAS)* on $\mathcal{S}$ if it is IS and

$$\lim_{t \to \infty} d(\phi(t, x^1), \phi(t, x^2)) = 0$$

for any $(x^1, x^2) \in \mathcal{S} \times \mathcal{S}$;

3) *incrementally exponentially stable (IES)* on $\mathcal{S}$ if there exist positive constants k and λ such that

$$d(\phi(t, x^1), \phi(t, x^2)) \leq k e^{-\lambda t} d(x^1, x^2), \ \forall t \in \mathbb{R}_+$$

for any $(x^1, x^2) \in \mathcal{S} \times \mathcal{S}$. ◁

If $\mathcal{S}$ contains an equilibrium point, IS, IAS and IES respectively imply the Lyapunov, asymptotic, and exponential stability of the equilibrium point. In contraction analysis, the paper [17] has established a connection between incremental stability and the prolongation of the system Σ via a Finsler-Lyapunov function. On the other hand, as mentioned in the previous subsection, for stability analysis of positive LTI systems, separable functions are commonly used. In this paper, the concepts of Finsler-Lyapunov functions and separable functions will be combined.

Consider the vector-valued function $v : \mathcal{S} \to \mathbb{R}_+^n$ such that $v(x) \succ 0$ on $\mathcal{S}$. In our analysis, we use the following non-negative functions defined on $\mathcal{S} \times \mathbb{R}^n$:

- the sum-separable function

$$M_1(x, \delta x) = \sum_{i \in \mathcal{I}_n} v_i(x) |\delta x_i|; \tag{7}$$

[1] A non-negative function $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$ is said to be a *distance* if it satisfies 1) $d(x^1, x^2) = d(x^2, x^1) = 0$ if and only if $x^1 = x^2$ for any $x^1, x^2 \in \mathcal{S}$, and 2) $d(x^1, x^3) \leq d(x^1, x^2) + d(x^2, x^3)$ for any $x^1, x^2, x^3 \in \mathcal{S}$. Note that symmetry is not required.

- the diagonal Riemannian metric

$$M_2(x, \delta x) = \left(\sum_{i \in \mathcal{I}_n} (v_i(x) \delta x_i)^2 \right)^{1/2}; \tag{8}$$

- and the max-separable function

$$M_\infty(x, \delta x) = \max_{i \in \mathcal{I}_n} (v_i(x) |\delta x_i|). \tag{9}$$

At each $(x, \delta x) \in \mathcal{S} \times \mathbb{R}^n$, these functions can be viewed as the 1-, 2-and ∞-norms of the vector $[v_1(x)\delta x_1 \cdots v_n(x)\delta x_n]^{\mathrm{T}}$, respectively, where we recall that $v_i(x) > 0$ on $\mathcal{S}$ for any $i \in \mathcal{I}_n$.

Next, we show the definitions of the distances induced by these functions. For any pair $(x^1, x^2) \in \mathcal{S} \times \mathcal{S}$, let $\Gamma(x^1, x^2)$ be the collection of piecewise C^1 paths $\gamma : [0, 1] \to \mathcal{S}$ such that $\gamma(0) = x^1$ and $\gamma(1) = x^2$. Then, we define non-negative functions $d_i : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$, $i = 1, 2, \infty$ as

$$d_i(x^1, x^2) := \inf_{\gamma \in \Gamma(x^1, x^2)} \int_0^1 M_i \left(\gamma(s), \frac{d\gamma(s)}{ds} \right) ds. \tag{10}$$

As d_1, d_2, and d_∞ all satisfy the definition of distance, they will be referred to as induced distances.

Note that $M_2(x, \delta x)$ is a Riemannian metric, and thus a Finsler metric. If $\mathcal{S}$ is connected, by applying the Hopf-Rinow theorem [36], [37], it can be shown that for any $(x^1, x^2) \in \mathcal{S} \times \mathcal{S}$ there exists a geodesic $\gamma^* \in \Gamma(x^1, x^2)$ on $\mathcal{S} \times \mathcal{S}$, i.e.,

$$d_2(x^1, x^2) = \int_0^1 M_2 \left(\gamma^*(s), \frac{d\gamma^*(s)}{ds} \right) ds. \tag{11}$$

Moreover, since γ^* is piecewise C^1, for any given $\varepsilon > 0$ and $(x^1, x^2) \in \mathcal{S} \times \mathcal{S}$, there exists a class C^1 path $\bar{\gamma} \in \Gamma(x^1, x^2)$ such that

$$\int_0^1 M_2 \left(\bar{\gamma}(s), \frac{d\bar{\gamma}(s)}{ds} \right) ds \leq (1 + \varepsilon) d_2(x^1, x^2). \tag{12}$$

In our incremental stability analysis, we will use this class C^1 path $\bar{\gamma}(s)$ as is done for systems that are not necessarily monotone in [17]. In contrast to $d_2(\cdot, \cdot)$, it is not clear whether the Hopf-Rinow theorem, a sufficient condition for the existence of a geodesic, is applicable for $d_1(\cdot, \cdot)$ or $d_\infty(\cdot, \cdot)$ because they do not satisfy the definition of the Finsler metric [36], [37] (in particular, strict convexity). The existence of the geodesic is purely a mathematical question and is not the target of this paper. Not to spend efforts on this question, we establish the following equivalence for the induced distances, the proof of which is given in Appendix A-A.

Proposition 2.6: Let $\mathcal{S} \subset \mathbb{R}^n$ be connected. Consider the induced distances d_i, $i = 1, 2, \infty$ by $v(x) \succ 0$ on $\mathcal{S}$. Then, for any $i, j \in \{1, 2, \infty\}$, there exist positive constants $\underline{a}_{i,j}$ and $\overline{a}_{i,j}$ such that

$$\underline{a}_{i,j} d_i(x^1, x^2) \leq d_j(x^1, x^2) \leq \overline{a}_{i,j} d_i(x^1, x^2) \tag{13}$$

for all $(x^1, x^2) \in \mathcal{S} \times \mathcal{S}$.

This proposition implies that incremental stability properties with respect to the three distances are equivalent. Exploiting this equivalence, one can use the path $\bar{\gamma}(s)$ also for incremental stability analysis with respect to $d_1(\cdot, \cdot)$ or $d_\infty(\cdot, \cdot)$.

III. SUFFICIENT CONDITIONS FOR INCREMENTAL ASYMPTOTIC STABILITY

A. Sum-Separable Conditions

In this section, we derive sufficient conditions for IAS of monotone systems based on sum-and max-separable functions. Then, in the next section, we focus on IES.

First, we extend the relation 1) $\implies$ 2) in Proposition 2.3 to nonlinear monotone systems by using the sum-separable function (without taking absolute values)

$$v^{\mathrm{T}}(x)\delta x = \sum_{i \in \mathcal{I}_n} v_i(x)\delta x_i.$$

This is a differential one-form. In differential geometry, the Lie derivative of a differential one-form along the vector field f, defined as

$$L_f(v^{\mathrm{T}}(x)\delta x) := \left(v^{\mathrm{T}}(x)\frac{\partial f(x)}{\partial x} + \left(\frac{\partial v(x)}{\partial x}f(x)\right)^{\mathrm{T}}\right)\delta x, \quad (14)$$

plays an essential role for analyzing the system Σ, especially its observability [23], [24].

Now, we establish a connection between contraction analysis and the Lie derivatives of differential one-forms (14) for monotone systems. The proof is given in Appendix A-B.

Theorem 3.1: Let $\mathcal{S} \subset \mathcal{X}$ be a connected and forward invariant set of the monotone system Σ in (1) on a convex set $\mathcal{X}$. Let $v : \mathcal{S} \to \mathbb{R}_+^n$ be a class C^1 vector-valued function such that

1) $v(x) \succ 0$ for any $x \in \mathcal{S}$.

Then, the monotone system Σ is IAS on $\mathcal{S}$ with respect to $d_1(\cdot, \cdot)$ induced by the sum-separable function $v^{\mathrm{T}}(x)|\delta x|$ if

2) there exists a continuous positive definite function α such that

$$L_f(v^{\mathrm{T}}(x)\delta x) \le -\alpha(v^{\mathrm{T}}(x)\delta x) \quad (15)$$

for any $(x, \delta x) \in \mathcal{S} \times \mathbb{R}_+^n$.

Loosely speaking, the condition (15) implies that $v^{\mathrm{T}}(x)|\delta x|$ can be regarded as a Lyapunov function for contraction analysis. In [17, Theorem 2], the LaSalle invariance principle (for not-necessarily monotone systems), e.g., [30, Theorem 4.4], has been extended to contraction analysis, which can also be applied in our case. The proof of the following corollary is omitted because it follows from [17, Theorem 2] and our Theorem 3.1.

Corollary 3.2: Let $\mathcal{S} \subset \mathcal{X}$ be a connected and forward invariant set of the monotone system Σ in (1) on a convex set $\mathcal{X}$. Also, let $v : \mathcal{S} \to \mathbb{R}_+^n$ be a class C^1 vector-valued function satisfying Condition 1) in Theorem 3.1. If there exists a non-negative function $\bar{\alpha} : \mathcal{S} \times \mathbb{R}_+^n \to \mathbb{R}_+$ such that

$$L_f(v^{\mathrm{T}}(x)\delta x) \le -\bar{\alpha}(x, \delta x) \quad (16)$$

on $\mathcal{S} \times \mathbb{R}_+^n$, then the monotone system Σ is IS on $\mathcal{S} \times \mathcal{S}$. Suppose in addition that Σ has at least one bounded solution in $\mathcal{S}$. Then, any solution $(\phi(t, x_0), \Phi(t, x_0)\delta x_0)$ of its prolonged system starting from $(x_0, \delta x_0) \in \mathcal{S} \times \mathbb{R}_+^n$ converges to the largest invariant set Δ in

$$\Pi := \{(x, \delta x) \in \mathcal{S} \times \mathbb{R}_+^n : \bar{\alpha}(x, \delta x) = 0\}. \quad (17)$$

If $\Delta = \mathcal{S} \times \{0\}$, the system Σ is IAS on $\mathcal{S}$ with respect to the distance $d_1(x^1, x^2)$ induced by $v^{\mathrm{T}}(x)|\delta x|$. ◁

Remark 3.3: We note that condition (15) in Theorem 3.1 as well as condition (16) in Corollary 3.2 only needs to be verified for states δx (of the variational system) in the positive orthant. Despite not considering the entire state space, the incremental stability properties in Theorem 3.1 and Corollary 3.2 hold for *any* (pair of) trajectories of the system Σ. ◁

Based on this invariance principle, it is possible to give a sufficient condition for IAS in terms of local observability [23], [24]. Suppose that f is of class C^∞, and there exists a class C^∞ function $h : \mathbb{R}^n \to \mathbb{R}_+$ such that (16) holds for

$$\bar{\alpha}(x, \delta x) = \frac{\partial h(x)}{\partial x}\delta x,$$

where $\partial h(x)/\partial x \succeq 0$ on $\mathcal{S}$. Then, the largest invariant set contained in Π defined in (17) is

$$\Delta = \{(x, \delta x) \in \mathcal{S} \times \mathbb{R}_+^n : d(L_f^i h(x)) = 0, \ i = 0, 1, \dots\},$$

where $d(L_f^i h(x)) = (\partial L_f^i h(x)/\partial x)\delta x$ and with $L_f^0 h(x) := h(x)$, $L_f^{i+1}h(x) := (\partial L_f^i h(x)/\partial x)f(x)$, $i = 0, 1, \dots$. Therefore, $\Pi = \mathcal{S} \times \{0\}$ if and only if the system Σ with output

$$y = h(x)$$

satisfies the observability rank condition [23], [24] at each $x \in \mathcal{S}$. This observation extends the well-known connection between the Lyapunov equation and observability of linear systems.

