CHAPTER 4

Partial cross ownership and tacit collusion under cost asymmetries*

4.1 Introduction

There are many cases in which firms acquire their rivals’ stock as passive investments that give them a share in the rivals’ profits but not in the rivals’ decision making. These investments are often multilateral; examples of industries that feature complex webs of partial cross ownerships are the Japanese and the US automobile industries (Alley, 1997), the global airline industry (Airline Business, 1998), the Dutch financial sector (Dietzenbacher et al., 2000), the Nordic power market (Amundsen and Bergman, 2002), and the global steel industry (Gilo et al., 2006). While horizontal mergers are subject to substantial antitrust scrutiny and are often opposed by antitrust authorities, passive investments in rivals were either granted a de facto exemption from antitrust liability or have gone unchallenged by antitrust agencies in recent cases (Gilo, 2000).1 This lenient approach towards passive in-

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1 For example, to the best of our knowledge, Microsoft’s investments in the nonvoting stocks of Apple and Inprise/Borland Corp. were not challenged by antitrust agencies while Gillette’s 22.9% stake in Wilkinson Sword was approved by the US Department of Justice (DOJ) after the DOJ was assured that this stake would be passive (see United States v. Gillette Co. 55 Federal Register at 28312). The US Federal Trade Commission (FTC) approved Tele-Communications Inc.’s (TCI’s) 9.0% stake in Time Warner which at the time was TCI’s main rival in the cable TV industry and even allowed TCI to raise its stake in Time Warner to 14.99% in the future, after being assured that TCI’s stake would be completely
vestments in rivals stems from the courts’ interpretation of the exemption for stock acquisitions “solely for investment” included in Section 7 of the Clayton Act.

In an earlier work Gilo et al. (2006) began to investigate the merits of this lenient approach of courts and antitrust agencies towards passive investments in rivals. They showed that partial cross ownership (PCO) arrangements can facilitate tacit collusion among rival firms though cases exist in which such investments have no effect on the incentive of firms to collude. In particular it was shown that when firm $r$ increases its stake in a rival firm $s$, then collusion is never hindered, and that it will be surely facilitated if and only if (i) each firm in the industry holds a stake in at least one rival, (ii) the maverick firm in the industry (the firm with the strongest incentive to deviate from a collusive agreement)\(^2\) has a direct or an indirect stake in firm $r$, \(^3\) and (iii) firm $s$ is not the industry maverick. These results were established, however, under the assumption that firms are symmetric and have the same marginal cost functions. In the current study, we relax this assumption and examine the effect of PCO on the incentives of asymmetric firms to collude. This is obviously an important question since most industries feature cost asymmetries among firms.

To address this question we posit an infinitely repeated Bertrand oligopoly model in which firms have asymmetric marginal costs and they acquire some of their rivals’ (nonvoting) shares. This simple setting allows us to deal with the complexity generated by multilateral PCO. This complexity arises since, in general, multilateral PCO arrangements create multiplier effects so the profit of each firm, both under collusion as well as under deviation from collusion, depends on the whole set of PCO in the industry and not only on the firm’s own stake in rivals. Another advantage of this model is that PCO does not affect the equilibrium in the one shot case and therefore does not have any unilateral competitive effects. This allows us to focus on the effect of PCO on the ability of firms to engage in tacit collusion. We say that PCO arrangements facilitate tacit collusion if they expand the range of discount factors for which tacit collusion can be sustained.

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\(^2\) The Horizontal Merger Guidelines of the US Department of Justice and FTC define maverick firms as “firms that have a greater economic incentive to deviate from the terms of coordination than do most of their rivals,” see http://www.usdoj.gov/atr/public/guidelines/hmg.htm. For an excellent discussion of the role that the concept of maverick firms plays in the analysis of coordinated competitive effects, see Baker (2002).

\(^3\) Firm $i$ has an indirect stake in firm $r$ if it either has a stake in a firm that has a stake in firm $r$, or if it has a stake in a firm that has a stake in a firm that has a stake in firm $r$, and so on.
In the first part of this study we consider the case where only the most efficient firm in the industry invests in rivals. We show that even unilateral PCO by this firm may facilitate a market-sharing scheme in which all firms charge the same collusive price and divide the market between them. Unlike the case where firms have the same marginal costs, here firms have different monopoly prices on which they wish to collude. We assume that the collusive price is a compromise between the monopoly prices of the different firms. We show that when the most efficient firm invests in rivals, the collusive price would increase relative to the case where there are no PCO arrangements. Moreover, we show that the most efficient firm in the industry prefers to first invest in its most efficient rival both because this is the most effective way to promote tacit collusion and because such investment leads to a collusive price that is closer to the most efficient firm’s monopoly price. Only if investment in the most efficient rival is insufficient to sustain a market-sharing scheme, then the most efficient firm begins to invest in less efficient rivals.

In the second part of this chapter, we turn to multilateral PCO arrangements. In that case, cost asymmetries raise the complexity of the analysis considerably because the most efficient firm earns a positive profit even after the collusive agreement breaks down. Consequently, an increase in a firm $i$'s direct or indirect stake in the most efficient firm has conflicting effects on firm $i$'s incentive to collude. On the one hand, a larger (direct or indirect) stake in the most efficient firm makes firm $i$ less eager to deviate from collusion, because firm $i$ obtains a larger share in the collusive profit of the most efficient firm. But on the other hand, the increased stake of firm $i$ in the most efficient firm also gives it a larger share in the profit of the most efficient firm once the collusive agreement breaks down. This second effect weakens the incentive of firm $i$ to collude.

Despite these complications, we are able to show that an increase in the stake of firm $r$ in firm $s$ never hinders collusion and it will strictly facilitate collusion if and only if (i) the industry maverick has a direct or indirect stake in firm $r$, and (ii) firm $s$ is not the industry maverick. When either (i) or (ii) fails to hold, the increase in firm $r$'s stake in firm $s$ does not affect tacit collusion. These results extend the earlier findings in Gilo et al. (2006) and show that the results when firms have symmetric cost functions generalize to the asymmetric costs case.

Apart from Gilo et al. (2006), we are aware of only one other paper, Malueg (1992), that studies the coordinated effects of PCO. His paper differs from ours in several ways as he considers a repeated symmetric Cournot game in which firms hold identical stakes in one another, and moreover, in his paper, it is effectively
the controllers rather than the firms that hold stakes in rivals. This difference is important because investments by controllers do not feature the complex chain-effect interaction between the profits of rival firms which is a main focus of our study. Other papers that look at the competitive effects of PCO include Reynolds and Snapp (1986), Bolle and Güth (1992), Flath (1991, 1992a), Reitman (1994), and Dietzenbacher et al. (2000). These papers, however, examine the unilateral effects of PCO arrangements in the context of static oligopoly models.\footnote{See also Bresnahan and Salop (1986) and Kwoka (1992) for a related analysis of static models of horizontal joint ventures. Alley (1997) and Parker and Röller (1997) provide empirical evidence on the effect of PCO on collusion. Alley (1997) finds that failure to account for PCO leads to misleading estimates of the degree of tacit collusion in the Japanese and US automobile industries (see also Chapter 3 for similar conclusion in the case of the Japanese banking sector). Parker and Röller (1997) find that cellular telephone companies in the US tend to collude more in one market if they have a joint venture in another market.}

The rest of the chapter is organized as follows. Section 4.2 examines the ability of firms to achieve the fully collusive outcome in the context of an infinitely repeated Bertrand model with asymmetric firms without PCO. Section 4.3 examines the case where only the most efficient firm in the industry invests in rivals. Section 4.4 examines multilateral PCO arrangements. Conclusions and final remarks are given in Section 4.5. All technical proofs are given in Appendix 4.A.

### 4.2 Tacit collusion absent PCO

We examine the coordinated competitive effects of PCO in the context of an infinitely repeated Bertrand oligopoly model with \( n \geq 2 \) firms. We assume that the \( n \) firms produce a homogenous product using a constant returns to scale technology and face a downward sloping demand function \( Q(p) \). In every period, the \( n \) firms simultaneously choose prices and the lowest price firm captures the entire market. In case of a tie, the set of lowest price firms get equal shares of the total sales. The firms, however, have different marginal costs: let \( c_i \) be the (constant) marginal cost of firm \( i \) and assume \( c_1 < c_2 < \ldots < c_n \). That is, higher indices represent higher cost firms. The profit of firm \( i \) when it serves the entire market at a price \( p \) is given by

\[
y_i(p) = Q(p)(p - c_i).
\]

We shall make the following assumptions on \( y_i(p) \).

**Assumption 1:** \( y_i(p) \) has a unique global maximizer, \( p^{\text{m}}_i \).
**Assumption 2:** $p_1^m > c_n$ and $y_1(c_2) > y_1(c_j)/(j - 1)$ for all $j = 3, \ldots, n$.

