Strategic differentiation in finite network games

Masked, I advance.

René Descartes

Simple decisions or actions taken by interacting individuals can lead to surprisingly complex and unpredictable population-level outcomes. In particular, when individual decisions or actions are based on personal interest, the long run collective behavior, characterized by these selfish decisions, can be detrimental for the population as a whole. Mathematical models of such systems require trade-offs between the complexity of micro-dynamics and the accuracy with which the model can describe a macro-behavior.

Evolutionary game theory has proven to be a valuable tool in providing mathematical models for such complex dynamical systems. In its original application to evolving biological populations, rational reasoning or decision-making is not needed; rather, competing strategies propagate through a population via natural selection. Economists later adapted this evolutionary game model for the mathematical modeling of individual decision-makers with bounded rationality [41]. To reach decisions they are satisfied with, players may thus rely on simple rules. Relative best responses and rational imitations, that we have studied in the previous chapters, are examples of such myopic decision rules. As we have seen, these decision-making models can be extended to include groupwise interactions [96], that are important to study because many
biological and social interactions involve more than two individuals whose collective decisions can have a variety of behaviors even in well-mixed populations [103].

A common assumption in the existing models for finite network games is that players do not distinguish between their opponents. In some sense the opponents are anonymous and hence, there is no difference in the actions employed against each of them. Indeed, in the previous chapters, it was assumed that all players employ the same action against their opponents. However, to create a competitive advantage, in real life competitive settings, it is often crucial to identify the rivals [104]. Avoiding ‘blindspots’ in a competitive decision-making process, i.e. those decisions that require taking into account the decisions of competitors, is a major topic in the strategic decision-making literature [105]. In such competitive environments, decision-makers are likely to distinguish their opponents, and consequently, they may employ different actions against them.

In this chapter, the mechanism of strategic differentiation is introduced through which a subset of players in the network, called differentiators, can employ different actions against different opponents. We will connect strategic differentiation to the theory of potential games and their generalizations and show that for the class of weighted potential games the effect of strategic differentiation on any network topology can be studied analytically using the potential function of the original game. In the following, we will distinguish groupwise and pairwise games on networks. To make the difference clear we introduce some additional notation. Let \( \pi_{ij}(\sigma_i, \sigma_j) \) denote the payoff that player \( i \) obtains from action \( \sigma_i \in A_i \) in the pairwise interaction against opponent \( j \in N_i \) with action \( \sigma_j \in A_j \). The total payoff that player \( i \) obtains in a single round of play with pairwise interactions is

\[
\pi_i(\sigma_i, \sigma_{-i}) = \sum_{j \in N_i} w_{ij} \pi_{ij}(\sigma_i, \sigma_j),
\]

with \( w_{ij} \in \mathbb{R} \) denoting the weight associated to the local interaction between \( i \) and \( j \). We refer to a non-cooperative game with a payoff function of the form Eq. (5.1) as a pairwise network game. An example of such a game is the famous spatial prisoner’s dilemma game. As we have seen in the previous chapter, players may also interact in groups with a size greater than two, and thus the local interactions form a multiplayer game. The spatial public goods game studied in the previous chapter is an example of such a groupwise network game. In general, the payoffs of groupwise network games cannot be represented by the corresponding sum of pairwise interactions, however the total payoff that player \( i \) obtains in a single round of play is again a weighted sum of the local payoffs,

\[
\pi_i(\sigma_i, \sigma_{-i}) = \sum_{j \in \bar{N}_i} w_{j} \pi_{ij}(\sigma_i, \sigma_{-i}),
\]

(5.2)
with \( w_j \in \mathbb{R} \) denoting the weight associated to the multiplayer game with exactly those players in \( \mathcal{N}_j \). Note that in Eq. (5.2) the single round local payoffs depend on \( |\mathcal{N}_j + 1| \geq 2 \) actions and the network structure imposes an interdependence in the payoffs of players that are connected via an undirected path with length two, sometimes referred to as the 2-hop neighbors.