Remark 3.4: In [16], a condition similar to that in Theorem 3.1 is obtained based on matrix measures. However, in contrast to [16, Theorem 1], Theorem 3.1 does not require that $v(x) > \underline{c}\mathbb{1}_n$ on $\mathcal{S}$ for some constant $\underline{c} > 0$. In addition, our v_i is not restricted to depend on x_i only. If v_i depends on only x_i, the differential one-form $v^{\mathrm{T}}(x)\delta x$ is integrable. In this case, a path integral does not depend on the path, i.e., every piecewise C^1 path is a geodesic. This makes analysis simpler because one can avoid the discussions preceding Proposition 2.6. At the same time, our result Theorem 3.1 has a direct extension using LaSalle's invariance principle, which is difficult to obtain on the basis of matrix measures as in [16]. Finally, the paper [16] does not proceed with the converse analysis. In contrast, we provide the converse proof for IES in the next section. ◁

The following simple example demonstrates the difference between the results in [16] and ours.

Example 3.5: Consider the one-dimensional system

$$\dot{x} = -x^3.$$

It is monotone on $\mathbb{R}$, and the origin is globally asymptotically stable. Take $\mathcal{S} = \mathbb{R}$ and compute

$$L_f(v(x)\delta x) = -\left(\frac{dv(x)}{dx}x^3 + v(x)x^2\right)\delta x.$$

Regardless of the choice of $v(\cdot)$, $L_f(v(x)\delta x) = 0$ at $x = 0$. As a result, the conditions in [16, Theorem 1] do not hold. Instead, we apply Corollary 3.2. For $v(x) = 1$, we have

$$L_f(v(x)\delta x) = -x^2\delta x \quad (18)$$

and Π is

$$\Pi = (\{0\} \times \mathbb{R}_+) \cup (\mathcal{S} \times \{0\}).$$

By the LaSalle invariance principle, the omega limit set of a trajectory of the prolonged system, denoted by L^+, satisfies $L^+ \subset \Pi$. In addition, we have that $\lim_{t\to\infty} v(x(t))\delta x(t) = c$ for some $c \in \mathbb{R}_+$ and with $v(x) = 1$, again by LaSalle's invariance principle.

According to Corollary 3.2, the system is IS, and the origin is a unique equilibrium point, which means that all solutions of the original system starting from $\mathcal{S}$ are bounded. We denote the omega limit set of the original system by $X^+ \subset \mathcal{S}$. Then, $L^+ = X^+ \times \{c\} \subset \Pi$. From the structure of Π, if $c \neq 0$ then $X^+ \times \{c\} \subset \{0\} \times \mathbb{R}_+$. Therefore, for any trajectory of the prolonged system of which the omega limit point is in $X^+ \times \{c\}$, $c \neq 0$, the trajectory of the original system converges to the origin. On the other hand, if $c = 0$ then $X^+ \times \{0\} \subset \mathcal{S} \times \{0\}$. In this case, Corollary 3.2 (or the argument of [17, the proof of Theorem 2]) is applicable. Since the origin is an equilibrium point, any trajectory converges to the origin. In summary, the origin is asymptotically stable. $\triangleleft$

B. Max-Separable Conditions

In this section, we extend relation 3) $\implies$ 1) in Proposition 2.3 by using the max-separable function (without taking absolute values)

$$W(x, \delta x) := \max_{i \in \mathcal{I}_n} \frac{\delta x_i}{w_i(x)}. \tag{19}$$

Since W is not differentiable, we use its upper-right Dini derivative along trajectories of the prolonged system of Σ, defined as

$$D^+W(x, \delta x)$$
$$:= \limsup_{h \to 0^+} \frac{W\left(x + hf(x), \delta x + h\frac{\partial f(x)}{\partial x}\delta x\right) - W(x, \delta x)}{h}.$$

Specifically, if each $w_i(x)$ ($i \in \mathcal{I}_n$) is of class C^1, this upper-right Dini derivative satisfies (see [38])

$$D^+W(x, \delta x)$$
$$= \max_{j \in \mathcal{J}(x, \delta x)} \frac{1}{w_j^2(x)} \left(\frac{\partial f_j(x)}{\partial x} w_j(x)\delta x - \frac{\partial w_j(x)}{\partial x} f(x)\delta x_j \right), \tag{20}$$

with $\mathcal{J}(x, \delta x) = \{j \in \mathcal{I}_n : \delta x_j/w_j(x) = W(x, \delta x)\}$.

In the previous section, we have derived the incremental stability conditions in terms of the Lie derivatives of differential one-forms. Its dual is given by the following Lie bracket [23], [24] along a vector field f,

$$[w, f](x) := \frac{\partial f(x)}{\partial x} w(x) - \frac{\partial w(x)}{\partial x} f(x). \tag{21}$$

For ease of notation, we denote the j-th element of the vector-valued function $[w, f](x)$ by $[w, f]_j(x)$. The above Lie brackets are typically used for accessibility analysis [23], [24].

In the following theorem, we use the Lie bracket for IAS analysis, similar to what was done through the Lie derivative of the differential one-form. The proof is given in Appendix A-C.

Theorem 3.6: Let $\mathcal{S} \subset \mathcal{X}$ be a connected and forward invariant set of the monotone system Σ in (1) on a convex set $\mathcal{X}$. Let $w : \mathcal{S} \to \mathbb{R}_+^n$ be a class C^1 vector-valued function such that

1) $w(x) \succ 0$ for any $x \in \mathcal{S}$.

Then, the monotone system Σ is IAS on $\mathcal{S}$ with respect to $d_\infty(\cdot, \cdot)$ induced by the max-separable function $\max_{i \in \mathcal{I}_n} |\delta x_i|/w_i(x)$ if

2) there exists a continuous positive definite function α such that

$$\frac{\delta x_i}{w_i^2(x)}[w, f]_i(x) \leq -\alpha\left(\frac{\delta x_i}{w_i(x)}\right), \ \forall i \in \mathcal{I}_n \tag{22}$$

for any $(x, \delta x) \in \mathcal{S} \times \mathbb{R}_+^n$.

Remark 3.7: The LaSalle invariance principle is also applicable in the max-separable case. If there exists a non-negative function $\bar{\alpha} : \mathcal{S} \times \mathbb{R}_+^n \to \mathbb{R}_+$ such that

$$\frac{\delta x_i}{w_i^2(x)}[w, f]_i(x) \leq -\bar{\alpha}(x, \delta x)$$

for all $i \in \mathcal{I}_n$ and $(x, \delta x) \in \mathcal{S} \times \mathbb{R}_+^n$, then the monotone system Σ is IS. Suppose that the monotone system Σ has at least one bounded solution in $\mathcal{S}$. Then, any solution to its prolonged system starting from $\mathcal{S} \times \mathbb{R}_+^n$ converges to the largest invariant Δ set in

$$\Pi := \{(x, \delta x) \in \mathcal{S} \times \mathbb{R}_+^n : \bar{\alpha}(x, \delta x) = 0\}.$$

If $\Delta = \mathcal{S} \times \{0\}$, the monotone system Σ is IAS on $\mathcal{S}$. $\triangleleft$

Example 3.8: Inspired by a Lorenz chaotic system and an example for incremental stability analysis in [35], consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\beta x_1 + x_2 x_3 \\ -\sigma x_2 + \sigma x_3 \\ -x_3 \end{bmatrix},$$

where $\beta, \sigma > 0$. This system is monotone on the positive orthant and we take $\mathcal{S} = \mathbb{R}_+^3$. It can be seen that the system is in an upper triangular form, and the subsystem consisting of the second and third elements,

$$\dot{x}_{23} = \begin{bmatrix} -\sigma & \sigma \\ 0 & -1 \end{bmatrix} x_{23},$$

is linear. In order to find $w(x)$ and α satisfying Conditions 1) and 2) in Theorem 3.6, we consider to use an eigenvalue and eigenvector of the linear part. An eigenvalue is -1, and a corresponding eigenvector is $[\sigma/(\sigma - 1) \ 1]^\mathrm{T}$. Now, if there exists $w_1(x)$ such that

$$\begin{bmatrix} -\beta & x_3 & x_2 \\ 0 & -\sigma & \sigma \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} w_1(x) \\ \sigma/(\sigma - 1) \\ 1 \end{bmatrix}$$
$$- \begin{bmatrix} \frac{\partial w_1(x)}{\partial x_1} & \frac{\partial w_1(x)}{\partial x_2} & \frac{\partial w_1(x)}{\partial x_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\beta x_1 + x_2 x_3 \\ -\sigma x_2 + \sigma x_3 \\ -x_3 \end{bmatrix}$$
$$\preceq - \begin{bmatrix} w_1(x) \\ \sigma/(\sigma - 1) \\ 1 \end{bmatrix}, \tag{23}$$

then the conditions hold for $\alpha(z) = z$ and

$$w(x) = \begin{bmatrix} w_1(x) \\ \sigma/(\sigma-1) \\ 1 \end{bmatrix}.$$

It can be verified that such $w_1(x)$ is obtained as

$$w_1(x) = 1 + \frac{(\sigma-1)(\beta-2)x_2 + \sigma(\beta-2\sigma)x_3}{(\sigma-1)(\beta-2)(\beta-\sigma-1)}.$$

In fact, $w_1(x) - 1$ is a solution to (23) with the equality. In order to satisfy the requirement $w(\cdot) \succ 0$, we use $w_1(x)$ instead of $w_1(x) - 1$. For $\beta > \sigma + 1$ and $\sigma > 1$, all the conditions are satisfied, and therefore, the system is IAS on $\mathbb{R}^3_+$ (In fact, Proposition 4.6 in the next section concludes IES). Note that w_1 depends on x_2 and x_3, which demonstrates the utility of considering each w_i as a function of x instead of only x_i as in [16]. ◁

IV. EXPONENTIAL STABILITY

A. Incremental Exponential Stability for Monotone Systems

In this section, we focus on IES of monotone systems. To establish necessary conditions for IES in terms of sum- or max-separable functions, we need the following technical conditions on the set $\mathcal{S} \subset \mathcal{X}$.

Assumption 4.1: The set $\mathcal{S}$ is convex and both forward and backward invariant for (1) , i.e., $\phi(t, x_0) \in \mathcal{S}$ for all $t \in \mathbb{R}$ for any $x_0 \in \mathcal{S}$. In addition, for any $x_0 \in \mathcal{S}$, there exists $a \neq 0$ such that $x_0 + ae_i \in \mathcal{S}$ for all $i \in \mathcal{I}_n$. ◁

Remark 4.2: We note that the latter assumption on the set $\mathcal{S}$ is trivially satisfied when $\mathcal{S}$ is open. However, as the sign of a can be chosen freely, this condition also holds when $\mathcal{S}$ is a cone, e.g., the positive orthant $\mathbb{R}^n_+$. ◁

In Appendix B, we prove the following necessary and sufficient condition for IES as an extension of Proposition 2.3 towards nonlinear monotone systems. This result can also be viewed as an extension of the Lyapunov converse theorem [30, Theorem 4.14] for exponential stability to IES of monotone systems.