Assumption 1 is standard and holds whenever the demand function is either concave or not too convex. Since $c_1 < c_2 < \ldots < c_n$, then $p_1^m < p_2^m < \ldots < p_n^m$, where $p_i^m \equiv \arg\max_p y_i(p)$ is the monopoly price from firm $i$’s point of view.\(^5\) That is, higher cost firms prefer higher monopoly prices. The first part of Assumption 2 ensures that all firms are effective competitors because it states that the monopoly price of the most efficient firm exceeds the marginal cost of the least efficient firm. The second part of Assumption 2 implies that in a static Bertrand game, firm 1 will prefer to set a price slightly below $c_2$ and capture the entire market than share the market with firm 2 at a price slightly below $c_3$, or share the market with firms 2 and 3 at a price slightly below $c_4$, and so on. Given this assumption, it is clear that absent collusion, firm 1 will prefer to monopolize the market by charging a price slightly below $c_2$.\(^6\)

When the stage game is infinitely repeated, firms may be able to engage in tacit collusion. The fact that different firms have different monopoly prices raises the obvious question of which price would they coordinate on in a collusive equilibrium? If side payments were possible, firms would clearly let firm 1, which is the most efficient firm, serve the entire market at a price $p_1^m$ (e.g., firms 2, ..., $n$ would all set prices above $p_1^m$ and would make no sales). The firms will then use side payments to share the monopoly profit

$$y_1^m \equiv Q(p_1^m)(p_1^m - c_1). \quad (4.1)$$

We rule out this possibility by assuming that side payments are not feasible, say due to the fear of antitrust prosecution.

Instead, we consider a collusive scheme led by firm 1. According to this scheme, firm 1 sets a price $\hat{p}$, which is some compromise between the monopoly prices of the various firms, i.e., $p_1^m \leq \hat{p} \leq p_n^m$. All firms adopt $\hat{p}$ and consumers randomize between them.\(^7\) Consequently, each firm $i$ serves $1/n$ of the market and its profit in

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\(^5\)By revealed preferences, the fact that $y_i(\cdot)$ has a unique maximizer implies that $Q(p_1^m)(p_1^m - c_i) > Q(p_i^m)(p_i^m - c_i)$, and $Q(p_i^m)(p_i^m - c_i) > Q(p_1^m)(p_1^m - c_i)$. Summing up the two inequalities and simplifying, yields $Q(p_1^m)(c_i - c_i) > Q(p_i^m)(c_i - c_i)$. Assuming without loss of generality that $j > i$, and noting that $Q(\cdot) < 0$, it follows that $p_i^m > p_1^m$.

\(^6\)For example, the case of a linear demand function $Q(p) = a - bp$ and all $c_j$’s at equal distance (i.e., $c_{j+1} - c_j = \theta > 0$ for all $j = 1, \ldots, n - 1$) satisfies Assumption 2, since then $y_1(c_2) = Q(c_2)\theta > Q(c_j)\theta = y_1(c_j)/(j - 1)$ for all $j = 3, \ldots, n$.

\(^7\)That is, we study “pure” price fixing. A more elaborate collusive scheme might also involve market division in which case the market shares need not be equal. Such a scheme, however, will be in general much harder to enforce and easier for antitrust authorities to detect.
every period is $\hat{y}_i/n$, where

$$\hat{y}_i \equiv Q(\hat{p})(\hat{p} - c_i), \quad i = 1, \ldots, n. \quad (4.2)$$

Although $\hat{p}$ can exceed firm 1’s monopoly price, $p_{1m}$, it cannot exceed it by too much. To see why, note that firm 1 can always ensure itself a profit of $y_1(c_2)$ by setting a price slightly below $c_2$ and capturing the entire market. Hence, to ensure that firm 1 has an incentive to collude at $\hat{p}$, it must be the case that $\hat{y}_1/n \geq y_1(c_2)$. Since by Assumption 2, $c_2 < p_{1m} \leq \hat{p}$, it follows that $\hat{p}$ is bounded from above by $\overline{p}$, where $\overline{p}$ is implicitly defined by $y_1(\overline{p})/n = y_1(c_2)$ (see Figure 4.1). If this is not the case, i.e., if $\hat{p} > \overline{p}$, then firm 1 would be better off deviating to $c_2$ and capturing the entire market than colluding at $\hat{p}$. Before proceeding, we add the following assumption which is illustrated in Figure 4.1:

**Assumption 3**: $\overline{p} < p_{2m}$, where $\overline{p}$ is implicitly defined by $y_1(\overline{p})/n = y_1(c_2)$.

![Figure 4.1: Illustrating Assumption 3](image)

Recalling that $p_{1m} < p_{2m}^n < \ldots < p_{nm}$, Assumption 3 implies that $\overline{p} < p_{im}$ for all $i = 2, \ldots, n$. Since $\hat{p} \leq \overline{p}$, it follows that $\hat{p} < p_{im}$ for all $i = 2, \ldots, n$: the collusive price is below the monopoly prices of all firms but 1. This implies in turn that the optimal deviation for firm $i = 2, \ldots, n$ is to set a price slightly below $\hat{p}$, while the optimal deviation for firm 1 is to set a price $p_{1m}$. Following any deviation from the
collusive scheme (including a deviation by firm 1), firm 1 charges a price slightly below \( c_2 \) forever after and captures the entire market.

Recall that we have assumed that firm 1 prefers to set a price of \( c_j \) and share the market with firms \( i = 2, \ldots, j - 1 \) than set a price of \( c_{j+1} \) and share the market with firms \( i = 2, \ldots, j \) (second part of Assumption 2). Recalling that on the equilibrium path, it must be the case that \( \hat{y}_1 / n \geq y_1 (c_2) \) (otherwise firm 1 does not wish to collude), this implies that firm 1 prefers to collude with all \( n - 1 \) rivals at \( \hat{p} \) than collude with only \( j \) firms by setting a price just below \( c_{j+1} \).

We assume that the pricing decisions of each firm are effectively made by its controller (i.e., a controlling shareholder) whose ownership stake is \( \gamma_{ii} \). We are now interested in finding conditions that will ensure that in a subgame perfect equilibrium of the infinitely repeated game, every controller will set \( \hat{p} \) in every period.

Using \( \delta \) to denote the intertemporal discount factor, the condition that ensures that the controller of firm \( i = 2, \ldots, n \) does not wish to deviate from the collusive scheme is given by

\[
\gamma_{ii} \frac{\hat{y}_i}{n(1 - \delta)} \geq \gamma_{ii} \hat{y}_i, \quad i = 2, \ldots, n.
\]

The left-hand side of (4.3) is the infinite discounted payoff of firm \( i \)'s controller which consists of his share in firm \( i \)'s collusive profit. The right-hand side of (4.3) is the controller’s share in the one-time profit that firm \( i \) earns in the period in which it undercuts its rivals slightly and captures the entire market. Condition (4.3) can be rewritten as

\[
\delta \geq \hat{\delta} \equiv 1 - \frac{1}{n}.
\]

That is, the controllers of firms \( 2, \ldots, n \) have an incentive to participate in the collusive scheme provided that they are sufficiently patient. This condition is identical to the well-known condition for tacit collusion in the context of an infinitely repeated Bertrand model with \( n \) identical firms (see e.g., Tirole, 1988, Ch. 6.3.2.1).\(^8\)

As for firm 1, then its controller does not wish to deviate from the collusive scheme provided that

\[^8\text{To see how realistic this condition is, one can use the identity } \delta = 1/(1 + r), \text{ where } r \text{ is an interest rate. Then } \delta > 1 - 1/n \text{ is equivalent to } r < 1/(n - 1). \text{ Thus, for } n = 10 \text{ collusive scheme is sustainable if } r < 11.1\%.\]
where \( y_1^m \) is the one-time profit of firm 1 in the period in which it deviates and captures the entire market while charging \( p_1^m \), and \( y_1(c_2) \) is the per-period profit of firm 1 in all subsequent periods. Condition (4.4) can be rewritten as

\[
\delta \geq \hat{\delta}_1(\hat{p}) \equiv \frac{y_1^m - \hat{y}_1/n}{y_1^m - y_1(c_2)}. \tag{4.5}
\]

Note that

\[
\hat{\delta}_1(\hat{p}) > \frac{y_1^m - \hat{y}_1/n}{y_1^m} \geq 1 - \frac{1}{n} \equiv \hat{\delta},
\]

where the weak inequality follows because \( y_1^m \geq \hat{y}_1 \). Since \( \hat{\delta}_1(\hat{p}) > \hat{\delta} \), it is clear that if firm 1 wishes to collude then all other firms surely wish to collude. That is, firm 1 is the maverick firm in the industry, i.e., the firm with the strongest incentive to deviate from a collusive agreement. Hence, (4.5) is a necessary and sufficient condition for the collusive scheme led by firm 1 to be sustained as a subgame perfect equilibrium of the infinitely repeated game. Moreover, since \( \hat{p} \geq p_1^m \), it follows that \( \hat{y}_1 \) increases as \( \hat{p} \) is lowered towards \( p_1^m \). As a result, firm 1’s controller would prefer to set \( \hat{p} = p_1^m \) and thereby maximize his infinite discounted stream of collusive profits while relaxing constraint (4.5). Hence,

**Theorem 4.1.** Absent PCO by firms, firm 1 is the industry maverick and its controller would like to set the collusive price equal to \( p_1^m \). Collusion at \( p_1^m \) can be sustained as a subgame perfect equilibrium of the infinitely repeated game provided that \( \delta \geq \hat{\delta}_1(p_1^m) \).