### 5.1 Strategic Differentiation

In a network game with strategic differentiation, a differentiator can employ a separate pure action for each neighbor; see figure 5.1 for an example of a pairwise game with a single differentiator on a ring network. Let \( \mathcal{D} \) be a non-empty subset of \( \mathcal{V} \) denoting the set of differentiators in the network, and let \( \mathcal{F} := \mathcal{V} \setminus \mathcal{D} \) denote the set of non-differentiators. In a groupwise network game, the actions of a player \( i \in \mathcal{D} \) is a vector \( s_i \in S_i := \mathcal{A}_{|\mathcal{N}_i|+1} \): a separate action can be chosen in each of the multiplayer game with the closed neighborhoods that the player belongs to. The action space of differentiators is indicated by \( \mathcal{S}_\mathcal{D} := \prod_{i \in \mathcal{D}} S_i \). When the game interactions are pairwise, the dimension of the action vector of player \( i \) is reduced by one because in this case players only interact with their \( |\mathcal{N}_i| \) neighbors. For some \( j \in \mathcal{N}_i \), we indicate by \( s_{ij} \in s_i \) the action that player \( i \in \mathcal{D} \) employs in the local pairwise (resp. groupwise) game played against \( j \) (resp. \( \mathcal{N}_j \)). Note that for all \( i \in \mathcal{D} \) and \( j \in \mathcal{N}_i \) we assume that \( s_{ij} \in \mathcal{A}_i \), i.e. each action employed by a differentiator is in its own action set. The action space of players who do not differentiate is indicated by \( \mathcal{A}_\mathcal{F} := \prod_{j \in \mathcal{F}} \mathcal{A}_j \). Without loss of generality, label differentiators by \( \mathcal{D} = \{1, \ldots, |\mathcal{D}|\} \) and the non-differentiators by \( \mathcal{F} = \{|\mathcal{D} + 1|, \ldots, n\} \). Then the action space of the

![Figure 5.1: Graphical interpretation of a pairwise non-cooperative game on a network with strategic differentiation. The label of outgoing edges indicates the action played in the local pairwise interaction. In this example \( \mathcal{D} = \{1\} \) and \( \mathcal{F} = \{2, 3, 4\} \).](image)

networked game with strategic differentiation is given by

\[ S = S_D \times A_F. \]

An action profile in the action space of a game with strategic differentiation is indicated by \( s \in S \). As before, \( s_{-i} \in S_{-i} \) indicates the action profile of all players except player \( i \). In a game with strategic differentiation we denote the local payoff function for the interaction between \( i \) and (the neighbors of) \( j \in \bar{N}_i \) by \( u_{ij} : S \to \mathbb{R} \).

Similarly, \( u : S \to \mathbb{R}^n \) denotes the combined payoff vector of the game with strategic differentiation. For pairwise interactions the payoffs of a differentiator \( i \in D \) is

\[
u_i(s_i, s_{-i}) = \sum_{j \in \bar{N}_i \cap D} w_{ij} u_{ij}(s_{ij}, s_{ji}) + \sum_{h \in \bar{N}_i \cap F} w_{ih} u_{ih}(s_{ih}, \sigma_h). \tag{5.3}
\]

The payoff of a non-differentiator \( k \in F \) is

\[
u_k(s_k, s_{-k}) = \sum_{m \in \bar{N}_k \cap D} w_{km} u_{km}(\sigma_k, s_{mk}) + \sum_{v \in \bar{N}_k \cap F} w_{kv} u_{kv}(\sigma_k, \sigma_v) \tag{5.4}
\]

For games with strategic differentiation and groupwise interactions, as in Eq. (5.2), the local payoff function \( u_{ij} \) will depend on more than two actions. For \( i \in D \),

\[
u_i(s_i, s_{-i}) = \sum_{k \in \bar{N}_i} w_{ik} u_{ik}(s_{ik}, s_{-i}). \tag{5.5}
\]

And the payoff of a non-differentiator \( j \in F \) is,

\[
u_j(s_j, s_{-j}) = \sum_{k \in \bar{N}_j} w_{jk} u_{jk}(\sigma_j, s_{-j}). \tag{5.6}
\]

We are now ready to formally define a network game with strategic differentiation.

**Definition 17** (Strategically differentiated game). A network game with strategic differentiation is defined by the triplet \( \Xi := (G, S, u) \). If \( \pi_{ij} = u_{ij} \) for all \((i, j) \in E\), then \( \Xi \) is said to be the strategically differentiated version of \( \Gamma = (G, A, \pi) \).