Theorem 4.3: Let $\mathcal{S}$ satisfy Assumption 4.1 and consider the monotone system Σ in (1) on $\mathcal{S}$. Suppose that $\partial f(x)/\partial x$ is bounded on $\mathcal{S}$. Then, the following statements are equivalent:

a) the monotone system Σ is IES on $\mathcal{S}$ with respect to one of the distances $d_1(x^1, x^2) = |x^1 - x^2|_1$, $d_2(x^1, x^2) = |x^1 - x^2|_2$, and $d_\infty(x^1, x^2) = |x^1 - x^2|_\infty$;

b) there exist a class C^1 vector-valued function $v : \mathcal{S} \to \mathbb{R}^n_+$ and positive constants $\underline{c}$, $\bar{c}$ and c_v such that $\underline{c}\mathbb{1}_n \preceq v(x) \preceq \bar{c}\mathbb{1}_n$ and

$$L_f(v^{\mathrm{T}}(x)\delta x) \leq -c_v v^{\mathrm{T}}(x)\delta x \qquad (24)$$

on $\mathcal{S} \times \mathbb{R}^n_+$;

c) there exists a class C^1 vector-valued function $w : \mathcal{S} \to \mathbb{R}^n_+$ and positive constants $\underline{c}$, $\bar{c}$ and c_w such that $\underline{c}\mathbb{1}_n \preceq w(x) \preceq \bar{c}\mathbb{1}_n$ and

$$[w, f](x) \preceq -c_w w(x) \qquad (25)$$

on $\mathcal{S}$;

d) there exist a class C^1 vector-valued function $p : \mathcal{S} \to \mathbb{R}^n_+$ and positive constants $\underline{c}$, $\bar{c}$ and c_p such that

$$\underline{c}\mathbb{1}_n \preceq p(x) \preceq \bar{c}\mathbb{1}_n \qquad (26)$$

on $\mathcal{S}$, and $V(x, \delta x) := \sum_{i \in \mathcal{I}_n} p_i(x)\delta x_i^2$ satisfies

$$\frac{\partial V(x, \delta x)}{\partial x} f(x) + \frac{\partial V(x, \delta x)}{\partial \delta x} \frac{\partial f(x)}{\partial x} \delta x$$
$$\leq -c_p V(x, \delta x) \qquad (27)$$

on $\mathcal{S} \times \mathbb{R}^n_+$.

Note that in Condition a), IES with respect to $d_1(\cdot, \cdot)$, $d_2(\cdot, \cdot)$, and $d_\infty(\cdot, \cdot)$ are equivalent owing to the equivalence of norms on $\mathbb{R}^n$ (recall that $\mathcal{S} \subset \mathbb{R}^n$).

Remark 4.4: For not-necessarily monotone systems and not-necessarily diagonal Riemannian metrics, a similar relation to a) $\iff$ d) is shown in a different way in [21, Proposition 4] for a class C^3 vector field f. In Theorem 4.3 however, owing to monotonicity, the existence of a diagonal Riemannian metric is guaranteed for the class C^2 vector field f. In fact, the paper [15] gives a different proof for d) $\implies$ a) based on results on positive LTV systems [39]. However, the papers [15], [39] have not studied the sum- or max-separable condition. In fact, we provide an alternative proof for d) $\implies$ a) based on these sum- and max-separable functions, which is a natural extension of the proof for positive LTI system, see [10, Theorem 15]. ◁

In Theorems 3.1 and 3.6, sufficient conditions for IAS are provided without assuming backward invariance of $\mathcal{S}$ or boundedness of $\partial f(x)/\partial x$, $v(x)$, or $w(x)$. In fact, it is possible to derive sufficient conditions for IES without these assumptions, which are formally stated as the following propositions. The proofs are given in Appendices A-B and A-C.

Proposition 4.5: Let $\mathcal{S} \subset \mathcal{X}$ be a connected and forward invariant set of the monotone system Σ in (1) on a convex set $\mathcal{X}$. Let $v : \mathcal{S} \to \mathbb{R}^n_+$ be a class C^1 vector-valued function which satisfies Condition 1) in Theorem 3.1. Then, the monotone system Σ is IES on $\mathcal{S}$ with respect to $d_1(\cdot, \cdot)$ induced by the sum-separable function $v^{\mathrm{T}}(x)|\delta x|$ if (24) holds for any $(x, \delta x) \in \mathcal{S} \times \mathbb{R}^n_+$.

Proposition 4.6: Let $\mathcal{S} \subset \mathcal{X}$ be a connected and forward invariant set of the monotone system Σ in (1) on a convex set $\mathcal{X}$. Let $w : \mathcal{S} \to \mathbb{R}^n_+$ be a class C^1 vector-valued function which satisfies Condition 1) in Theorem 3.6. Then, the monotone system Σ is IES on $\mathcal{S}$ with respect to $d_\infty(\cdot, \cdot)$ induced by the max-separable function $\max_{i \in \mathcal{I}_n} |\delta x_i|/w_i(x)$ if (25) holds for any $(x, \delta x) \in \mathcal{S} \times \mathbb{R}^n_+$.

Boundedness of $\partial f(x)/\partial x$ is the standard assumption for the converse proof of exponential stability; see, for instance, [30, Theorem 4.14]. As shown in Appendix B, under this standard assumption, IES of monotone systems implies the existence of the bounded functions $v(x)$, $w(x)$, and $p(x)$. Therefore, boundedness of $\partial f(x)/\partial x$ and $v(x)$ are imposed for establishing the converse statements.

B. Connection with Differentially Positive Systems

In [18], [29], an infinitesimal approach to monotonicity is introduced leading to the notion of differential positivity as a

generalization of monotonicity. In this section, we discuss the relation between the results in [18], [29] and our incremental stability analysis.

Differential positivity relies on the introduction of a so-called cone field, which associates a cone to every point in the state space. A differentially positive system is then the one that makes this cone field invariant. For the system Σ in (1) and a *constant* cone field $\mathbb{R}^n_+$, this amounts to $\Phi(t, x_0)\mathbb{R}^n_+ \subset \mathbb{R}^n_+$ for any $t \geq 0$ and $x_0 \in S$, i.e., condition (6) holds and differential positivity and monotonicity are equivalent (see [18, Theorem 1]).

Even though [18] does not explicitly analyze incremental stability properties, the asymptotic behavior of trajectories is studied under the stronger notion of *strict* differential positivity. Again the constant cone field $\mathbb{R}^n_+$ as an example, this property can be stated as $\Phi(t, x_0)\mathbb{R}^n_+ \subset \mathcal{R}$ for any $t \geq 0$ and $x_0 \in S$ and with $\mathcal{R}$ a cone satisfying $\mathcal{R} \subset \text{int}(\mathbb{R}^n_+) \cup \{0\}$ (for simplicity, $\mathcal{R}$ is taken to be constant here). Loosely speaking, strict differential positivity calls for trajectories of the variational system to converge to the interior of the cone; a property that is made explicit in [18] as the contraction of the so-called Hilbert metric. In fact, [18, Theorem 2] shows that such trajectories satisfy (assuming backward invariance of $\mathcal{S}$)

$$\lim_{t \to \infty} \Phi\big(0, \phi(-t, x)\big)\delta x \in \{\lambda w(x) : \lambda \geq 0\}, \qquad (28)$$

for any $(x, \delta x) \in \mathcal{S} \times \mathbb{R}^n_+$, where w is the *Perron-Frobenius vector field* which satisfies $w(x) \in \text{int}(\mathbb{R}^n_+)$ for all $x \in \mathcal{S}$. For linear systems, the Perron-Frobenius vector field reduces to the well-known Perron-Frobenius vector of the system matrix. As the trajectories of the variational system converge to the Perron-Frobenius vector field, this can be thought of as a type of partial (or horizaontal) contraction.

In the current paper, however, we study incremental stability properties without requiring strict differential positivity. Moreover, as shown in Appendices A-B and A-C, our IAS conditions in Theorems 3.1 and 3.6 guarantee the convergence property to the *origin* for the variational systems (rather than to the Perron-Frobenius vector).

C. Nonlinear Eigenvalues

As mentioned in Remark 2.4, Proposition 2.3 for stability analysis of positive LTI systems has a strong connection with the Perron-Frobenius theorem and the dominant eigenvalue. In this subsection, we establish a similar connection among nonlinear eigenvalues [25]–[27], Perron-Frobenius vector fields (as in Section IV-B), and contraction analysis.

Eigenvalues and eigenvectors of the system matrix play an important role in the analysis and control of LTI systems. In order to generalize these results, the concept of eigenvalues and eigenvectors are extended to nonlinear systems and used for stability, controllability, and observability analysis, as well as for solving a so called differential Riccati equation which appears in controller design [25]–[27]. Extensions of eigenvalues and eigenvectors are based on the interpretation that an eigenvector of a matrix A is an element of a one-dimensional A-invariant subspace. In differential geometry, the concept of A-invariant subspace is extended as an invariant

codistribution or distribution [23], [24]. Therefore, one can interpret an element of a one-dimensional invariant codistribution or distribution as a nonlinear version of an eigenvector.

A vector-valued function $v : \mathbb{R}^n \to \mathbb{R}^n$ corresponds to the one-dimensional invariant codistribution if there exists a function $\lambda_v : \mathbb{R}^n \to \mathbb{R}$ such that

$$L_f(v^\text{T}(x)\delta x) = \lambda_v(x)v^\text{T}(x)\delta x, \qquad (29)$$

where $\text{span}\{v^\text{T}(x)\delta x\}$ is a one-dimensional invariant codistribution. We call the above $\lambda_v(\cdot)$ and $v(\cdot)$ a *nonlinear left eigenvalue and eigenvector*, respectively [25]–[27]. Similarly, a vector valued function $w : \mathbb{R}^n \to \mathbb{R}^n$ corresponds to a one-dimensional invariant distribution if there exists a function $\lambda_w : \mathbb{R}^n \to \mathbb{R}$ such that

$$[w, f](x) = \lambda_w(x)w(x), \qquad (30)$$

where $\text{span}\{w(x)\}$ is a one-dimensional invariant distribution. We call the above $\lambda_w(\cdot)$ and $w(\cdot)$ a *nonlinear right eigenvalue and eigenvector*, respectively [25]–[27]. In the linear case, the definition (29) and (30) corresponds to the eigenvalue and left and right eigenvector, respectively. As expected as a property of eigenvalues, nonlinear eigenvalues are invariant under a nonlinear change of coordinates [25]–[27].

Now, we are ready to investigate the connection among nonlinear eigenvalues, Perron-Frobenius vector fields, and contraction analysis. First, as shown in [18], the Perron-Frobenius vector field of a strictly differential positive system always satisfies (30) for some $\lambda_w(x)$. That is, the Perron-Frobenius vector field is a nonlinear eigenvector. Recall that the Perron-Frobenius vector field $w(x)$ is in $\text{int}(\mathbb{R}^n_+)$, i.e., $w(\cdot) \succ 0$. Therefore, by comparing (30) and (25), it can be seen that if $\lambda_w(x)$ is uniformly negative, then Proposition 4.6 concludes IES. That is, one can evaluate IES of a strictly differentially positive system by checking the nonlinear eigenvalue corresponding to the Perron-Frobenius vector field.

Remark 4.7: Although there are various applications of nonlinear eigenvalues, their computation remains challenging as it requires solving a nonlinear partial differential equation. However, it may be possible to use numerical computational results of the Koopman operator [40]. A Koopman eigenvalue and eigenfunction can be respectively viewed as a complex number $\lambda \in \mathbb{C}$ and scalar valued function $\phi : S \to \mathbb{C}$ satisfying

$$\frac{\partial \phi(x)}{\partial x} f(x) = \lambda \phi(x).$$

By computing partial derivatives with respect to x, we have

$$\left(\frac{\partial v(x)}{\partial x} f(x) \right)^\text{T} + v^\text{T}(x)f(x) = \lambda v^\text{T}(x),$$

where $v^\text{T}(x) := \partial \phi(x)/\partial x$. This is nothing but (29); recall (14). Therefore, λ and $\partial \phi(x)/\partial x$ are respectively a nonlinear left eigenvalue and eigenvector, and it is possible to apply numerical methods for Koopman eigenvalues and eigenfunctions to a nonlinear eigenvalue and eigenvector. $\lhd$

D. Remark on Positive Linear Time-Varying Systems

As mentioned in Remark 2.2, the variational system (2) of the monotone system Σ in (1) can be viewed as a positive LTV system. Therefore, Theorem 4.3 is applicable for exponential stability analysis of positive LTV systems.