### 4.3 Tacit collusion with unilateral partial ownership by firm 1

In this section we will only examine the competitive effects of unilateral partial ownership (PO) investments by firm 1 in rival firms. The competitive effects of multilateral PCO arrangements are considered in Section 4.4. We will now use \( \hat{\delta}_1(p_1^m) \) (the critical discount factor above which the collusive scheme characterized in the previous section can be sustained) as our measure of the ease of collusion;
accordingly, we will say that PCO facilitates tacit collusion if it lowers $\hat{\delta}_1(p_1^m)$, and will say that PCO hinders tacit collusion if it raises $\hat{\delta}_1(p_1^m)$.

Specifically, assume that firm 1 invests in rivals and let $w_{12}, \ldots, w_{1n}$ be its ownership stakes in firms 2, \ldots, $n$. Since the collusive profit of each firm $i$ is $\bar{y}_i/n$, it follows that firm 1’s infinite discounted stream of profits under collusion is

$$\frac{\bar{y}_1 + \sum_{i \neq 1} w_{1i}\bar{y}_i}{n(1-\delta)}.$$ 

If firm 1’s controller deviates from the collusive scheme, all rivals make zero profits, so firm 1’s payoff is

$$y_1^m + \frac{\delta y_1(c_2)}{1-\delta},$$

exactly as in the absence of PO. Consequently, the condition that ensures that firm 1’s controller does not wish to deviate from the collusive scheme is now given by

$$\gamma_{11} \left( \frac{\bar{y}_1 + \sum_{i \neq 1} w_{1i}\bar{y}_i}{n(1-\delta)} \right) \geq \gamma_{11} \left( y_1^m + \frac{\delta y_1(c_2)}{1-\delta} \right), \quad (4.6)$$

or

$$\delta \geq \hat{\delta}_{11}(\hat{\delta}) \equiv \frac{y_1^m - (\bar{y}_1 + \sum_{i \neq 1} w_{1i}\bar{y}_i) / n}{y_1^m - y_1(c_2)}. \quad (4.7)$$

Notice that $\hat{\delta}_{11}(\hat{\delta})$ is decreasing with each $w_{1i}$: the larger the stakes of firm 1 in rival firms, the stronger is firm 1’s incentive to collude. The reason for this is that the collusive payoff of firm 1 increases when it invests in rivals, while its payoff under deviation is unaffected because rival firms make a profit of 0 in the period in which firm 1 deviates as well as in all future periods. Clearly, firm 1 does not have an incentive to invest in rivals up to the point where $\hat{\delta}_{11}(\hat{\delta})$ drops below $\hat{\delta}$ since then it might very well happen that firm 1 is no longer the industry maverick (and thus firm 1’s stakes in rivals no longer facilitate tacit collusion). Hence, we shall assume in the rest of this section that firm 1 remains an industry maverick even when it holds PO stakes in rivals. A sufficient condition for that to be the case is that firm 1’s profit when the collusive agreement breaks down, $y_1(c_2)$, is at

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9 Of course, the infinitely repeated game admits multiple subgame perfect equilibria. We restrict attention to the most collusive equilibrium and focus on $\hat{\delta}_1(p_1^m)$ because this is a standard way to capture the notion of “ease of collusion”.

10 Hence, for PCO holdings we adopt the same notation as in Chapter 3.
least as large as firm 1’s average stake in the profits of rival firms under collusion, 
\[(\sum_{i \neq 1} w_i \hat{y}_i) / (n - 1),\]  
because then,  
\[
\hat{\delta}_1^{po}(\hat{\rho}) \geq \frac{y_m - (y_m + \sum_{i \neq 1} w_i \hat{y}_i)}{y_m - y_1(c_2)} \geq \frac{y_m - (y_m + (n - 1)y_1(c_2))}{y_m - y_1(c_2)} = 1 - \frac{1}{n} \equiv \hat{\delta},
\]  
where the first weak inequality follows because \(y_m \geq \hat{y}_1\).

Assuming then that firm 1 is the industry maverick, firm 1’s controller selects \(\hat{\rho}\) to maximize the infinite discounted sum of firm 1’s collusive profits given by the left-hand side of (4.6) subject to (4.7). The following result follows (see Appendix 4.A).

**Theorem 4.2.** Suppose that firm 1 invests in rivals but still remains the industry maverick. Using \(\hat{\rho}^*\) to denote the optimal collusive price from firm 1’s perspective, the following holds:

(i) \(\hat{\rho}^*\) is increasing with each \(w_{1i}\) and is above firm 1’s monopoly price: \(\hat{\rho}^* > \hat{p}_m^1\).

(ii) \(\hat{\delta}_1^{po}(\hat{\rho}^*)\) is decreasing with each \(w_{1i}\) and is below \(\hat{\delta}_1(\hat{p}_m^1)\) – the critical discount factor above which collusion can be sustained absent PO.

(iii) PO in an efficient rival raises \(\hat{\rho}^*\) by less and lowers \(\hat{\delta}_1^{po}(\hat{\rho}^*)\) by more than a similar PO in a less efficient rival.

Theorem 4.2 implies that investments by firm 1 in rivals do not only facilitate tacit collusion by lowering the critical discount factor above which tacit collusion can be sustained, but also lead to a higher collusive price. The latter result arises because, due to its investment in rivals, firm 1 is interested in maximizing a weighted average of its own profit and the profits of the firms it invests in. The higher firm 1’s investments in rivals, the higher the weight that firm 1’s assigns to the rivals’ profits in its objective function. Maximizing the rivals’ profits requires a higher monopoly price than the monopoly price from firm 1’s own perspective.

The theorem suggests that to the extent that firm 1 invests in rivals, it always prefers to invest in its most efficient rival first, since this leads to a collusive price that is closer to firm 1’s monopoly price and also expands the range of discount factors above which collusion can be sustained. This also implies that firm 1 will have an incentive to minimize its investments in rivals subject to being able to facilitate tacit collusion. If investment in the most efficient rival is not sufficient to sustain collusion, then firm 1 invests in the next efficient rival.
4.4 Tacit collusion with multilateral PCO

In this section we turn to the case where all firms potentially invest in rivals. To this end, let $w_{ij}$ be firm $i$’s partial cross ownership stake in firm $j$ and define the following $n \times n$ PCO matrix:

$$W = \begin{pmatrix}
0 & w_{12} & \cdots & w_{1n} \\
& 0 & \cdots & \cdots \\
& \vdots & \ddots & \vdots \\
& w_{n1} & w_{n2} & \cdots & 0
\end{pmatrix}.$$ 

Row $i$ in the matrix $W$ specifies the stakes that firm $i$ has in all rival firms, while column $j$ in the matrix $W$ specifies the stakes that rival firms hold in firm $j$. Since apart from rival firms each firm is also held by its controller and possibly by outside stakeholders, the sum of each column of $W$ is strictly less than 1. It is also assumed that a firm cannot own shares in itself, i.e., all diagonal terms in the matrix $W$ are equal to 0.

4.4.1 The accounting profits under PCO

When firms hold stakes in each other, the profit of each firm potentially depends on the profits of all other firms in the industry. For instance, firm 1 may get a share $w_{12}$ of firm 2’s profit while at the same time firm 2 owns a share $w_{23}$ in the profit of firm 3, which in turn holds a share $w_{31}$ in the profit of firm 1. Hence, we potentially have a multiplier effect that drives a wedge between the direct profit of each firm and its overall profit that also includes the firm’s share in the profits of rival firms. Therefore, before characterizing the conditions that ensure that a collusive scheme can be supported as a subgame perfect equilibrium of an infinitely repeated game, we first need to express the profit of each firm under collusion and in the case of a deviation from collusion.

Under collusion, all firms charge the same price, $\hat{p}$. Since the products are homogeneous, consumers choose which firm to buy from at random, so the market share of each firm is $1/n$. Hence, the direct profit of each firm $i$ (excluding its share in the profits of rivals) is $\hat{y}_i/n$, where $\hat{y}_i$ is given by equation (4.2). Since by assumption, $c_1 < c_2 < \ldots < c_n$, we have $\hat{y}_1 > \hat{y}_2 > \ldots > \hat{y}_n$: the direct profit of firm 1 exceeds that of firm 2, which in turn exceeds that of firm 3, and so on. In addition to its direct profit, each firm $i$ also gets a share in its rivals’ profits due to
its cross ownership stake in these firms. Hence, the (column) vector of collusive profits, \( \pi = (\pi_1, \pi_2, \ldots, \pi_n)' \), is given by the solution to the following system of \( n \) equations:

\[
\pi = (1/n)y + W\pi,
\]

where \( y \equiv (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n)' \).