**Example 6** (Strategic differentiation in a groupwise game). As an example of a groupwise game with strategic differentiation consider a linear public goods game in which players need to determine whether or not to contribute to a public good that their opponents can profit from. This decision is modeled by the pure action set \( A_i = \{0, 1\} \) for all \( i \in V \). A differentiator \( i \in D \) may choose to contribute to one good but withhold from contributing to another. Hence, \( s_i \in \{0, 1\}^{\left| N_i \right| + 1} \). For some \( j \in \bar{N}_i \), when \( s_{ij} = 1 \) (resp. \( s_{ij} = 0 \)) player \( i \) is cooperating (resp. defecting) in the local game against the group of opponent players \( \bar{N}_j \). Let \( c_{ij} \in \mathbb{R}_{>0} \) denote the contribution of a cooperating
player \( i \) in the game against the neighbors of \( N_j \subseteq D \). In a public goods game, the contributions get multiplied by an enhancement factor \( r \in [1,n] \), which can be seen as a benefit-to-cost ratio of the local interaction. The payoff that player \( i \) in this local game is,

\[
    u_{ij}(s_{ij}, s_{-i}) = \frac{r(\sum_{k \in N_j} s_{kj} c_{kj} + s_{jj} c_{jj})}{d_j + 1} - c_{ij} s_{ij}.
\]

### 5.2 Rationality in games with strategic differentiation

A best response of a differentiator is a vector of actions for which each element is a best response in the corresponding local game.

**Definition 18** (Differentiated Best Response). For a differentiator \( i \in D \), the action \( s_i \in S_i \) is a strategically differentiated pure best response against \( s_{-i} \) if for all \( s_{ik} \in s_i \)

\[
    s_{ik} \in \arg \max_{x \in A_i} u_{ik}(x, s_{-i})
\]

Based on the definition of a differentiated best response, a Nash equilibrium in a strategically differentiated game is naturally defined as follows.

**Definition 19** (Differentiated Pure Nash equilibrium). An action profile \( s \in S \) is a differentiated pure Nash equilibrium of \( \Xi \) if for all \( j \in F \), \( \sigma_j \in s \) is a best response and for all \( i \in D \), \( s_i \in s \) is a strategically differentiated pure best response.

When \( D = \emptyset \) the original definition of a pure Nash equilibrium is recovered. Best responses of differentiators are thus defined as vectors of actions for which each element is locally optimal. Herein lies the main distinguishing feature of best replies in games without strategic differentiation: a best reply \( x^*_i \) over the aggregated payoff \( \pi_i(x_i, x_{-i}) \) might not optimize the payoffs of each separate local game with payoff \( \pi_{ij} \). Hence, a strategically differentiated Nash equilibrium can contain actions that are not present in the Nash equilibrium of the game without strategic differentiation.

Let us now consider myopic best response dynamics in games with strategic differentiation: the action \( s_{ij} \) is chosen such that it maximizes \( u_{ij}, \) ceteris paribus.

**Definition 20** (Differentiated myopic best response dynamics). If a player \( i \) updates using differentiated best responses the resulting (myopic) best response dynamics are

\[
    s_{ij}(t + 1) \in \arg \max_{y \in A_i} u_{ik}(y, s_{-i}(t)).
\]
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When all differentiators update their actions according to the differentiated best response dynamics Eq. (5.9) and the others according to myopic best response dynamics, we indicate the “evolutionary” game with strategic differentiation by \((\Xi, \beta)\).

Remark 6. Differentiated myopic best response dynamics are an unconstrained version of myopic best responses in the sense that the local actions are optimized over the local payoffs without requiring that the employed actions are equal. It follows that when \(D = V\), for innovative update dynamics like myopic best response the effect of the network structure on the equilibria of the network game is lost. That is, the equilibrium action profiles of the networked game would, in this case, correspond to a collection of separate Nash equilibria of the local games played on the network. When \(D \subset V\) the network structure remains important to the myopic best response dynamics. Moreover, the differentiators may obtain an advantage over their opponents that are not able to differentiate their actions because for each \(\sigma_i \in A_i, s_{-i} \in S_{-i}\)

\[\exists s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(\sigma_i, s_{-i}).\]

Hence, in terms of payoffs, players that differentiate their actions rationally are always at least as successful as they would have been not differentiating. The benefit that differentiators can get over non-differentiators implies that especially for evolutionary update dynamics in which the most successful players are imitated, the existence of differentiators can have a significant impact on the evolution of the actions in the network.