To make this explicit, consider the positive LTV system

$$\dot{x}(t) = A(t)x(t), \tag{31}$$

where $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ is continuous. Suppose that each off-diagonal element of $A(t)$ is non-negative for all $t \in \mathbb{R}$. From Theorem 4.3, we have the following extension of Proposition 2.3, where Conditions 2) and 3) are our contribution while Condition 4) is established in a different way in [39]. The proof is similar to that of Theorem 4.3; for more details, see Appendix C.

Corollary 4.8: Suppose that $A(t)$ is bounded on $\mathbb{R}$. The following statements are equivalent:

1) the positive LTV system (31) is exponentially stable;
2) there exist a class C^1 vector-valued function $v : \mathbb{R} \to \mathbb{R}^n_+$ and positive constants $\underline{c}$, $\bar{c}$ and c_v such that $\underline{c}\mathbb{1}_n \preceq v(t) \preceq \bar{c}\mathbb{1}_n$ and

$$v^{\mathrm{T}}(t)A(t) + \frac{dv^{\mathrm{T}}(t)}{dt} \preceq -c_v v^{\mathrm{T}}(t)$$

for any $t \in \mathbb{R}$;
3) there exist a class C^1 vector-valued function $w : \mathbb{R} \to \mathbb{R}^n_+$ and positive constants $\underline{c}$, $\bar{c}$ and c_w such that $\underline{c}\mathbb{1}_n \preceq w(t) \preceq \bar{c}\mathbb{1}_n$ and

$$A(t)w(t) - \frac{dw(t)}{dt} \preceq -c_w w(t)$$

for any $t \in \mathbb{R}$;
4) there exists a class C^1 vector-valued function $p : \mathbb{R} \to \mathbb{R}^n_+$ and positive constants $\underline{c}$, $\bar{c}$ and c_w such that

$$\underline{c}\mathbb{1}_n \preceq p(t) \preceq \bar{c}\mathbb{1}_n$$

for any $t \in \mathbb{R}$, and $V(t,x) := \sum_{i \in \mathcal{I}_n} p_i(t)x_i^2$ satisfies

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x}A(t)x \leq -c_p V(t,x)$$

for any $t \in \mathbb{R}$ and $x \in \mathbb{R}^n_+$. ◁

V. EXAMPLES AND APPLICATIONS

A. Example

Consider a nonlinear single-integrator multi-agent system on an undirected graph given by

$$\dot{x}_1 = -\mathcal{L}_1(x_1) + \sum_{j \in \mathcal{N}_1} \mathcal{L}_{1,j}(x_j - x_1),$$

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} \mathcal{L}_{i,j}(x_j - x_i), \ i \neq 1,$$

where $x_i \in \mathbb{R}_+$ with $i \in \mathcal{I}_n$, $\mathcal{N}_i$ is the set of neighbors of agent i, and $i \notin \mathcal{N}_i$. The first agent will be referred to as the leader and $d\mathcal{L}_1(x_1)/dx_1 > 0$ for any $x_1 \in \mathbb{R}_+$. The coupling functions satisfy $\mathcal{L}_{i,j}(0) = 0$ and $\partial \mathcal{L}_{i,j}(\bar{x})/\partial \bar{x} \geq 0$ for all $i \in \mathcal{I}_n, j \in \mathcal{N}_i$. In addition, they are taken to be pairwise odd, meaning that $\mathcal{L}_{i,j}(x) = -\mathcal{L}_{j,i}(-x)$ for all $x \in \mathbb{R}$, $i \in \mathcal{I}_n$, and

$j \in \mathcal{N}_i$ [41]. We note that these assumptions are automatically satisfied for linear multi-agent systems on undirected graphs.

We study incremental stability under the assumption that the system with output $y = \mathcal{L}_1(x_1)$ satisfies the observability rank condition (see the discussion below Corollary 3.2) at each $x \in \mathbb{R}^n_+ \setminus D$, where

$$D := \{x \in S : x_1 = x_2 = \cdots = x_n\}. \tag{32}$$

Denote the compact form of the system by

$$\dot{x} = -\mathcal{L}(x).$$

Then, from the assumption for the couplings and $\mathcal{L}_1(\cdot)$, the system is monotone on $\mathbb{R}^n_+$ and satisfies

$$-\mathbb{1}_n^{\mathrm{T}}\frac{\partial \mathcal{L}(x)}{\partial x} = -\left[\begin{array}{cccc} \frac{d\mathcal{L}_1(x_1)}{dx_1} & 0 & \cdots & 0 \end{array} \right] \preceq 0.$$

Therefore, (16) holds for $v(x) = \mathbb{1}_n$ and $\bar{\alpha}(x, \delta x) = (d\mathcal{L}_1(x_1)/dx_1)\delta x_1$. From Corollary 3.2, the system is IS.

Next, suppose that the system has a bounded solution in $\mathbb{R}^n_+$. For instance, this is true if the system has an equilibrium point in $\mathbb{R}^n_+$. From the observability assumption, the largest invariant set contained in Π of (17) is

$$\Delta = (\mathbb{R}^n_+ \times \{0\}) \cup (D \times \{\delta x \in \mathbb{R}^n_+ : \delta x_1 = 0\}),$$

given as the union of two sets. In a similar manner as Example 3.5, stability analysis can be decomposed into analysis of two different subset. For the first set, i.e., $(\mathbb{R}^n_+ \times \{0\})$, we have for any $(x^1, x^2) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ that

$$\lim_{t \to \infty} d_1(\phi(t, x^1), \phi(t, x^2)) = 0.$$

For the second set, from the definition of D in (32), for any $(x^1, x^2) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$, we have

$$\lim_{t \to \infty} \phi_1(t, x^1) = \lim_{t \to \infty} \phi_2(t, x^1) = \cdots = \lim_{t \to \infty} \phi_n(t, x^1),$$

$$\lim_{t \to \infty} \phi_1(t, x^2) = \lim_{t \to \infty} \phi_2(t, x^2) = \cdots = \lim_{t \to \infty} \phi_n(t, x^2),$$

$$\lim_{t \to \infty} d_1([\begin{array}{cccc} \phi_1(t, x^1) & 0 & \cdots & 0 \end{array}]^{\mathrm{T}},$$

$$[\begin{array}{cccc} \phi_1(t, x^2) & 0 & \cdots & 0 \end{array}]^{\mathrm{T}}) = 0.$$

In both cases, any pair of trajectories converge to each other, i.e., the system is IAS on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ but not necessarily IES.

B. Stabilizing Controller Design

In the contraction framework, the stabilization problem is studied by using so-called control contraction metrics in [31], [32], and these techniques have recently been applied to distributed control of monotone systems with the diagonal Riemannian metric [42]. The concept of control contraction metrics can be extended to sum- or max-separable functions of monotone systems using the results of Sections III and IV, which also suggests the distributed controller design based on separable functions as future work. To make this explicit, we consider the closed-loop system

$$\dot{x}(t) = f(x(t)) + Bk(x(t)), \tag{33}$$

where f is the same function considered for the monotone system Σ on convex $\mathcal{X}$, and $B \in \mathbb{R}^{n \times m}_+$. Our objective is

to characterize feedback control laws $k : \mathbb{R}^n \to \mathbb{R}^m$ that guarantee IES of the closed-loop system. In a similar manner to [31, Theorem 1], we have the following proposition for stabilizability of monotone system.

Proposition 5.1: Consider a monotone system Σ in (1) on a convex set $\mathcal{X}$. Let $\mathcal{S} \subset \mathcal{X}$ be connected. Also, let $v : \mathcal{S} \to \mathbb{R}^n_+$ be a class C^1 vector-valued function such that

1) $v(x) \succ 0$ for any $x \in \mathcal{S}$;
2) there exist a class C^1 matrix-valued function $K : \mathbb{R}^n \to \mathbb{R}^{m \times n}$ and positive constant c_v such that each off-diagonal element of $BK(x)$ is non-negative for any $x \in \mathcal{X}$, and

$$v^{\mathrm{T}}(x)\frac{\partial f(x)}{\partial x} + BK(x) + \left(\frac{\partial v(x)}{\partial x}f(x)\right)^{\mathrm{T}} \leq -c_v v^{\mathrm{T}}(x),$$
$$\frac{\partial v(x)}{\partial x}B = 0,$$

for any $x \in \mathcal{S}$;
3) $K(x)$ is integrable, i.e., there exists a vector-valued function $k : \mathbb{R}^n \to \mathbb{R}^m$ such that $\partial k(x)/\partial x = K(x)$.

If $\mathcal{S}$ is a forward invariant set of the closed-loop system (33) then the closed loop system is monotone on $\mathcal{X}$ and is IES on $\mathcal{S}$.

Proof: First, we show that the closed-loop system (33) is monotone. To this end, note that

$$\frac{\partial(f(x) + Bk(x))}{\partial x} = \frac{\partial f(x)}{\partial x} + BK(x).$$

From monotonicity of Σ, f satisfies the Kamke condition (5). Combining this with the requirement for $BK(\cdot)$ in Condition 2), it can be concluded that the closed-loop system (33) satisfies the Kamke condition in Proposition 50. Thus, the closed-loop system is monotone on $\mathcal{X}$.

Next, Conditions 1) and 2) in the statement of this theorem imply that the closed-loop system (33) satisfies all conditions of Proposition 2.1. As a result, the closed-loop system is IES on $\mathcal{S}$. ∎

As in [31], the integrability condition $\partial k(x)/\partial x = K(x)$ can be relaxed by using a path integral. We also note that non-negativity for each off-diagonal element of $BK(x)$ is only a sufficient condition to preserve monotonicity. In fact, as long as the closed-loop system is monotone, Proposition 5.1 is valid. In other worlds, even if the original system is not monotone, Proposition 5.1 may still apply.

One can extend Proposition 5.1 to IAS based on Theorem 3.1. Also, the LaSalle invariance principle argument can be extended as demonstrated by the following example.

Example 5.2 (Revisit the example in Section V-A): Consider the controlled single-integrator multi-agent system

$$\dot{x}_1 = \sum_{j \in \mathcal{N}_1} \mathcal{L}_{1,j}(x_j - x_1) + u,$$
$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} \mathcal{L}_{i,j}(x_j - x_i), \ i \neq 2,$$

where each function and symbol are defined in the example in Section V-A except for the input variable $u \in \mathbb{R}$. Its compact form is denoted by

$$\dot{x} = -\overline{\mathcal{L}} + bu.$$

Now, we consider stabilizing controller design. For $v(x) = \mathbb{1}_n$ and $K = -[1 \ 0 \ \cdots \ 0]$, the conditions in Proposition 5.1 hold except for the first inequality of Condition 2). Instead of this inequality, we have

$$-\mathbb{1}_n^{\mathrm{T}}\left(\frac{\partial\overline{\mathcal{L}}(x)}{\partial x} + bK\right) = -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \preceq 0.$$

By applying the LaSalle invariance principle argument, it is possible to show that the closed-loop system with $k(x) = -x_1$ is IAS on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$. ◁

It is also possible to obtain a max-separable version of the stabilizability condition based on Proposition 4.6. Since the proof is similar, it is omitted.

Proposition 5.3: Consider the monotone system Σ in (1) on a convex set $\mathcal{X}$. Let $\mathcal{S} \subset \mathcal{X}$ be connected. Also, let $w : \mathcal{S} \to \mathbb{R}^n_+$ be a class C^1 vector-valued function such that

1) $w(x) \succ 0$ for any $x \in \mathcal{S}$;
2) there exist a class C^1 matrix-valued function $K : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and positive constant c_w such that each off-diagonal elements of $BK(x)$ is non-negative for any $x \in \mathcal{X}$, and

$$\left(\frac{\partial f(x)}{\partial x} + BK(x)\right)w(x) - \frac{\partial w(x)}{\partial x}f(x) \leq -c_w w^{\mathrm{T}}(x),$$
$$[w, b_i] = 0, \ i \in \mathcal{I}_m,$$

for any $x \in \mathcal{S}$, where $B = [b_1 \ \cdots \ b_m]$;
3) $K(x)$ is integrable, i.e., there exists a vector-valued function $k : \mathbb{R}^n \to \mathbb{R}^m$ such that $\partial k(x)/\partial x = K(x)$.