Next, we consider what happens when the controller of firm \( i \) deviates from the collusive scheme. If \( i \neq 1 \), then firm \( i \) will slightly undercut \( \hat{p} \), so the direct profit of all firms but \( i \) will be 0, while the direct profit of firm \( i \) will be arbitrarily close to \( \hat{y}_i \). Consequently, the vector of current profits, \( \pi^{d_i} = (\pi^{d_i}_1, \pi^{d_i}_2, \ldots, \pi^{d_i}_n)' \), is defined by the solution to the following system:

\[
\pi^{d_i} = y^{d_i} + W\pi^{d_i}, \quad \text{for all} \quad i = 2, ..., n,
\]

where \( y^{d_i} \equiv (0, \ldots, 0, \hat{y}_i, 0, \ldots, 0)' \) is an \( n \)-dimensional vector with \( \hat{y}_i \) in the \( i \)-th entry and 0's elsewhere.

If the deviant is firm 1 (\( i = 1 \)), then it charges \( p^{m}_1 \) and its profit in the current period will be \( y^{m}_1 \) (see equation (4.1)). The current direct profits of all other firms will be 0. Hence, the vector of profits in period in which firm 1’s controller deviates, \( \pi^{d_1} = (\pi^{d_1}_1, \pi^{d_1}_2, \ldots, \pi^{d_1}_n)' \), is defined by the solution to the system

\[
\pi^{d_1} = y^{d_1} + W\pi^{d_1},
\]

where \( y^{d_1} \equiv (y^{m}_1, 0, \ldots, 0)' \) is an \( n \)-dimensional vector with \( y^{m}_1 \) in the first entry and 0's elsewhere.

Once the collusive agreement breaks down, firm 1 will charge a price slightly below \( c_2 \) in every period and will capture the entire market. Hence the vector of profits following a breakdown of the collusive agreement, \( \pi^f = (\pi^f_1, \pi^f_2, \ldots, \pi^f_n)' \), is defined by the solution to the following system:

\[
\pi^f = y^f + W\pi^f,
\]

where \( y^f \equiv (y_1(c_2), 0, \ldots, 0)' \) is an \( n \)-dimensional vector with \( y_1(c_2) \) in the first entry and 0's elsewhere.

To solve systems (4.8)-(4.11), note that since the PCO matrix, \( W \), is nonnegative and the sum of each of its columns is strictly less than 1, systems (4.8)-(4.11) are Leontief systems and have unique nonnegative solutions (see Sydsæter et al., 2005,
Chapter 22; see also discussions after equation (3.4) in Chapter 3) defined by

\[
\pi(\hat{p}; \mathbf{W}) = (1/n)\mathbf{L}\mathbf{y},
\]

\[
\pi_{il}^{d}(\hat{p}; \mathbf{W}) = \mathbf{L}\mathbf{y}_{i}, \quad i = 1, \ldots, n,
\]

\[
\pi_{f}(c_{2}; \mathbf{W}) = \mathbf{L}_{f},
\]

where \(\mathbf{L} \equiv (\mathbf{I} - \mathbf{W})^{-1}\) is the inverse Leontief matrix that specifies the aggregate imputed shares of “real” equityholders (i.e., outside equityholders that are not part of the \(n\) firms) in the accounting profits of the \(n\) firms.\(^{11}\) That is, the \(ij\)-th entry in the matrix \(\mathbf{L}\), denoted \(l_{ij}\), is the aggregate imputed share that the real equityholders of firm \(i\) have in the accounting profit of firm \(j\). Equation (4.12) implies that the accounting collusive profit of firm \(i \neq 1\) is \(\pi_{i}(\hat{p}; \mathbf{W}) = (\sum_{j=1}^{n} l_{ij}\hat{y}_{j}) / n\), its one-time profit in the period in which it deviates from the collusive scheme is \(\pi_{il}^{d}(\hat{p}; \mathbf{W}) = l_{i1}\hat{y}_{i}\), and its profit in any subsequent period is \(\pi_{il}^{f}(c_{2}; \mathbf{W}) = l_{i1}y_{1}(c_{2})\). The corresponding accounting profits of firm 1 are \(\pi_{1}(\hat{p}; \mathbf{W}) = (\sum_{j=1}^{n} l_{1j}\hat{y}_{j}) / n\), \(\pi_{1}^{d}(\hat{p}; \mathbf{W}) = l_{11}y_{1}^{m}\), and \(\pi_{1}^{f}(c_{2}; \mathbf{W}) = l_{11}y_{1}(c_{2})\).

Given the important role that the aggregate imputed shares matrix, \(\mathbf{L}\), plays in our analysis, we state the following result whose proof appears in Gilo et al. (2006).

**Lemma 4.1.** The aggregate imputed shares matrix \(\mathbf{L}\) has the following properties:

(i) \(l_{ii} \geq 1\) for all \(i\), and \(0 \leq l_{ij} < l_{ii}\) for all \(i\) and all \(j \neq i\).

(ii) Let \(i\) and \(j\) be two distinct firms. Then, \(l_{ij} = 0\) if and only if firm \(i\) does not have a direct and an indirect stake in firm \(j\).\(^{12}\)

(iii) \(l_{ii} > 1\) if and only if firm \(i\) has a direct or an indirect stake in some firm \(j\) which in turn has a direct or an indirect stake in firm \(i\) (i.e., \(l_{ij} > 0\) and \(l_{ji} > 0\)).

(iv) \(\mathbf{L}_{i} \equiv \sum_{j=1}^{n} (1 - \sum_{k \neq j} w_{kj}) l_{ij} = 1\) for all \(i\).

To interpret Lemma 4.1, recall that \(l_{ij}\) is the aggregate imputed share that the real equityholders of firm \(i\) have in the accounting profit of firm \(j \neq i\) through the direct or indirect cross ownership of firm \(i\) in firm \(j\) and \(l_{ii}\) is the aggregate imputed share that the real equityholders of firm \(i\) have in the accounting profit of their own firm. Part (i) of Lemma 4.1 says that a 1% stake in firm \(i\) may give the real equityholders of firm \(i\) more than a 1% imputed share in the firm’s profit (i.e., \(l_{ii} \geq 1\)), and the real equityholders always have larger imputed shares in their own firm’s profit than in the profits of rival firms (i.e., \(l_{ij} < l_{ii}\) for all \(i\) and all \(j \neq i\)). Part (ii) of the

\(^{11}\)The terminology “imputed shares” is due to Dorofeenko et al. (2008).

\(^{12}\)We will say that firm \(i\) has no indirect stake in firm \(j\), if it has no stake in a firm that has a stake in firm \(j\), and has no stake in a firm that has a stake in a firm that has a stake in firm \(j\) and so on.
lemma says that the real equityholders of firm $i$ will get a share in the profit of a rival firm $j$ if and only if firm $i$ has a direct and/or indirect stake in firm $j$. Part (iii) of the lemma says that if the real equityholders of firm $i$ have a direct or an indirect stake in some rival firm $j$ and this firm’s real equityholders in turn have a direct or an indirect stake in firm $i$, then the aggregate imputed share that a real equityholder of firm $i$ will have in firm $i$ will exceed 1. In other words, a 1% stake in firm $i$ will give a “real” equityholder of firm $i$ more than a 1% share in the firm’s profit. The reason for this surprising property is that multilateral cross ownership arrangements create a multiplier effect that results in an overstatement of the firms’ cash flows.\footnote{See Dietzenbacher et al. (2000), Dorofeenko et al. (2008), and Chapter 3 of this thesis for additional discussion of this effect of PCO.} Part (iv) of the lemma ensures, however, that the aggregate effective shares of “real” equityholders in each firm $i$ sum up to 1. Hence, while the accounting profits of firms will overstate the total cash flows, the aggregate payoff of all real equityholders will sum up exactly to the total cash flows.

### 4.4.2 Collusion with multilateral PCO

Given the accounting profits of the $n$ firms under collusion and following a deviation from the fully collusive scheme, the condition that ensures that the collusive outcome can be sustained as a subgame perfect equilibrium is

$$\frac{\gamma_{ii}\pi_i(\hat{p};W)}{1 - \delta} \geq \gamma_{ii} \left( \pi^d_i(\hat{p};W) + \delta \pi^f_i(c_2;W) \right), \quad i = 1, \ldots, n. \tag{4.13}$$

The left-hand side of (4.13) is the infinite discounted payoff of firm $i$’s controller under collusion, consisting of the controller’s share in firm $i$’s collusive profit. The right-hand side of (4.13) is the controller’s share in the profit that firm $i$ earns when it undercuts its rivals slightly (the one-time profit $\pi^d_i(\hat{p};W)$ in the period in which firm $i$ deviates and $\pi^f_i(c_2;W)$ in all subsequent periods). If (4.13) holds, no controller wishes to unilaterally deviate from the fully collusive scheme.