5.3 Potential functions for network games with strategic differentiation

In this section, we describe conditions on the local interactions of network games that ensure that their strategically differentiated versions have pure Nash equilibria and convergence of differentiated myopic best response dynamics is guaranteed. For this, we apply the theory of potential games to strategically differentiated games. Consider the following definition derived from ordinal potential games [42].

Definition 21 (Differentiated ordinal potential game). \(\Xi\) is a strategically differentiated ordinal potential game if there exists an ordinal potential function \(P : S \rightarrow \mathbb{R}\) such that for all \(\sigma_j, \sigma'_j \in A_j, s_i, s'_i \in S_i, s_{-i} \in S_{-i}\) and \(s_{-j} \in S_{-j}\) the following holds:

\[u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) > 0 \Leftrightarrow P(s_i, s_{-i}) - P(s'_i, s_{-i}) > 0, \quad \forall i \in D,\]

\[u_j(\sigma_j, s_{-j}) - u_j(\sigma'_j, s_{-j}) > 0 \Leftrightarrow P(\sigma_j, s_{-j}) - P(\sigma'_j, s_{-j}) > 0, \quad \forall j \in F.\]
Note that if $D = \emptyset$, the original definition of an ordinal potential game introduced in [42] is recovered from Definition 21. It is well known that every finite ordinal potential game has a pure Nash equilibrium. This property is generalized to strategically differentiated games in the following lemma.

**Lemma 8.** Every finite differentiated ordinal potential game possesses a differentiated pure Nash equilibrium.

**Proof.** First assume $s^* \in S$ is a differentiated Nash equilibrium for $\Xi$. Then, for all $i \in D$, $s^*_i \in s^*$ is such that

$$u_i(s^*_i, s^*_{-i}) - u_i(s_i, s^*_{-i}) \geq 0, \forall s_i \in S_i.$$

For all $j \in F$ the actions of the non-differentiators $\sigma^*_j \in s^*$ are such that

$$u_i(\sigma^*_j, s^*_{-j}) - u_i(\sigma_j, s^*_{-j}) \geq 0, \forall \sigma_j \in A_j.$$

By the definition 21 of a differentiated ordinal potential game it follows that

$$P(s^*_i, s^*_{-i}) - P(s_i, s^*_{-i}) \geq 0, \forall i \in D \text{ and } \forall s_i \in S_i,$$

$$P(\sigma^*_j, s^*_{-j}) - P(\sigma_j, s^*_{-j}) \geq 0, \forall j \in F \text{ and } \forall \sigma_j \in A_j.$$

Hence $s^*$ is also a maximum point in $P$. Similarly one can show that each maximum point in $P$ is a differentiated Nash equilibrium of $\Xi$. Since for finite games $S$ is bounded, a maximum of $P$ always exists. This completes the proof.

One can show that if $\Xi$ is a differentiated ordinal potential game, then $(\Xi, \beta)$ will always terminate in a differentiated Nash equilibrium. Instead, we now focus on finding conditions on the local interactions in groupwise games on networks that ensure the convergence properties of $\Gamma$ under best responses are preserved in its strategically differentiated version $\Xi$. This is especially useful when one already has a potential function for the original game on a network and is interested in comparing the behavior of the game with strategic differentiation. Before doing so, consider the following definition.

**Definition 22** (Differentiated weighted potential games). $\Xi$ is a strategically differentiated weighted potential game if there exists a potential function $\hat{P} : S \to \mathbb{R}$ and weights $\alpha_i, \alpha_j \in \mathbb{R}_{>0}$, such that for all $\sigma_j, \sigma'_j \in A_j$, $s_i, s'_i \in S_i$, $s_{-i} \in S_{-i}$, and $s_{-j} \in S_{-j}$ the following holds:

$$u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) = \alpha_i \left[ \hat{P}(s_i, s_{-i}) - \hat{P}(s'_i, s_{-i}) \right], \quad \forall i \in D$$

$$u_j(\sigma_j, s_{-j}) - u_j(\sigma'_j, s_{-j}) = \alpha_j \left[ \hat{P}(\sigma_j, s_{-j}) - \hat{P}(\sigma'_j, s_{-j}) \right], \quad \forall j \in F.$$

Note that if $D = \emptyset$, the original definition of a weighted potential game introduced in [42] is recovered.
The following result relates the fundamental properties of weighted potential games to their strategically differentiated version.