If $\mathcal{S}$ is a forward invariant set of the closed-loop system (33), then the closed loop system is monotone on $\mathcal{X}$ and is IES on $\mathcal{S}$. ◁

C. Partial Incremental Stability

The concept of partial stability [43] has been extended towards contraction analysis in the notions of partial contraction [33], [34] or horizontal contraction [17]. For instance, partial contraction is applied to observer design [19], [21], [44] and tracking control [45], [46]. Propositions 4.5 and 4.6 can be extended for partial contraction analysis.

Consider the following system with the state z and the input x,

$$\dot{z}(t) = g(z(t), x(t)), \tag{34}$$

where $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is of class C^2, and $g(x, x) = f(x)$ for all $x \in \mathbb{R}^n$. The latter implies that, if the initial state is $z_0 = x_0$, then $z(t) = x(t)$ for all $t \in \mathbb{R}_+$. Therefore, if the system (34) is IES with respect to $z(t)$, then $z(t) \to x(t)$ as $t \to \infty$. For instance, if $g(z, x) = f(z) + k(z, h(x))$ with $k(x, h(x)) = 0$, then the system (34) becomes an observer for the system Σ with output $y = h(x)$. The system (34) is called a virtual system of Σ. The unique solution $z(t)$ to the virtual system (34) at time $t \in \mathbb{R}_+$ starting from $z(0) = z_0$ with input $x(t) = \phi(t, x_0)$ is denoted by $\bar{\phi}(t, z_0, \phi)$.

Suppose that this system is monotone on convex $\mathcal{X} \times \mathcal{X}$ with the state z and the input $u(t) = \phi(t, x_0)$ [1], i.e., the

implication

$$z^1 \preceq z^2, \ u^1 \preceq u^2$$
$$\implies \bar{\phi}(t, z^1, u^1) \preceq \bar{\phi}(t, z^2, u^2), \ \forall t \in \mathbb{R}_+ \quad (35)$$

holds for any $z^1, z^2 \in \mathcal{X}$ and continuous $u^1(t), u^2(t) \in \mathcal{X}$ for all $t \in \mathbb{R}_+$. In (35), $u^1 \preceq u^2$ means $u^1(t) \preceq u^2(t)$ for all $t \in \mathbb{R}_+$. Note that since the system Σ is monotone on convex $\mathcal{X}$, $x^1 \preceq x^2$ implies $\phi(\cdot, x^1) \preceq \phi(\cdot, x^2)$. Consequently, if $z^1 \preceq z^2$ and $x^1 \preceq x^2$ then $\bar{\phi}(\cdot, z^1, \phi^1) \preceq \bar{\phi}(\cdot, z^2, \phi^2)$, where $\phi^i := \phi(\cdot, x^i)$, $i = 1, 2$.

Now, we formally define incremental stability of the virtual system.

Definition 5.4: The virtual system (34) is said to be IES on $\mathcal{S}$ with respect to z uniformly in x if there exist positive constants k and λ such that

$$d(\bar{\phi}(t, z^1, \phi^1), \bar{\phi}(t, z^2, \phi^2)) \leq ke^{-\lambda t}d(z^1, z^2), \ \forall t \in \mathbb{R}_+$$

for any $(z^1, z^2), (x^1, x^2) \in \mathcal{S} \times \mathcal{S}$. ◁

Definition 5.4 is an extension of partial stability [43] to IES, and a similar property is called partial contraction [33], [34]. As simple applications of Propositions 4.5 and 4.6, we have the following conditions for partial stability. Since the proofs are similar to those propositions, they are omitted.

Proposition 5.5: Let $\mathcal{S} \subset \mathcal{X}$ and $\mathcal{S} \times \mathcal{S}$ be connected and forward invariant sets of the monotone system (1) on convex $\mathcal{X}$ and the virtual monotone system (34) on convex $\mathcal{X} \times \mathcal{X}$, respectively. Let $v : \mathcal{S} \to \mathbb{R}_+^n$ be a class C^1 vector-valued function such that

1) $v(z) \succ 0$ for any $z \in \mathcal{S}$;
2) there exists a positive constant c_v such that

$$v^\mathrm{T}(z)\frac{\partial g(z, x)}{\partial z} + \left(\frac{\partial v(z)}{\partial z}g(z, x)\right)^\mathrm{T} \leq -c_v v^\mathrm{T}(z)$$

for any $x, z \in \mathcal{S}$.

Then, the monotone virtual system (34) is IES on $\mathcal{S}$ with respect to z uniformly in x. ◁

Proposition 5.6: Let $\mathcal{S} \subset \mathcal{X}$ and $\mathcal{S} \times \mathcal{S}$ be connected and forward invariant sets of the monotone system (1) on convex $\mathcal{X}$ and the virtual monotone system (34) on convex $\mathcal{X} \times \mathcal{X}$, respectively. Let $w : \mathcal{S} \to \mathbb{R}_+^n$ be a class C^1 vector-valued function such that

1) $w(z) \succ 0$ for any $z \in \mathcal{S}$;
2) there exists a positive constant c_w such that

$$\frac{\partial g(z, x)}{\partial z}w(z) - \frac{\partial w(z)}{\partial z}g(z, x) \preceq -c_w w(z)$$

for any $z, x \in \mathcal{S}$.

Then, the monotone virtual system (34) is IES on $\mathcal{S}$ with respect to z uniformly in x. ◁

In a similar manner, one can define IS and IAS with respect to z uniformly in x and extend Theorems 3.1 and 3.6, which is demonstrated by the following example.

Example 5.7 (Revisit the example in Section V-A): We consider the design of an observer for the single-integrator multi-agent system in Example 5.2 with $u = 0$ and output $y = x_1$, i.e.,

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} \mathcal{L}_{i,j}(x_j - x_i), \ i \in \mathcal{I}_n,$$
$$y = x_1.$$

Then, from Corollary 3.2, the system is IS on $\mathbb{R}_+^n$ with $v(x) = \mathbb{1}_n$. Suppose that the system has a bounded solution. Then, from IS, any solution to the system $\phi(t, x_0)$ is bounded for any $t \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}_+^n$.

Here, we consider the following virtual system

$$\dot{z}_1 = \sum_{j \in \mathcal{N}_1} \mathcal{L}_{1,j}(z_j - z_1) - (z_1 + y),$$
$$\dot{z}_i = \sum_{j \in \mathcal{N}_i} \mathcal{L}_{i,j}(z_j - z_i), \ i \neq 1.$$

According to Example 5.2, this system with $y = 0$ is IAS on $\mathbb{R}_+^n \times \mathbb{R}_+^n$. Note that $\partial g(z, x)/\partial z$ does not depend on y and thus x. Therefore, for any bounded y, it is possible to conclude that the virtual system is IAS on $\mathbb{R}_+^n \times \mathbb{R}_+^n$ with respect to z uniformly in x. This implies that the virtual system is an observer of the multi-agent system. ◁

VI. Conclusion

In this paper, we have presented sum- and max-separable incremental stability conditions for monotone systems in terms of Lie derivatives of differential one-forms and Lie brackets of vector fields. In particular, for incremental asymptotic stability, we have given sufficient conditions, and for incremental exponential stability, we have provided necessary and sufficient conditions under several assumptions. Moreover, we have shown that a diagonal contraction Riemannian metric can be constructed on the basis of those sum- and max-separable functions. In other words, we have provided natural extensions of separable conditions for stability of positive LTI systems. Finally, based on the fact that variational systems can be viewed as LTV systems, we have established separable exponential stability conditions for positive LTV systems. Future work includes the converse analysis of IAS, contraction analysis on general cones, and the development of methods for finding vector-valued functions that satisfy sum- or max-separable conditions.

Appendix A
Proofs for IAS

Here, our goal is proving Theorems 3.1 and 3.6. The proofs are based on Proposition 2.6. Therefore, first we prove this proposition and then each theorem.

A. Proof of Proposition 2.6

Proof: We only prove for the case $i = 2$, $j = 1$ as the proofs for the other combinations are similar.

Recall that, for a given $(x, \delta x) \in \mathcal{S} \times \mathbb{R}^n$, each separable function can be viewed as a norm of the vector $[v_1(x)\delta x_1 \cdots v_n(x)\delta x_n]^\mathrm{T} \in \mathbb{R}^n$. In fact, $M_1(x, \delta x)$ and $M_2(x, \delta x)$ correspond to its 1 and 2 vector norms, respectively.

From the equivalence of norms on $\mathbb{R}^n$, there exist positive constants $\underline{a}_{2,1}$ and $\overline{a}_{2,1}$ such that, for all $(x, \delta x) \in \mathcal{S} \times \mathbb{R}^n$,

$$\underline{a}_{2,1} M_2(x, \delta x) \leq M_1(x, \delta x) \leq \overline{a}_{2,1} M_2(x, \delta x). \tag{36}$$

To show that these inequalities are preserved for the induced distances d_1 and d_2, let

$$E_1(x^1, x^2) := \left\{ \int_0^1 M_1\left(\gamma(s), \frac{d\gamma(s)}{ds}\right) ds : \gamma \in \Gamma(x^1, x^2) \right\}.$$

Then, by the above definition, for any $\varepsilon \in E_1(x^1, x^2)$, there exists a piecewise C^1 path $\gamma \in \Gamma(x^1, x^2)$ such that

$$\varepsilon = \int_0^1 M_1\left(\gamma(s), \frac{d\gamma(s)}{ds}\right) ds. \tag{37}$$

For this γ, we obtain from (36) that

$$\underline{a}_{2,1} M_2\left(\gamma(s), \frac{d\gamma(s)}{ds}\right) \leq M_1\left(\gamma(s), \frac{d\gamma(s)}{ds}\right), \tag{38}$$

for any $s \in [0, 1]$. Integrating the above (for s ranging over the interval $[0, 1]$) preserves the inequality. Moreover, we note that the path γ does not necessarily achieve the infimum in the induced distance d_2, see (10) for $i = 2$. As a result, (38) and (37) lead to

$$\underline{a}_{2,1} d_2(x^1, x^2) \leq \varepsilon.$$

Since ε is arbitrary, $\underline{a}_{2,1} d_2(x^1, x^2)$ is a lower bound for $E_1(x^1, x^2)$. Therefore, $\underline{a}_{2,1} d_2(x^1, x^2) \leq d_1(x^1, x^2)$. In a similar manner, it is possible to show the second inequality of (13) for $i = 2$ and $j = 1$. ∎

B. Proofs of Theorem 3.1 and Proposition 4.5

Proof: As a preliminary step, we analyze stability of the prolongation of Σ. From (14) and (15), the time-derivative of $v^{\mathrm{T}}(x(t))\delta x(t)$ along solutions to the prolonged system of Σ satisfies

$$\frac{d(v^{\mathrm{T}}(x(t))\delta x(t))}{dt} = L_f(v^{\mathrm{T}}(x(t))\delta x(t))$$
$$\leq -\alpha(v^{\mathrm{T}}(x(t))\delta x(t)) \tag{39}$$

for any $(x(t), \delta x(t)) \in \mathcal{S} \times \mathbb{R}_+^n$ and $t \in \mathbb{R}_+$. Then, the use of the comparison principle (e.g., [30, Lemma 3.4]) leads to

$$v^{\mathrm{T}}(x(t))\delta x(t) \leq v^{\mathrm{T}}(x_0)\delta x_0, \tag{40}$$

for any $t \in \mathbb{R}_+$ and $(x_0, \delta x_0) \in \mathcal{S} \times \mathbb{R}_+^n$. In fact, as α is positive definite, it follows from [30, Lemma 4.4] that (39) implies the existence of a class $\mathcal{KL}$ function β such that

$$v^{\mathrm{T}}(x(t))\delta x(t) \leq \beta(v^{\mathrm{T}}(x_0)\delta x_0, t) \tag{41}$$

for any $t \in \mathbb{R}_+$ and $(x_0, \delta x_0) \in \mathcal{S} \times \mathbb{R}_+^n$. Note that (40) and (41) do not necessarily hold for any $(x_0, \delta x_0) \in \mathcal{S} \times \mathbb{R}^n$, i.e., δx_0 is required to be in $\mathbb{R}_+^n$.