Recalling that $\pi_i(\hat{p};W) = (\sum_{j=1}^n l_{ij}\tilde{y}_j)/n$ and $\pi^f_i(c_2;W) = l_{1i}\tilde{y}_1(c_2)$ for all $i$, $\pi^d_1(\hat{p};W) = l_{11}\tilde{y}_1^m$, and $\pi^d_i(\hat{p};W) = l_{ii}\tilde{y}_i$ for all $i \neq 1$, and using $z_{ij} \equiv l_{ij}/l_{ii}$ to denote the relative imputed share that the equityholders of firm $i$ have in firm $j$ (relative to their imputed share in their “own” firm $i$), the necessary condition (4.13) for collusion can be rewritten as
\[ \delta (y_1^n - y_1(c_2)) \geq y_1^n - \frac{1}{n} \sum_{j=1}^{n} z_{1j} \tilde{y}_j, \quad \text{and} \quad (4.14) \]

\[ \delta (\tilde{y}_i - z_{i1} y_1(c_2)) \geq \tilde{y}_i - \frac{1}{n} \sum_{j=1}^{n} z_{ij} \tilde{y}_j, \quad i = 2, \ldots, n. \quad (4.15) \]

Notice that by definition, \( y_1^n > \tilde{y}_1 / n \geq y_1(c_2) \) (see Figure 4.1) and recall that \( \tilde{y}_1 > \tilde{y}_2 > \ldots > \tilde{y}_n \). Since, part (i) of Lemma 4.1 implies that \( z_{ii} = 1 \) for all \( i \) and \( z_{ij} < 1 \) for all \( i \) and all \( j \neq i \), it follows that both sides of (4.14) are positive. Moreover, \( \tilde{y}_1 / n \geq y_1(c_2) \) implies that \( (\sum_{j=1}^{n} z_{ij} \tilde{y}_j) / n = z_{1i} \tilde{y}_1 / n + (\sum_{j \neq 1}^{n} z_{ij} \tilde{y}_j) / n \geq z_{i1} y_1(c_2) \), with strict inequality when \( z_{ij} > 0 \) for some \( j \). Hence, \( \tilde{y}_i - z_{i1} y_1(c_2) \geq \tilde{y}_i - (\sum_{j=1}^{n} z_{ij} \tilde{y}_j) / n \). Before proceeding we impose the following assumption on \( \tilde{y}_i \):

**Assumption 4:** \( \tilde{y}_i > (\sum_{j=1}^{n} z_{ij} \tilde{y}_j) / n \) for all \( i \neq 1 \).

Assumption 4 implies that each firm \( i \neq 1 \) earns more money when it unilaterally deviates from a collusive scheme than it earns under collusion. This assumption ensures that both sides of (4.15) are positive. Notice that in the presence of PCO this need not be the case because under collusion, firm \( i \) gets a share in the profits of its rivals, while under deviation it does not. If firm \( i \) is relatively inefficient, then its profit under collusion may exceed its profit when it deviates even though in the latter case the firm serves the entire market while under collusion it serves only \( 1/n \) of the market (but it gets a share in the profits of its rivals).

With Assumption 4 in place, (4.14) and (4.15) imply the following result.

**Lemma 4.2.** Let \( z_{ij} \equiv l_{ij} / l_{ii} \) be the relative imputed share that the equityholders of firm \( i \) have in firm \( j \) (relative to their imputed share in their “own” firm \( i \)). Then, the fully collusive outcome can be sustained as a subgame perfect equilibrium of the infinitely repeated game provided that

\[ \delta \geq \hat{\delta}^{po}(W) \equiv \max \left\{ \hat{\delta}_1(W), \ldots, \hat{\delta}_n(W) \right\}, \]

where

\[ \hat{\delta}_i(W) \equiv \frac{y_1^n - \frac{1}{n} \sum_{j=1}^{n} z_{1j} \tilde{y}_j}{y_1^n - y_1(c_2)}, \quad (4.16) \]
and
\[
\hat{\delta}_i(W) = \frac{\hat{y}_i - \frac{1}{n} \sum_{j=1}^{n} z_{ij} \hat{y}_j}{\hat{y}_i - z_{i1} y_1(c_2)}, \quad i = 2, ..., n. \tag{4.17}
\]

It is easy to see that the incentives of firms to collude depend on cross ownership only through the matrix \(Z\) whose characteristic element is \(z_{ij}\). In what follows we shall therefore examine how changes in cross ownership affect the matrix \(Z\) and consequently the critical discount factors above which firms wish to collude.

### 4.4.3 A firm increases its stake in a rival firm by buying shares from an outsider or from the rival’s controller

Now, suppose that firm \(r\) increases its stake in firm \(s\), \(w_{rs}\) by \(\omega > 0\). The resulting new PCO matrix is \(W_\omega\); it differs from the original PCO matrix only in that its \(rs\)-th entry is \(w_{rs} + \omega\) rather than \(w_{rs}\). Our main question is whether \(\hat{\delta}_i(W_\omega)\) is higher or lower than \(\hat{\delta}_i(W)\).

To address this question, note from equation (4.16) that \(\frac{\partial \hat{\delta}_1(W)}{\partial z_{1}} < 0\) for all \(j\), and note from equation (4.17) that \(\frac{\partial \hat{\delta}_i(W)}{\partial z_{ij}} < 0\) for all \(i \neq 1\) and all \(j \neq 1\). Moreover, from equation (4.17) it follows that
\[
\frac{\partial \hat{\delta}_i(W)}{\partial z_{i1}} = -\frac{-\hat{y}_i}{\hat{y}_i - z_{i1} y_1(c_2)} + \frac{y_1(c_2)}{(\hat{y}_i - z_{i1} y_1(c_2))^2} \left( \hat{y}_i - \frac{1}{n} \sum_{j=1}^{n} z_{ij} \hat{y}_j \right)

= -\frac{\hat{y}_i}{\hat{y}_i - z_{i1} y_1(c_2)} + \frac{y_1(c_2)}{(\hat{y}_i - z_{i1} y_1(c_2))^2} \left( z_{i1} \hat{y}_1 - \sum_{j \neq 1}^{n} z_{ij} \hat{y}_j \right)

= -\frac{\hat{y}_i}{\hat{y}_i - z_{i1} y_1(c_2)} + \frac{y_1(c_2)}{(\hat{y}_i - z_{i1} y_1(c_2))^2} \sum_{j \neq 1}^{n} z_{ij} \hat{y}_j < 0,
\]

where the inequality follows because by assumption \(\hat{y}_1 / n \geq y_1(c_2)\) (otherwise firm 1 has no incentive to collude) and \(\sum_{j \neq 1}^{n} z_{ij} \hat{y}_j = z_{i1} \hat{y}_i + \sum_{j \neq 1}^{n} z_{ij} \hat{y}_j \geq \hat{y}_i > 0\) (recall that \(z_{ii} = 1\)). Hence,

**Lemma 4.3.** \(\frac{\partial \hat{\delta}_i(W)}{\partial z_{ij}} < 0\) for all \(i\) and all \(j\): the critical discount factor above which firm \(i\) wishes to collude is a strictly decreasing function of each of firm \(i\)'s relative imputed shares in rival firms.

Lemma 4.3 implies that in order to determine the effect of the increase in firm \(r\)'s stake in firm \(s\) by \(\omega\) on firm \(i\)'s incentive to collude, we only need to know how it affects the \(i\)'th row in matrix \(Z\), which specifies the relative imputed shares of firm
i in all rival firms. To this end, note from Lemma A1 in Gilo et al. (2006) that

\[ z_{ij}^\omega \equiv \frac{l_{ij}^\omega}{l_{ii}^\omega} = \frac{l_{ij} + \varepsilon_i l_{sj}}{l_{ii} + \varepsilon_i l_{si}} \quad \varepsilon_i = \frac{\omega l_{ir}}{1 - \omega l_{sr}} \geq 0, \quad (4.18) \]

where \( l_{ij}^\omega \) is the typical element of the new matrix of aggregate imputed shares \( L^\omega = (I - W^\omega)^{-1} \).

Straightforward differentiation yields

\[ \frac{\partial z_{ij}^\omega}{\partial \omega} = \frac{l_{ii} l_{sj} - l_{si} l_{ij}}{(l_{ii} + \varepsilon_i l_{si})^2} \times \frac{l_{ir}}{(1 - \omega l_{sr})^2}. \quad (4.19) \]

Using this equation we are able to prove the following result, which generalizes Theorem 1 in Gilo et al. (2006) to the case of asymmetric firms (see Appendix 4.A).

**Theorem 4.3.** Starting with a PCO matrix \( W \), suppose that firm \( r \) increases its stake in firm \( s \) by some \( \omega > 0 \), so that the new PCO matrix \( W^\omega \) differs from \( W \) only with respect to the \( rs \)-th entry which is increased by \( \omega \). Then

(i) \( \hat{\delta}_s(W^\omega) = \hat{\delta}_s(W) \),

(ii) \( \hat{\delta}_i(W^\omega) = \hat{\delta}_i(W) \) if \( l_{ir} = 0 \) (firm \( i \) has no direct and indirect stake in the acquiring firm \( r \)), and

(iii) \( \hat{\delta}_i(W^\omega) < \hat{\delta}_i(W) \) otherwise, i.e., for all \( i \neq s \) and \( l_{ir} > 0 \).

Theorem 4.3 shows that an increase in the stake of firm \( r \) in firm \( s \) never hinders collusion. In fact, the theorem shows that there are only two special cases in which collusion is not strictly facilitated: one case arises when the maverick firm is the target firm (firm \( s \)). Collusion is not facilitated in this case because the incentive of the target firm to collude is not affected by the fact that firm \( r \) has increased its stake in firm \( s \). Intuitively, when firm \( r \) increases its stake in firm \( s \), the relative imputed shares of firm \( s \) do not change because the imputed shares of firm \( s \) in all \( j \), \( l_{sj} \), must change by the same constant proportion.\(^{14}\) This is because \( l_{sj} \) will change in this case if and only if the target firm \( s \) has a direct or an indirect stake in the acquiring firm \( r \) (i.e., when \( l_{sr} > 0 \)), thus a change in \( l_{sj} \) for any \( j \) is only due to the link of firm \( s \) to firm \( r \). The second special case arises when the maverick firm has no direct or indirect stake in the acquiring firm (firm \( r \)). Then, the increase in \( w_{rs} \) does not affect the relative imputed shares of the maverick firm in any way and hence its incentive to collude are not affected either. In all other cases collusion is strictly facilitated. We summarize these conclusions in the next corollary.

\(^{14}\) In fact, from Lemma A1 in Gilo et al. (2006) it follows that \( l_{sj}^\omega = \left[ \frac{1}{1 - \omega l_{sr}} \right] l_{sj} \) for all \( j \).
**Corollary 4.1.** An increase in firm r’s cross ownership stake in firm s never hinders tacit collusion and surely facilitates it if and only if (i) each industry maverick has a direct or an indirect stake in firm r, and (ii) firm s is not an industry maverick.

The proof of Theorem 4.3 provides a simpler proof for Theorem 1 in Gilo et al. (2006). To see why, note that in the special case where all firms have the same marginal cost (the case considered in Gilo et al., 2006), $\hat{y}_1 = \ldots = \hat{y}_n = y_i^{m}$ and $y_1(c_2) = 0$. Hence, equations (4.16) and (4.17) imply that $\frac{\partial \hat{\delta}_i(W)}{\partial z_{ij}} < 0$ for all $i$ and all $j$. Then, the proof of Theorem 4.3 implies immediately that $\hat{\delta}_i(W) \geq \hat{\delta}_i(W^{ai})$ with strict equality if and only if $i = s$ or $l_{ir} = 0$.

To get some intuition for Theorem 4.3, note that the key step in the proof is the observation that $l_{ii}l_{sj} - l_{si}l_{ij} \geq 0$ for all $j$ with strict inequality for $j = s$. In order to find an interpretation for $l_{ii}l_{sj} - l_{si}l_{ij} \geq 0$, consider the following two cases of firms’ direct shareholdings as depicted in Figure 4.2.

**Figure 4.2:** PCO structure resulting in $l_{sj} = l_{si}l_{ij}/l_{ii}$

Note: The arrows are directed from a firm-holder of direct stakes to owned firm(s). These links can be mutual.

Case (a) applies when only firm $i$ has a direct stake in firm $j$, and case (b) applies when only firm $s$ holds a direct stake in firm $i$ that has a direct or an indirect stake in firm $j$. In both cases the absence of firm $i$ would immediately imply that firm $s$ has no shares in firm $j$. Hence, in such cases we say that firm $s$ has a share (direct and/or indirect) in firm $j$ only due to firm $i$.

To quantify such “dependence” of firm $s$ ownership in firm $j$ on the presence of an intermediate firm $i$, let $W^{-i}$ be the modified PCO matrix derived from $W$ by setting its $i$-th row and $i$-th column to zero, and let $L^{-i} = (I - W^{-i})^{-1}$ be the associated matrix of imputed shares. Theorem 1 in Zeng (2001) implies that the

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For the sake of simplicity, in Figure 4.2 we did not draw double-sided arrows as we are primary interested in the ownership paths that stem from firm $s$ and end at firm $j$. Notice that the number of such paths are infinite if there are cycles present, and potentially all firms can participate in such indirect links infinite number of times.
sj-th element of $L^{-i}, l_{sj}^{-i}$, is equal to

$$l_{sj}^{-i} = \frac{l_{ii} l_{sj} - l_{si} l_{ij}}{l_{ii}}, \quad \text{for all } i \neq s. \quad (4.20)$$

Therefore, the part of firm $s$’s imputed share in firm $j$ which is due to (the presence of) firm $i$ ($\neq s$) is equal to

$$l_{sj} - l_{sj}^{-i} = \frac{l_{si} l_{ij}}{l_{ii}}.$$

This also implies that if firm $s$ has a stake in firm $j$ only due to firm $i$ (i.e., in the absence of firm $i$ there is no direct and/or indirect share of firm $s$ in firm $j, l_{sj}^{-i} = 0$) then it must be true that $l_{sj} = l_{si} l_{ij}/l_{ii}$, which holds for any $i \neq s$. This exactly corresponds to the cases depicted in Figure 4.2.

With respect to the intuition behind the inequality $l_{ii} l_{sj} - l_{si} l_{ij} \geq 0$, note that it is equivalent to $l_{sj} \geq l_{si} l_{ij}/l_{ii}$. Here, $l_{sj}$ is the imputed share of firm $s$ in firm $j$ and $l_{si} l_{ij}/l_{ii}$ is the part of firm $s$’s imputed share in firm $j$ which is due to (the presence of) firm $i$. It is then intuitively clear that $l_{sj} \geq l_{si} l_{ij}/l_{ii}$, because $l_{si} l_{ij}/l_{ii}$ takes into account only part of the imputed share of firm $s$ in firm $j$.

**Example:** To illustrate the case of $l_{sj} = l_{si} l_{ij}/l_{ii}$ and Theorem 4.3, we will now examine the following example. Consider an industry with 3 firms where the PCO matrix is

$$W = \begin{pmatrix} 0 & \alpha & 0 \\ \beta & 0 & \beta \\ \eta & 0 & 0 \end{pmatrix}.$$

The associated matrix of imputed shares is given by

$$L = (I - W)^{-1} = \frac{1}{1 - \alpha \beta (1 + \eta)} \begin{pmatrix} 1 & \alpha & \alpha \beta \\ \beta (1 + \eta) & 1 & \beta \\ \eta & \alpha \eta & 1 - \alpha \beta \end{pmatrix}.$$

Note that without firm 2, firm 1 has no stake in firm 3, so $l_{13}^{-2} = 0$. Since the total stake of firm 1 in firm 3 due to (the presence of) firm 2 is given by $l_{13} - l_{13}^{-2} = l_{12} l_{23}/l_{22}$, it follows that $l_{13} = l_{12} l_{23}/l_{22}$, which can be verified to hold in the example. Likewise, firm 3 has a stake in firm 2 only through firm 1, so $l_{32}^{-1} = 0$; hence, $l_{32} = l_{31} l_{12}/l_{11}$, which again can be verified to hold in the example.
Now assume that firm 1 increases its stake in firm 2 by $\omega > 0$ (hence $r = 1$ and $s = 2$). Then simple algebra shows that

$$Z^\omega - Z = \begin{pmatrix} 0 & \omega & \beta \omega \\ 0 & 0 & 0 \\ \frac{\alpha \eta \omega}{F} & \frac{\eta \omega (1 + \alpha^2 - \alpha \beta)}{F} & 0 \end{pmatrix},$$

where $F \equiv (1 - \alpha \beta)(1 - \alpha(\beta + \omega)) > 0$. Note that the second row in the matrix contains zeros. Hence, $z_{2j}$ for all $j$ does not change, implying that the collusive incentive of the target firm (firm 2) is not affected by the increase in firm 1’s stake in firm 2 (case (i) of Theorem 4.3). Moreover, if firm 3 does not have a stake in firm 1 (the acquirer), i.e., $\eta = 0$, then $l_{31} = 0$, and $z_{3j}$ does not change as well for all $j$ (case (ii) of Theorem 4.3). Finally, so long as $\eta > 0$ and $l_{i1} > 0$ for $i = 1, 3$ (i.e., $l_{ir} > 0$ for $i = 1, 3$), then the above result for $Z^\omega - Z$ shows that $z_{12}, z_{13}, z_{31}$, and $z_{32}$ increase as case (iii) of Theorem 4.3 predicts.

### 4.4.4 A firm increases its stake in a rival firm by buying shares from another rival firm

Theorem 4.3 assumes implicitly that when firm $r$ increases its stake in firm $s$, it buys additional shares from the outside investors or the controller of firm $s$. However, cases exist in which one firm buys shares in a rival firm from another rival. A case in point is a recent transaction in the global steel industry, where Luxemburg-based Arcelor has increased its stake in the Brazilian steelmaker CST from 18.6% to 27.95% by buying shares from Acesita which is also based in Brazil. To examine the effect of such ownership transfers on the incentives to collude, suppose that firm $r$ increases its stake in firm $s$ by buying an ownership stake $\phi$ from firm $k$. The resulting PCO matrix $W^{\phi}$ is obtained from the original PCO matrix $W$ by increasing the $rs$-th entry in $W$ by $\phi$ and lowering the $ks$-th entry by $\phi$. Equation (2) in Zeng (2001) shows that in this case,

$$z_{ij}^{\phi} = \frac{l_{ij}^{\phi}}{l_{ii}^{\phi}} = l_{ij} + \varepsilon_i^{\phi} l_{sj}, \quad \varepsilon_i^{\phi} = \frac{\phi(l_{ir} - l_{ik})}{1 - \phi(l_{sr} - l_{sk})}.$$  

Note that (4.21) is somewhat similar to the expression we used earlier for $z_{ij}^\omega$ in

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16 Acesita sold its entire 18.7% stake in CST to Arcelor and to CVRD which is a large Brazilian miner of iron and ore. In addition to its stake in CST, Arcelor also owns stakes in Acesita and in Belgo-Mineira, which is another Brazilian steelmaker (see “CVRD, Arcelor Team up for CST”, The Daily Deal, December 28, 2002, M&A; “Minister: Steel Duties Still Under Study - Brazil”, Business News Americas, April 8, 2002.)
The main difference is that while $\epsilon_i \geq 0$, now $\epsilon_i^\phi \geq 0$ as $l_{ir} \geq l_{ik}$.