We are ready to prove Theorem 3.1 for IAS. Take any $\varepsilon > 0$ in (12). Since $v(x) \succ 0$ on $\mathcal{S}$, there exists a class C^1 path $\bar{\gamma}(s) \in \Gamma(x^1, x^2)$ satisfying (12) (note that the induced distance $d_2(\cdot, \cdot)$ is considered here). We choose

$(\bar{\gamma}(s), |d\bar{\gamma}(s)/ds|)$ as the initial state $(x_0, \delta x_0)$ of the prolongation of Σ. From (3), the corresponding solution is

$$(x(t), \delta x(t)) = \left(\phi(t, \bar{\gamma}(s)), \frac{\partial \phi(t, \bar{\gamma}(s))}{\partial x} \left| \frac{d\bar{\gamma}(s)}{ds} \right| \right). \tag{42}$$

Note that $(\bar{\gamma}(s), |d\bar{\gamma}(s)/ds|)$ is in $\mathcal{S} \times \mathbb{R}_+^n$ for any $(x^1, x^2) \in \mathcal{S} \times \mathcal{S}$ and $s \in [0, 1]$. According to Remark 2.2, the solution (42) is in $\mathcal{S} \times \mathbb{R}_+^n$ for any $t \in \mathbb{R}_+$, which implies $\delta x(t) = |\delta x(t)|$ and, consequently,

$$\frac{\partial \phi(t, \bar{\gamma}(s))}{\partial x} \left| \frac{d\bar{\gamma}(s)}{ds} \right| = \left| \frac{\partial \phi(t, \bar{\gamma}(s))}{\partial x} \left| \frac{d\bar{\gamma}(s)}{ds} \right| \right|. \tag{43}$$

At this point, observe that $\phi(t, \bar{\gamma}(s))$ (when regarded as a function of s) is a path in $\Gamma(\phi(t, x^1), \phi(t, x^2))$. Then, consider

$$\underline{a}_{2,1} M_2\left(\phi(t, \bar{\gamma}(s)), \frac{d\phi(t, \bar{\gamma}(s))}{ds} \right)$$
$$\leq M_1\left(\phi(t, \bar{\gamma}(s)), \frac{d\phi(t, \bar{\gamma}(s))}{ds} \right), \tag{44a}$$
$$= M_1\left(\phi(t, \bar{\gamma}(s)), \left| \frac{\partial \phi(t, \bar{\gamma}(s))}{\partial x} \frac{d\bar{\gamma}(s)}{ds} \right| \right), \tag{44b}$$
$$\leq M_1\left(\phi(t, \bar{\gamma}(s)), \left| \frac{\partial \phi(t, \bar{\gamma}(s))}{\partial x} \right| \left| \frac{d\bar{\gamma}(s)}{ds} \right| \right), \tag{44c}$$

where (44a) follows from (36) and (44b) is the result of the chain rule and the observation that $M_1(x, \delta x) = M_1(x, |\delta x|)$ for all $(x, \delta x) \in \mathcal{S} \times \mathbb{R}^n$. Finally, (44c) is a consequence of (6).

Observing that the arguments in M_1 in (44c) are $x(t)$ and $\delta x(t)$, respectively (recall (42) and (43)), it follows from (40) and $(x_0, \delta x_0) = (\bar{\gamma}(s), |d\bar{\gamma}(s)/ds|)$ that

$$\underline{a}_{2,1} M_2\left(\phi(t, \bar{\gamma}(s)), \frac{d\phi(t, \bar{\gamma}(s))}{ds} \right)$$
$$\leq M_1\left(\bar{\gamma}(s), \left| \frac{d\bar{\gamma}(s)}{ds} \right| \right), \tag{45a}$$
$$\leq \overline{a}_{2,1} M_2\left(\bar{\gamma}(s), \left| \frac{d\bar{\gamma}(s)}{ds} \right| \right), \tag{45b}$$

where (45a) again follows from (36). We recall that $\phi(t, \bar{\gamma}(s))$ is a path in $\Gamma(\phi(t, x^1), \phi(t, x^2))$, but not necessarily a geodesic. Thus, by taking path integrals in (45), hereby using (10) for $i = 2$ and (12), we have

$$d_2(\phi(t, x^1), \phi(t, x^2)) \leq (\overline{a}_{2,1}/\underline{a}_{2,1})(1 + \varepsilon)d_2(x^1, x^2)$$

for any $t \in \mathbb{R}_+$ and $(x^1, x^2) \in \mathcal{S} \times \mathcal{S}$. Thus, the monotone systems is IS.

Following a similar reasoning, but applying (41) (instead of (40)) after (44), we obtain

$$d_2(\phi(t, x^1), \phi(t, x^2)) \leq \frac{1}{\underline{a}_{2,1}} \int_0^1 \beta\left(v^{\mathrm{T}}(\bar{\gamma}(s)) \frac{d\bar{\gamma}(s)}{ds}, t \right) ds,$$

after taking path integrals. Consequently, as β is of class $\mathcal{KL}$, $\lim_{t \to \infty} d_2(\phi(t, x^1), \phi(t, x^2)) = 0$ and the monotone system Σ is IAS with respect to $d_2(\cdot, \cdot)$ on $\mathcal{S}$. From the equivalence of induced distances (recall Proposition 2.6), we also have IAS with respect to $d_1(\cdot, \cdot)$.

Finally, we prove Proposition 4.5 for IES. From (24),

$$v^{\mathrm{T}}(x(t))\delta x(t) \leq e^{-c_v t} v^{\mathrm{T}}(x_0)\delta x_0, \tag{46}$$

such that the application of this result to (44c) gives

$$\underline{a}_{2,1} M_2 \left(\phi(t, \bar{\gamma}(s)), \frac{d\phi(t, \bar{\gamma}(s))}{ds} \right)$$

$$\leq e^{-c_v t} M_1 \left(\bar{\gamma}(s), \left| \frac{d\bar{\gamma}(s)}{ds} \right| \right), \quad (47a)$$

$$\leq \bar{a}_{2,1} e^{-c_v t} M_2 \left(\bar{\gamma}(s), \left| \frac{d\bar{\gamma}(s)}{ds} \right| \right), \quad (47b)$$

where again (36) is used to obtain (47b). Then, taking path integrals, using (10) for $i = 2$ and (12) as in the proof of IS, we obtain

$$d_2(\phi(t, x^1), \phi(t, x^2)) \leq \frac{a_2(1 + \varepsilon)}{a_1} e^{-c_v t} d_2(x^1, x^2).$$

Therefore, the monotone system Σ is IES with respect to $d_2(\cdot, \cdot)$, and consequently $d_1(\cdot, \cdot)$ from Proposition 2.6. ∎

C. Proofs of Theorem 3.6 and Proposition 4.6

Proof: First, we consider Theorem 3.6 for IAS. From the definition of $\mathcal{J}(x, \delta x)$ below (20) and $w(x) \succ 0$, we have

$$\frac{\delta x_i}{w_i(x)} \leq \frac{\delta x_j}{w_j(x)} \iff w_j(x)\delta x_i \leq w_i(x)\delta x_j \quad (48)$$

on $\mathcal{S} \times \mathbb{R}_+^n$ for any $j \in \mathcal{J}(x, \delta x)$ and $i \in \mathcal{I}_n$. In addition, due to the Kamke condition for monotonicity (5), (48) leads to

$$\frac{\partial f_j(x)}{\partial x_i} w_j(x)\delta x_i \leq \frac{\partial f_j(x)}{\partial x_i} w_i(x)\delta x_j$$

on $\mathcal{S} \times \mathbb{R}_+^n$ for any $j \in \mathcal{J}(x, \delta x)$ and $i \in \mathcal{I}_n$. As a result, we obtain

$$\frac{\partial f_j(x)}{\partial x} w_j(x)\delta x = \sum_{i \in \mathcal{I}_n} \frac{\partial f_j(x)}{\partial x_i} w_j(x)\delta x_i,$$

$$\leq \sum_{i \in \mathcal{I}_n} \frac{\partial f_j(x)}{\partial x_i} w_i(x)\delta x_j$$

$$= \frac{\partial f_j(x)}{\partial x} w(x)\delta x_j \quad (49)$$

on $\mathcal{S} \times \mathbb{R}_+^n$ for any $j \in \mathcal{J}(x, \delta x)$.

In the evaluation of the Dini derivative of $W(\cdot, \cdot)$, let $l \in \mathcal{J}(x, \delta x)$ be an index that achieves the maximum in (20). Then, by recalling that $w(x) \succ 0$ by assumption and the use of (49), we obtain

$$D^+ W(x, \delta x) \leq \frac{\delta x_l}{w_l^2(x)} [w, f]_l(x), \quad (50)$$

where the definition of the Lie bracket in (21) is used. As a result of (22), this leads to

$$D^+ W(x, \delta x) \leq -\alpha \left(\frac{\delta x_l}{w_l(x)} \right) = -\alpha \big(W(x, \delta x) \big) \quad (51)$$

where the equality is a result of $l \in \mathcal{J}(x, \delta x)$. Then, by applying applying the comparison principles [30, Lemma 3.4] and [30, Lemma 4.4], respectively, we get

$$\max_{j \in \mathcal{I}_n} \frac{\delta x_j(t)}{w_j(x(t))} \leq \max_{i \in \mathcal{I}_n} \frac{\delta x_{0,i}}{w_i(x_0)}, \quad (52)$$

$$\max_{j \in \mathcal{I}_n} \frac{\delta x_j(t)}{w_j(x(t))} \leq \beta \left(\max_{i \in \mathcal{I}_n} \frac{\delta x_{0,i}}{w_i(x_0)}, t \right), \quad (53)$$

for any $t \in \mathbb{R}_+$ and $(x_0, \delta x_0) \in \mathcal{S} \times \mathbb{R}_+^n$. In the above, β is a function of class $\mathcal{KL}$. In a similar manner to the proof of Theorem 3.1, it is possible to show IS and IAS, respectively.

Next, we consider Proposition 4.6 for IES. By multiplying $\delta x_i / w_i^2(x)$ to each element of (25), we have

$$\frac{\delta x_i}{w_i^2(x)} [w, f]_i(x) \leq -c_w \frac{\delta x_i}{w_i(x)}, \ \forall i \in \mathcal{I}_n. \quad (54)$$

The use of this result in (50), together with the definition of $\mathcal{J}(x, \delta x)$ and (19), yield

$$D^+ W(x, \delta x) \leq -c_w \frac{\delta x_l}{w_l(x)} = -c_w W(x, \delta x)$$

for any $(x, \delta x) \in \mathcal{S} \times \mathbb{R}_+^n$. Thus, IES can be proven in a similar manner to the proof of Theorem 3.1. ∎

APPENDIX B
PROOFS FOR IES

We provide the proof of Theorem 4.3 in the following order:
a) $\implies$ both b) and c) $\implies$ d) $\implies$ a).

A. a) $\implies$ b)

The proof of a) $\implies$ b) is given below. Note that we do not assume backward invariance of $\mathcal{S}$.