Using (4.21) yields

$$\frac{\partial z_{ij}^\phi}{\partial \phi} = \frac{l_{ir}l_{sj} - l_{si}l_{ij}}{(l_{ii} + \epsilon_i^\phi l_{si})^2} \times \frac{l_{ir} - l_{ik}}{(1 - \phi(l_{is} - l_{sk}))^2}.$$  \tag{4.22}

Repeating the same steps as in Theorem 4.3, we obtain the following result.

**Theorem 4.4.** Starting with a PCO matrix $W$, suppose that firm $r$ buys a stake $\phi$ in firm $s$ from firm $k$, so that the new PCO matrix $W^\phi$ is obtained from $W$ by increasing the $rs$-th entry by $\phi$ and decreasing the $ks$-th by $\phi$. Then,

(i) $\hat{\delta}_s(W^\phi) = \hat{\delta}_s(W)$,

(ii) $\hat{\delta}_i(W^\phi) = \hat{\delta}_i(W)$ if $l_{ir} = l_{ik}$ (firm $i$ has the same imputed share in firms $r$ and $k$), and

(iii) $\hat{\delta}_i(W^\phi) \leq \hat{\delta}_i(W)$ for all $i \neq s$ as $l_{ir} \geq l_{ik}$.

Theorem 4.4 implies the following result:

**Corollary 4.2.** A transfer of partial cross ownership in firm $s$ from firm $k$ to firm $r$ does not affect tacit collusion if the industry maverick is firm $s$ or if, at the outset, the industry maverick has the same imputed share in firms $k$ and $r$. Otherwise, the transfer of partial cross ownership facilitates tacit collusion if the industry maverick has a larger imputed share in firm $r$ (the acquirer) than in firm $k$ (the seller) but hinders tacit collusion if the reverse holds.

Proposition 3 in Gilo et al. (2006) also considered the effects of a transfer of partial cross ownership in firm $s$ from one firm to another but under the special assumption that at the outset all firms hold the exact same ownership stakes in one another. In this case, the matrix $L$ is symmetric in the sense that its diagonal terms are all the same and its off-diagonal terms are all equal to each other. In particular, $l_{ir} = l_{ik}$ for all $i \neq r, k$, so part (ii) of Theorem 4.4 shows that $\hat{\delta}_i(W^\phi) = \hat{\delta}_i(W)$ for all $i \neq r, k$ (which includes part (i) of Theorem 4.4 if $i = s$). As for firms $r$ and $k$, then part (i) of Lemma 4.1 implies that $l_{rr} > l_{rk}$ and $l_{kr} < l_{kk}$. Hence, equation (4.22) shows that $\partial z_{ij}^\phi / \partial \phi \geq 0$ and $\partial z_{ij}^\phi / \partial \phi \leq 0$ for all $j$ with strict inequality for $j = s$. Hence, by Lemma 4.3, $\hat{\delta}_r(W^\phi) < \hat{\delta}_r(W)$ and $\hat{\delta}_k(W^\phi) > \hat{\delta}_k(W)$, implying that the transfer of partial cross ownership in firm $s$ from firm $r$ to firm $k$ strengthens the incentive of firm $r$ to collude, weakens the incentive of firm $k$ to collude and has no effect on the incentives of other firms to collude. In the symmetric case considered by Gilo et al. (2006), $\hat{\delta}_1(W) = \ldots = \hat{\delta}_n(W)$, so the incentives of all firms to collude before the transfer of ownership are the same. Hence, the transfer of partial ownership turns
firm $k$ (the seller) into a maverick firm and since $\hat{\delta}_k(W^\phi) > \hat{\delta}_k(W)$, tacit collusion is hindered.

In the present case where firms have asymmetric marginal costs, any firm can potentially be the maverick firm. In particular, Corollary 4.2 shows that collusion is hindered when the maverick is firm $k$ and is facilitated if the maverick is firm $r$.

4.4.5 Conditions for firm 1 to be the maverick

Recall from Section 4.3 that when only firm 1 invests in a rival, firm 1 is the industry maverick. In the following theorem, we provide sufficient (but not necessary) conditions that ensure that firm 1 continues to be the industry maverick even in the presence of multilateral PCO arrangements (in the sense that $\hat{\delta}_1(W) > \hat{\delta}_i(W)$ for all $i \neq 1$).

Theorem 4.5. Sufficient (but not necessary) conditions for firm 1 (the most efficient firm in the industry) to be the industry maverick is that (i) $z_{1j} \leq z_{ij}$ for all $i, j \neq 1$, and (ii) $l_{ii}\hat{y}_i \leq l_{i1}y_1(c_2)$ for all $i \neq 1$.

Recall from the profits definitions in (4.12) that for all $i \neq 1$ we have $\pi^d_i(\hat{p}; W) = l_{ii}\hat{y}_i$ and $\pi^f_i(c_2; W) = l_{i1}y_1(c_2)$. Hence, Theorem 4.5 ensures that the most efficient firm (firm 1) is the industry maverick in the multilateral PCO setting if (i) its relative imputed share in each other firm $j (\neq 1)$ is no greater than the relative share of any other firm in firm $j$, and (ii) the deviation profit of any other firm $i (\neq 1)$ is no greater than $i$’s profit after the failure of collusion.

4.5 Conclusion

Acquisitions of one firm’s stock by a rival firm have been traditionally treated under Section 7 of the Clayton Act which condemns such acquisitions when their effect “may be substantially to lessen competition.” However, the third paragraph of this section effectively exempts passive investments made “solely for investment.” As argued in Gilo (2000), antitrust agencies and courts, when applying this exemption, did not conduct full-blown examinations as to whether such passive investments may substantially lessen competition.\(^{17}\)

\(^{17}\)We are aware of only two cases in which the ability of passive investments to lessen competition was acknowledged: the FTC’s decision in *Golden Grain Macaroni Co.* (78 F.T.C. 63, 1971), and the consent decree reached with the DOJ regarding US West’s acquisition of Continental Cablevision (this decree was approved by the district court in United states v. US West Inc., 1997-1 Trade cases (CCH), 71,767, D.C., 1997).
In this study we showed that although there are cases in which passive investments in rivals (both at the expense of outside shareholders and through ownership transfer among rivals) have no effect on the ability of firms to engage in tacit collusion, an across the board lenient approach towards such investments may be misguided. This is because passive investments in rivals may well facilitate tacit collusion, especially when these investments are multilateral and in firms that are not industry mavericks. We believe that antitrust courts and agencies should take account of these factors when considering cases involving passive investments among rivals.

Throughout the paper we have focused exclusively on the effect of PCO on the ability of firms to engage in (tacit) price fixing. However, if in addition to price fixing firms can also divide the market among themselves, then they would clearly be able to sustain collusion for a larger set of discount factors since they would have more instruments (the collusive price and the market shares). In particular, it would be possible to relax the incentive constraints of maverick firms by increasing their market shares at the expense of firms with nonbinding incentive constraints. This suggests in turn that in the presence of market sharing schemes, firms may have an incentive to become industry mavericks in order to receive a larger share of the market. As our analysis shows, one way to become an industry maverick is to avoid investing in rivals. Interestingly, this implies that besides the fact that market sharing schemes are harder to enforce (firms need to commit to ration their sales) and are more susceptible to antitrust scrutiny, they have another drawback, which is that they provide firms with a disincentive to invest in rivals and thereby facilitate tacit collusion.
4.A  Proofs

Proof of Theorem 4.2. (i) Firm 1 chooses \( \hat{p} \) to maximize the left-hand side of (4.6). Assume that only \( w_{1i} > 0 \) for some \( i \neq 1 \). Then the corresponding first order condition (FOC) is \( \partial \hat{y}_1 / \partial \hat{p} + w_1 \partial \hat{y}_1 / \partial \hat{p} = 0 \). Since \( \partial \hat{y}_k / \partial \hat{p} = (\partial Q(\hat{p}) / \partial \hat{p})(\hat{p} - c_k) + Q(\hat{p}) \), the last FOC can be rewritten as

\[
w_{1i} = -\frac{\zeta \hat{p} - c_1}{\hat{p} - c_i} + 1 \equiv Y(\hat{p}, c_1, c_i),
\]