Proof of a) $\implies$ b): Let $\hat{\gamma}(s) = x^1 + s(x^2 - x^1)$, $s \in [0, 1]$ be a line segment that connects $x^1, x^2 \in \mathcal{S}$. For a given $s \in [0, 1]$, consider the solution $\phi(t, \hat{\gamma}(s))$ of the monotone system Σ and note that, due to convexity and forward invariance of $\mathcal{S}$, $\phi(t, \hat{\gamma}(s)) \in \mathcal{S}$ for any $t \in \mathbb{R}_+$. Since $\phi(t, \hat{\gamma}(s))$ is a (not necessarily straight) path connecting $\phi(t, x^2)$ and $\phi(t, x^1)$, for any $t \in \mathbb{R}_+$ and $x^1, x^2 \in \mathcal{S}$, we have

$$\phi(t, x^2) - \phi(t, x^1) = \int_0^1 \frac{d\phi(t, \hat{\gamma}(s))}{ds} ds,$$

$$= \int_0^1 \frac{\partial \phi(t, \hat{\gamma}(s))}{\partial x} \frac{d\hat{\gamma}(s)}{ds} ds,$$

$$= \int_0^1 \Phi(t, \hat{\gamma}(s))(x^2 - x^1) ds, \quad (55)$$

where (3) and the definition of the straight line $\hat{\gamma}(s)$ are used to obtain the last equality. The positive invariance of $\mathcal{S}$ and (6) imply $\phi(t, x^2) - \phi(t, x^1) \succeq 0$ for $x^2 - x^1 \succeq 0$. Then,

$$d_1(\phi(t, x^1), \phi(t, x^2)) = |\phi(t, x^2) - \phi(t, x^1)|_1$$

$$= \mathbb{1}_n^{\mathrm{T}} (\phi(t, x^2) - \phi(t, x^1)) \quad (56)$$

for any $t \in \mathbb{R}_+$ and $(x^1, x^2) \in \mathcal{S} \times \mathcal{S}$ such that $x^2 \succeq x^1$. Therefore, the definition of IES, (56) and (55) yield

$$\int_0^1 \mathbb{1}_n^{\mathrm{T}} \Phi(t, \hat{\gamma}(s))(x^2 - x^1) ds = \mathbb{1}_n^{\mathrm{T}} (\phi(t, x^2) - \phi(t, x^1))$$

$$\leq k e^{-\lambda t} \mathbb{1}_n^{\mathrm{T}} (x^2 - x^1). \quad (57)$$

Suppose that a in Assumption 4.1 is positive for some standard basis vector e_i, i.e. $x \in \mathcal{S}$ implies $x + ae_i \in \mathcal{S}$; the negative case will be discussed later. From convexity of

$\mathcal{S}$, $x + he_i \in \mathcal{S}$ for any $0 < h \leq a$. Then, by substituting $x^1 = x$ and $x^2 = x + he_i$ into (57), it follows that

$$\int_0^1 \mathbb{1}_n^{\mathrm{T}} \Phi(t, x + hse_i)e_i ds \leq ke^{-\lambda t} \mathbb{1}_n^{\mathrm{T}} e_i = ke^{-\lambda t}.$$

After the change of variables $\tilde{s} = hs$, this yields

$$\phi(h) := \frac{1}{h} \int_0^h \mathbb{1}_n^{\mathrm{T}} \Phi(t, x + \tilde{s}e_i)e_i d\tilde{s} \leq ke^{-\lambda t}. \quad (58)$$

Notice that this holds for any $0 < h \leq a$. Since $\mathbb{1}_n^{\mathrm{T}} \Phi(t, x + \tilde{s}e_i)e_i$ is continuous at $\tilde{s} = 0$, it follows from the fundamental theorem of calculus [47, Theorem 6.20] that

$$\phi(0) = \lim_{h \to 0} \frac{1}{h} \int_0^h \mathbb{1}_n^{\mathrm{T}} \Phi(t, x + \tilde{s}e_i)e_i d\tilde{s} = \mathbb{1}_n^{\mathrm{T}} \Phi(t, x)e_i.$$

Moreover, as a result of (58) and continuity of ϕ, we obtain

$$\mathbb{1}_n^{\mathrm{T}} \Phi(t, x)e_i \leq ke^{-\lambda t}.$$

Then, the monotonicity property (6) implies that we have $\Phi_{i,j}(t, x) \leq ke^{-\lambda t}$ for any $j \in \mathcal{I}_n$, $t \in \mathbb{R}_+$ and $x \in \mathcal{S}$. Next, even if a is negative, one has the same conclusion by substituting $x^1 = x + he_i$ for $a \leq h < 0$ and $x^2 = x$ into (57) and using the change of variables $\tilde{s} = h(1 - s)$. In summary, we obtain

$$\Phi_{i,j}(t, x) \leq ke^{-\lambda t}, \ \forall i, j \in \mathcal{I}_n \quad (59)$$

Now, we are ready to construct a function $v(\cdot)$ satisfying the conditions in this lemma. Inspired by the converse proof for IES of positive LTI systems in [10, Theorem 15], define

$$v(x) := \int_{t-\delta}^t \Phi^{\mathrm{T}}(t - \tau, x) \mathbb{1}_n d\tau, \quad (60)$$

for some $\delta > 0$. The definition (60) does not depend on t as, by changing variables $r = t - \tau$, we have

$$v(x) = \int_0^\delta \Phi^{\mathrm{T}}(r, x) \mathbb{1}_n dr. \quad (61)$$

From (61), it can be observed that $v(x)$ is of class C^1 (as a function of x) for any $\delta > 0$, since $\Phi^{\mathrm{T}}(r, x)$ is.

This $v(x)$ is upper bounded. From (59), we obtain

$$\int_0^\delta \Phi(r, x)dr \preceq \int_0^\delta ke^{-\lambda r} \mathbb{1}_n \mathbb{1}_n^{\mathrm{T}} dr$$

$$= \frac{k}{\lambda}(1 - e^{-\lambda\delta}) \mathbb{1}_n \mathbb{1}_n^{\mathrm{T}} \preceq \frac{k}{\lambda} \mathbb{1}_n \mathbb{1}_n^{\mathrm{T}}, \ \forall x \in S.$$

As a result, $v(x) \preceq \bar{c}\mathbb{1}_n$, $\bar{c} = kn/\lambda$ on $\mathcal{S}$ for any $\delta > 0$.

Next, we show the lower boundedness of $v(x)$ for sufficiently large $\delta > 0$. From the boundedness of $\partial f(x)/\partial x$ and the forward invariance of $\mathcal{S}$, there exists a positive constant c_f such that $-c_f \leq \partial f_i(\phi(t, x))/\partial x_j$ for any $i, j \in \mathcal{I}_n$, $x \in \mathcal{S}$ and $t \in \mathbb{R}_+$. From (3), (6) and this lower bound, we have

$$\mathbb{1}_n^{\top} \frac{d\Phi(t - \tau, x)}{d\tau} = -\mathbb{1}_n^{\top} \frac{\partial f(\phi(t - \tau, x))}{\partial x} \Phi(t - \tau, x)$$

$$\preceq c_f n \mathbb{1}_n^{\mathrm{T}} \Phi(t - \tau, x), \ t \geq \tau,$$

where $\preceq$ is the element-wise inequality for matrices. Integration with respect to τ from $t - \delta$ to t yields

$$\mathbb{1}_n^{\top} - \mathbb{1}_n^{\top} \Phi(\delta, x) \preceq c_f n v^{\top}(x), \quad (62)$$

where the definition (60) of $v(x)$ is used. From (59), the left hand side satisfies

$$\mathbb{1}_n^{\top} - nke^{\lambda\delta} \mathbb{1}_n^{\top} \preceq \mathbb{1}_n^{\top} - \mathbb{1}_n^{\top} \Phi(\delta, x). \quad (63)$$

Therefore, for any c satisfying $0 < c < 1$, there exists a sufficiently large $\delta > 0$ such that

$$c\mathbb{1}_n \preceq \mathbb{1}_n^{\top} - nke^{-\lambda\delta} \mathbb{1}_n^{\top}. \quad (64)$$

Thus, we have $\underline{c}\mathbb{1}_n \preceq v(x)$, $\underline{c} := c/c_f n$ on $\mathcal{S}$.

It remains to show that v satisfies (24) for sufficiently large $\delta > 0$. To do so, substitute $x = \phi(r, \bar{x})$ into (60) to obtain

$$v(\phi(r, \bar{x})) = \int_{t-\delta}^t \Phi^{\mathrm{T}}(t - \tau, \phi(r, \bar{x})) \mathbb{1}_n d\tau \quad (65)$$

for $r \in \mathbb{R}_+$ and $\bar{x} \in \mathcal{S}$. From (4) and non-singularity of the transition matrix, (65) can be written as

$$v(\phi(r, \bar{x})) = \Phi^{-\mathrm{T}}(r, \bar{x}) \int_{t-\delta}^t \Phi^{\mathrm{T}}(t + r - \tau, \bar{x}) \mathbb{1}_n d\tau. \quad (66)$$

To simplify notation, fix $\bar{x} \in \mathcal{S}$ and define

$$G_t(r) := \int_{t-\delta}^t \Phi(t + r - \tau, \bar{x})d\tau. \quad (67)$$

Then, taking derivatives with respect to r on both sides of (66) yields

$$\frac{\partial v(\phi(r, \bar{x}))}{\partial \bar{x}} f(\phi(r, \bar{x}))$$

$$= \frac{\partial \Phi^{-\mathrm{T}}(r, \bar{x})}{\partial r} G_t^{\mathrm{T}}(r) \mathbb{1}_n + \Phi^{-\mathrm{T}}(r, \bar{x}) \frac{dG_t^{\mathrm{T}}(r)}{dr} \mathbb{1}_n. \quad (68)$$

We will show that (68) implies (24) by rewriting the two terms on the right-hand side. For the first term, we will use that (3) leads to

$$\frac{\partial \Phi^{-1}(r, \bar{x})}{\partial r} = -\Phi^{-1}(r, \bar{x}) \frac{\partial \Phi(r, \bar{x})}{\partial r} \Phi^{-1}(r, \bar{x})$$

$$= -\Phi^{-1}(r, \bar{x}) \frac{\partial f(\phi(r, \bar{x}))}{\partial \bar{x}}. \quad (69)$$

For the second term, note that the use of (3) and (67) leads to

$$\frac{dG_t(r)}{dr} = \int_{t-\delta}^t \frac{\partial \Phi(t + r - \tau, \bar{x})}{\partial r} d\tau,$$

$$= -\int_{t-\delta}^t \frac{\partial \Phi(t + r - \tau, \bar{x})}{\partial \tau} d\tau,$$

$$= -\Phi(r, \bar{x}) + \Phi(r + \delta, \bar{x}). \quad (70)$$

Returning to (68), the substitution of the results (69) and (70), as well as the use of the definitions (66) and (67), can be shown to lead to

$$\frac{\partial v(\phi(r, \bar{x}))}{\partial \bar{x}} f(\phi(r, \bar{x})) = -\frac{\partial^{\mathrm{T}} f(\phi(r, \bar{x}))}{\partial \bar{x}} v(\phi(r, \bar{x})) - \mathbb{1}_n$$

$$+ \Phi^{-\mathrm{T}}(r, \bar{x}) \Phi^{\mathrm{T}}(r + \delta, \bar{x}) \mathbb{1}_n. \quad (71)$$

for any $r \in \mathbb{R}_+$ and $\bar{x} \in \mathcal{S}$. Especially, when $r = 0$, this simplifies (with $\Phi(0, \bar{x}) = I_n$) to

$$\frac{\partial v(\bar{x})}{\partial \bar{x}} f(\bar{x}) + \frac{\partial^{\mathrm{T}} f(\bar{x})}{\partial \bar{x}} v(\bar{x}) = -\mathbb{1}_n + \Phi^{\mathrm{T}}(\delta, \bar{x}) \mathbb{1}_n.$$

From (63) and (64), for any c satisfying $0 < c < 1$, there exists a sufficiently large $\delta > 0$ such that

$$\frac{\partial v(\bar{x})}{\partial \bar{x}} f(\bar{x}) + \frac{\partial^{\mathrm{T}} f(\bar{x})}{\partial \bar{x}} v(\bar{x}) \preceq -c\mathbb{1}_n. \tag{72}$$

Since $v(x) \preceq \bar{c}\mathbb{1}_n$, i.e., $-c\mathbb{1}_n \preceq -(c/\bar{c})v(x)$, the condition (24) holds for $c_v = c/\bar{c}$. ∎

B. a) $\implies$ c)

For the proof of a) $\implies$ c), we require the backward invariance of $\mathcal{S}$.