(4.A.1)

where the price elasticity of demand is \( \zeta = (\partial Q/\partial \hat{p})(\hat{p}/Q) < 0 \). Hence, if \( w_{1i} \) changes, the right-hand side (rhs) of (4.A.1), \( Y(\hat{p}, c_1, c_i) \), also has to change. Thus, for given \( c_1 \) and \( c_i \) the collusive price must change. Using the fact that \( \partial \zeta / \partial \hat{p} = \zeta (1 - \zeta^2) / \hat{p} + (\hat{p}/Q)(\partial^2 Q / \partial \hat{p}^2) \), the derivative of the rhs of (4.A.1) with respect to the collusive price after some simple mathematical transformations can be shown to be given by

\[
\frac{\partial Y(\hat{p}, c_1, c_i)}{\partial \hat{p}} = \frac{(c_i - c_1)(2\zeta^2 - \partial^2 Q/\hat{p}^2)}{(\zeta \hat{p} - c_i + 1)^2} \quad \text{for all } i = 2, \ldots, n.
\]

(4.A.2)

Assumption 1 implies that the rhs of (4.A.2) is positive: for concave demand functions \( \partial^2 Q / \partial \hat{p}^2 \leq 0 \). Note that Assumption 1 also allows for not too convex demand functions (with \( \partial^2 Q / \partial \hat{p}^2 \geq 0 \), which we interpret here as \( \partial^2 Q / \partial \hat{p}^2 > 0 \). Thus from (4.A.1) and (4.A.2) it follows that \( \hat{p}^* \) is increasing with \( w_{1i} \) and is above \( p^m_1 \) recalling that \( p^m_1 < p^m_2 < \ldots < p^m_n \). The economic intuition is simple: when firm 1 gives a positive (or more) weight to the direct profits of other firms that call for higher prices, the collusive price must go up.

(ii) Absent PCO, the critical discount factor above which collusion can be sustained is \( \hat{\delta}_1(p^m_1) \). Using (4.5) and (4.7) it is clear that,

\[
\hat{\delta}_1(p^m_1) > \frac{y^m_1 - (y^m_1 + \sum_{i \neq 1} w_{1i} y_1(p^m_1)) / n}{y^m_1 - y_1(c_2)} \geq \frac{y^m_1 - (\hat{y}^*_1 + \sum_{i \neq 1} w_{1i} \hat{y}_i) / n}{y^m_1 - y_1(c_2)} \equiv \delta_1^{\text{po}}(\hat{p}^*),
\]

where \( \hat{y}^*_i = Q(\hat{p}^*)(\hat{p}^* - c_i) \) and the weak inequality follows because \( \hat{p}^* \) maximizes \( \hat{y}_1 + \sum_{i \neq 1} w_{1i} \hat{y}_i \). To complete the proof, note that by the envelope theorem,

\[
\frac{d \delta_1^{\text{po}}(\hat{p}^*)}{dw_{1i}} = -\frac{\hat{y}_i / n}{y^m_1 - y_1(c_2)} < 0.
\]
Using (4.16) and (4.17) we obtain

(i) Equation (4.19) implies that if \( \hat{p}_2 > \ldots > \hat{p}_n \), implying that PCO by firm 1 in an efficient rival raises \( \hat{p}^* \) by less and lowers \( \hat{p}_1^{\text{po}} (\hat{p}^*) \) by more than does a similar investment in a less efficient rival. The (more) formal proof of the first statement is the fact that one can easily show that the rhs of (4.A.2) is increasing in \( c_i \), i.e., \( \frac{\partial Y(\hat{p}, c_1, c_i)}{\partial p_{dc_i}} > 0 \). That is, the higher \( c_i \) (the less efficient is firm \( i \)), the more price change is needed for the FOC in (4.A.1) to hold.

**Proof of Theorem 4.3.** (i) Equation (4.19) implies that if \( i = s \) (firm \( i \) is the target firm \( s \)), then \( \partial z_{s1}^\omega / \partial \omega = 0 \) for all \( j \). Hence, by Lemma 4.3, \( \hat{\delta}_s(W^\omega) = \hat{\delta}_s(W) \).

(ii) Equation (4.19) implies that if \( l_{ir} = 0 \) (firm \( i \) has no direct and indirect stake in the investing firm \( r \)), then \( \partial z_{ij}^\omega / \partial \omega = 0 \) for all \( j \). Again, by Lemma 4.3, \( \hat{\delta}_i(W^\omega) = \hat{\delta}_i(W) \).

(iii) Now suppose that \( i \neq s \) and \( l_{ir} > 0 \). Theorem 1 in Zeng (2001) ensures that \( l_{ij} l_{si} - l_{si} l_{ij} \geq 0 \) for all \( j \). When \( j = s \), the inequality is strict since then \( l_{ij} l_{si} - l_{si} l_{ij} > 0 \), where the inequality follows because part (i) of Lemma 4.1 establishes that \( l_{ij} < l_{ii} \) for all \( j \neq i \). Together with the fact that \( l_{ir} \geq 0 \), this implies that \( \partial z_{ij}^\omega / \partial \omega \geq 0 \) for all \( i \) and all \( j \), with a strict inequality for \( j = s \). Hence, by Lemma 4.3, \( \hat{\delta}_i(W^\omega) < \hat{\delta}_i(W) \) for all \( i \neq s \).

**Proof of Theorem 4.5.** Using (4.16) and (4.17) we obtain

\[
\hat{\delta}_1(W) - \hat{\delta}_1(W) = \left( y_1^m - \frac{1}{n} \sum_{j=1}^n z_{1j} \hat{y}_j \right) \left( \hat{y}_i - z_{i1} y_1(c_2) \right) - \left( \hat{y}_i - \frac{1}{n} \sum_{j=1}^n z_{ij} \hat{y}_j \right) \left( y_1^m - y_1(c_2) \right)
\]

\[
= \left( \hat{y}_i - z_{i1} y_1(c_2) \right) y_1(c_2) - \frac{1}{n} \sum_{j=1}^n z_{ij} \hat{y}_j \left( \hat{y}_i - z_{i1} y_1(c_2) \right) + \frac{1}{n} \sum_{j=1}^n z_{ij} \hat{y}_j \left( y_1^m - y_1(c_2) \right)
\]

\[
= \left( \hat{y}_i - z_{i1} y_1(c_2) \right) y_1(c_2) - \frac{1}{n} \sum_{j\neq1} z_{ij} \hat{y}_j \left( \hat{y}_i - z_{i1} y_1(c_2) \right) + \frac{1}{n} \sum_{j\neq1} z_{ij} \hat{y}_j \left( y_1^m - y_1(c_2) \right)
\]

where the last equality follows because by definition, \( z_{11} = 1 \). Adding and subtracting \( \frac{1}{n} \sum_{j\neq1} z_{ij} \hat{y}_j \left( \hat{y}_i - z_{i1} y_1(c_2) \right) \) and \( \frac{1}{n} \sum_{j\neq1} z_{ij} \hat{y}_j \left( y_1^m - y_1(c_2) \right) \) to the numerator and rearranging terms yields
\[ \hat{\delta}_1(W) - \hat{\delta}_i(W) \]

\[
= \frac{(z_{i1}y_1^m + \sum_{j \neq 1} z_{ij}\hat{y}_j - \hat{y}_i)}{\left(\hat{y}_i - z_{i1}y_1(c_2)\right)} \left(\frac{\tilde{y}_1}{n} - y_1(c_2)\right) + \frac{1}{n} \sum_{j \neq 1} (z_{ij} - z_{1j}) \hat{y}_j (\hat{y}_i - z_{i1}y_1(c_2)) \\
+ \frac{1}{n} \sum_{j \neq 1} z_{ij}\hat{y}_j \left(\left(\frac{y_m^m}{n} - y_1(c_2)\right) - (\hat{y}_i - z_{i1}y_1(c_2)) - (\hat{y}_i - ny_1(c_2))\right) \\
= \frac{(z_{i1}y_1^m + \sum_{j \neq 1} z_{ij}\hat{y}_j - \hat{y}_i)}{\left(\hat{y}_i - z_{i1}y_1(c_2)\right)} \left(\frac{\tilde{y}_1}{n} - y_1(c_2)\right) + \frac{1}{n} \sum_{j \neq 1} (z_{ij} - z_{1j}) \hat{y}_j (\hat{y}_i - z_{i1}y_1(c_2)) \\
+ \frac{\sum_{j \neq 1} z_{ij}\hat{y}_j \left(\frac{y_m^m}{n} - \hat{y}_1\right)}{\left(\hat{y}_i - z_{i1}y_1(c_2)\right)} \left(\frac{(n-1)y_1(c_2)}{n} + \frac{l_1y_1(c_2) - l_i\tilde{y}_i}{l_i n}\right),
\]

Recalling that \( z_{ii} = 1 \), it follows that \( \sum_{j \neq 1} z_{ij}\hat{y}_j > \hat{y}_i \). Moreover, \( \hat{y}_1/n \geq y_1(c_2) \) (see Figure 4.1). Hence, the first term is positive. Therefore, firm 1 is the maverick firm in the industry in the sense that \( \hat{\delta}_1(W) > \hat{\delta}_i(W) \) for all \( i \neq 1 \) if \( z_{1j} \leq z_{ij} \) for all \( i, j \neq 1 \), and \( l_{ii}\hat{y}_i \leq l_{11}y_1(c_2) \) for all \( i \neq 1 \). This completes the proof. \( \blacksquare \)