Proof of a) $\implies$ c): From the equivalence of distances, the monotone system Σ is IES with respect to the distance $d_1(x^1, x^2) = |x^1 - x^2|_1$. According to the proof of a) $\implies$ b), we have (59), i.e., the system

$$\delta\dot{\lambda}(t) = \frac{\partial f(\phi(t, x))}{\partial x} \delta\lambda(t)$$

is exponentially stable with respect to $\delta\lambda$. To construct a max-separable function, we use its dual LTV system [48] given as

$$\delta\dot{p}(t) = \frac{\partial f^{\mathrm{T}}(\phi(-t, x))}{\partial x} \delta p(t). \tag{73}$$

From the backward invariance assumption, this system is defined for any $x \in \mathcal{S}$, and its solution reads

$$\delta p(t) = \left(\frac{\partial \phi(-t, x_0)}{\partial x} \right)^{-\mathrm{T}} \delta p_0.$$

For LTV systems, it is known that exponential stability of a system and its dual are equivalent [48]. Therefore, the system (73) is exponentially stable, i.e., there exist positive constants k and λ such that

$$\left(\frac{\partial \phi(-t, x)}{\partial x} \right)^{-\mathrm{T}} \leq ke^{-\lambda t} \mathbb{1}_n \mathbb{1}_n^{\mathrm{T}}. \tag{74}$$

Now, define

$$w(x) := \int_{t-\delta}^{t} \left(\frac{\partial \phi(-t+\tau, x)}{\partial x} \right)^{-1} \mathbb{1}_n d\tau, \tag{75}$$

where $\delta > 0$. Note that Remark 2.2 is also applicable to system (73), i.e., each element of $(\partial \phi(-t, x_0)/\partial x)^{-1}$ is non-negative for any $t \in \mathbb{R}_+$. Therefore, similar to the proof of a) $\implies$ b), by changing variables $r = t - \tau$, one can show from (74) that this $w(\cdot)$ satisfies $\underline{c}\mathbb{1}_n \preceq w(x) \preceq \bar{c}\mathbb{1}_n$ for some $0 < \underline{c} \leq \bar{c}$ for sufficiently large $\delta > 0$. Moreover, $w(x)$ is of class C^1 for any $x \in S$ and $\delta > 0$.

Next, substitute $x = \phi(-r, \bar{x})$, $r \in \mathbb{R}_+$, $\bar{x} \in \mathcal{S}$ into (75) to obtain

$$w(\phi(-r, \bar{x})) = \int_{t-\delta}^{t} \left(\frac{\partial \phi(-t+\tau, x)}{\partial x} \right)^{-1} \Bigg|_{x = \phi(-r, \bar{x})} \mathbb{1}_n d\tau.$$

By computing its derivative with respect to r, again following a similar approach as in the proof of a) $\implies$ b), we have

$$-\frac{\partial w(\phi(-r, \bar{x}))}{\partial \bar{x}} f(\phi(-r, \bar{x}))$$
$$= -\frac{\partial f(\phi(-r, \bar{x}))}{\partial \bar{x}} w(\phi(-r, \bar{x})) - \mathbb{1}_n$$
$$+ \frac{\partial \phi(-r, x)}{\partial x} \left(\frac{\partial \phi(-r-\delta, x)}{\partial x} \right)^{-1}$$

for any $r \in \mathbb{R}_+$ and $\bar{x} \in \mathcal{S}$. Especially, for $r = 0$, this can be written as

$$[w, f](x) = -\mathbb{1}_n + \left(\frac{\partial \phi(-\delta, x)}{\partial x} \right)^{-1}, \ \forall x \in \mathcal{S},$$

In a similar manner as the proof of a) $\implies$ b), from exponential stability, one can conclude that for any $0 < c < 1$, there exists a sufficiently large $\delta > 0$ such that

$$[w, f](x) \preceq -c\mathbb{1}_n \tag{76}$$

where the definition (21) is used. The proof can then be completed by again following the same ideas as in the proof of a) $\implies$ b). ∎

C. b) and c) $\implies$ d)

Proof: Define $p_i(x) := v_i(x)/w_i(x)$ ($i \in \mathcal{I}_n$). Then, it is clear that there exist positive constants $\underline{c}$ and $\bar{c}$ such that $\underline{c}\mathbb{1}_n \preceq p(x) \preceq \bar{c}\mathbb{1}_n$. By using (72) and (76), it can be shown that

$$\frac{\partial V(x, \delta x)}{\partial x} f(x) + \frac{\partial V(x, \delta x)}{\partial \delta x} \frac{\partial f(x)}{\partial x} \delta x$$
$$= -\sum_{i \in \mathcal{I}_n} \left(\frac{v_i(x)}{w_i^2(x)} + \frac{1}{w_i(x)} \right) \delta x_i^2$$
$$- \sum_{i \neq j} w_i(x) v_j(x) \frac{\partial f_j}{\partial x_i} \left(\frac{\delta x_i}{w_i(x)} - \frac{\delta x_j}{w_j(x)} \right)^2$$
$$\leq -c \sum_{i \in \mathcal{I}_n} \left(\frac{v_i(x)}{w_i^2(x)} + \frac{1}{w_i(x)} \right) \delta x_i^2,$$

where (5) and $v(x), w(x) \succ 0$ are used to obtain the inequality. From boundedness and uniformly positive definiteness of $v(x)$, $w(x)$, and $p(x)$, it is clear that there exists a positive constant c_p such that (27) holds. ∎

D. d) $\implies$ a)

Proof: For not-necessarily monotone systems, the paper [17] shows that Condition d) implies IES with respect to the distance in (11) with $v(\cdot) = p(\cdot)$ in (8), which we denote by $\bar{d}_2(x^1, x^2)$. Next, from (26), it is possible to show that

$$\underline{c}|x^1 - x^2|_2 \leq \bar{d}_2(x^1, x^2) \leq \bar{c}|x^1 - x^2|_2$$

for any $(x^1, x^2) \in \mathcal{S} \times \mathcal{S}$. The proof is similar to that of Proposition 2.6 and thus is omitted. Therefore, IES with respect to $\bar{d}_2(x^1, x^2)$ and $|x^1 - x^2|_2$ are equivalent. In summary, we have d) $\implies$ a). ∎

APPENDIX C
PROOFS OF COROLLARY 4.8

The proof of Corollary 4.8 is similar to that of Theorem 4.3, and we prove it in the following order: 1) $\implies$ both 2) and 3) $\implies$ 4) $\implies$ 1). Note that 4) $\implies$ 1) is a specific case of a Lyapunov theorem for LTV systems; see, e.g. [30, Example 4.21], and thus the proof is omitted. Also, the proof of both 2) and 3) $\implies$ 4) is similar to the proof in Appendix B-C. Namely, $p(t)$ in Condition 4) can be constructed as $p_i(t) := v_i(t)/w_i(t)$ ($i \in \mathcal{I}_n$) from $v(t)$

and $w(t)$ in Conditions 2) and 3), respectively. Therefore, we also omit its proof. Then, it remains to show 1) $\implies$ both 2) and 3).

Proof of 1) $\implies$ both 2) and 3): Let use $\Psi(t, \tau)$ to denote the transition matrix of the positive LTV system (31). Since this system is exponentially stable, the following functions are defined.

$$v(t) = \int_t^{t+\delta} \Psi^\top(\tau, t) \mathbb{1}_n d\tau,$$

$$w(t) = \int_{t-\delta}^t \Psi(t, \tau) \mathbb{1}_n d\tau,$$

where $\delta > 0$. By taking their time derivatives, we have

$$\frac{dv(t)}{dt} = -A^\top(t)v(t) - \mathbb{1}_n + \Psi^\mathrm{T}(t+\delta, t)\mathbb{1}_n,$$

$$\frac{dw(t)}{dt} = A(t)w(t) + \mathbb{1}_n - \Psi(t, t-\delta)\mathbb{1}_n.$$

Similar to the proof of a) $\implies$ b) for IES, from the exponential stability of the LTV system and the boundedness of $A(t)$, it is possible to show that for sufficiently large $\delta > 0$, functions $v(t)$ and $w(t)$ satisfy all requirements in 2) and 3), respectively. ∎

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Yu Kawano (M'13) is currently Associate Professor in Department of Mechanical Systems Engineering at Hiroshima University. He received the M.S. and Ph.D. degrees in engineering from Osaka University, Japan, in 2011 and 2013, respectively. From October 2013 to November 2016, he was a Post-Doctoral researcher at both Kyoto University and JST CREST, Japan. From November 2016 to March 2019, he was a Post-Doctoral researcher at the University of Groningen, The Netherlands. He has held visiting research positions at Tallinn University of Technology, Estonia and the University of Groningen and served as a Research Fellow of the Japan Society for the Promotion Science. He is an Associate Editor for Systems and Control Letters. His research interests include nonlinear systems, complex networks, and model reduction.

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Bart Besselink received the M.Sc. (cum laude) degree in mechanical engineering in 2008 and the Ph.D. degree in 2012, both from Eindhoven University of Technology, Eindhoven, The Netherlands.

Since 2016, he has been an Assistant Professor with the Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, Groningen, The Netherlands. He was a short-term Visiting Researcher with the Tokyo Institute of Technology, Tokyo, Japan, in 2012. Between 2012 and 2016, he was a Postdoctoral Researcher with the ACCESS Linnaeus Centre and Department of Automatic Control, KTH Royal Institute of Technology, Stockholm, Sweden.

His main research interests include systems theory and model reduction for nonlinear dynamical systems and large-scale interconnected systems, with applications in the field of intelligent transportation systems.

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Ming Cao (SM'16) has since 2016 been a professor of systems and control with the Engineering and Technology Institute (ENTEG) at the University of Groningen, the Netherlands, where he started as a tenure-track Assistant Professor in 2008. He received the Bachelor degree in 1999 and the Master degree in 2002 from Tsinghua University, Beijing, China, and the Ph.D. degree in 2007 from Yale University, New Haven, CT, USA, all in Electrical Engineering. From September 2007 to August 2008, he was a Postdoctoral Research Associate with the Department of Mechanical and Aerospace Engineering at Princeton University, Princeton, NJ, USA. He worked as a research intern during the summer of 2006 with the Mathematical Sciences Department at the IBM T. J. Watson Research Center, NY, USA. He is the 2017 and inaugural recipient of the Manfred Thoma medal from the International Federation of Automatic Control (IFAC) and the 2016 recipient of the European Control Award sponsored by the European Control Association (EUCA). He is a Senior Editor for Systems and Control Letters, and an Associate Editor for IEEE Transactions on Automatic Control, IEEE Transactions on Circuits and Systems and IEEE Circuits and Systems Magazine. He is a vice chair of the IFAC Technical Committee on Large-Scale Complex Systems. His research interests include autonomous agents and multi-agent systems, complex networks and decision-making processes.