**Theorem 7.** In $\Gamma$, if for all players $i \in V$ there exists for each local payoff function $\pi_j : A \to \mathbb{R}$, $j \in N_i$, a weighted potential function $\rho_j : A \to \mathbb{R}$ with a common weight $\alpha_i \in \mathbb{R}_{>0}$ for player $i$, then $(\Xi, \beta)$ converges to a differentiated pure Nash equilibrium.

**Proof.** For all $j \in F$, let $s_j := (\sigma_j, \ldots, \sigma_j) \in A_j^{[N_j+1]}$ such that each element in $s_j$ is equal to $\sigma_j \in A_j$. For all differentiators $i \in D$ let $s_i := s_i$. Then, for all $j \in F$ the payoff in the strategically differentiated game can be written as

$$u_j(\sigma_j, s_{-j}) = \sum_{k \in N_j} w_k u_{jk}(\bar{s}_{jk}, \bar{s}_{-jk}),$$

where $\bar{s}_{-jk}$ denotes the set of actions in the game centered around $k$, chosen by the neighbors of $k$ different from $j$, i.e., $\bar{s}_{-jk} := \{s_{vk} \in A_v : v \neq j \land v \in N_k\}$. For differentiators $i \in D$, the payoff in the strategically differentiated game is

$$u_i(s_i, s_{-i}) = \sum_{k \in N_i} w_k u_{ij}(\bar{s}_{ik}, \bar{s}_{-ik}).$$

By assumption, in any local game with the neighbors of some $k \in V$, for all $i \in \hat{N}_k$ there exists a function $\rho_k : A \to \mathbb{R}$ and weights $\alpha_i \in \mathbb{R}_+$, such that for every $\sigma_i, \sigma'_i \in A_i$ and every $\sigma_{-i} \in A_{-i}$ the following equality holds,

$$\pi_{ik}(\sigma_i, \sigma_{-i}) - \pi_{ik}(\sigma'_i, \sigma_{-i}) = \alpha_i [\rho_k(\sigma_i, \sigma_{-i}) - \rho_k(\sigma'_i, \sigma_{-i})].$$

For all the non-differentiators $j \in \hat{N}_k \cap F$, it follows that for every $\sigma_j, \sigma'_j \in A_j$ and $\bar{s}_{-jk} \in \prod_{v \in \hat{N}_k \setminus \{j\}} A_v$ it holds that

$$u_{jk}(x_j, \bar{s}_{-jk}) - u_{jk}(x'_j, \bar{s}_{-jk}) = \alpha_j [\rho_k(\sigma_j, \bar{s}_{-jk}) - \rho_k(\sigma'_j, \bar{s}_{-jk})] = \alpha_j [\rho_k(\bar{s}_{jk}, \bar{s}_{-jk}) - \rho_k(s'_j, \bar{s}_{-jk})].$$

Similarly for the differentiators $i \in D \cap \hat{N}_k$, since $\bar{s}_{ik} \in A_i$ and $\bar{s}_{-ik} \in A_{-ik}$, from the existence of the local weighted potential function $\rho_k$ it follows that for all $\bar{s}_{ik}, \bar{s}'_{ik} \in A_i$, $\bar{s}_{-ik} \in A_{-ik}$

$$u_{ik}(\bar{s}_{ik}, \bar{s}_{-ik}) - u_{ik}(\bar{s}'_{ik}, \bar{s}_{-ik}) = \alpha_i [\rho_k(\bar{s}_{ik}, \bar{s}_{-ik}) - \rho_k(s'_{ik}, \bar{s}_{-ik})].$$

Let $\bar{s}_{\{N_k\}} := \{s_{kv} \in A_k : k \in \hat{N}_k\}$ denote the set of actions employed by the players in the local interaction with the closed neighborhood of $l$. The difference in the payoffs
of a unique deviator $i \in V$ switching from action $\bar{s}_i$ to $\bar{s}_i'$ is given by
\begin{equation}
    u_i(\bar{s}_i, \bar{s}_{-i}) - u_i(\bar{s}_i', \bar{s}_{-i}) = \sum_{j \in \bar{N}_i} w_j \left[ u_{ij}(\bar{s}_{ij}, \bar{s}_{-ij}) - u_{ij}(\bar{s}_{ij}', \bar{s}_{-ij}) \right]
    = \sum_{j \in N_i} w_j \left[ \alpha_i \left( \rho_j(\bar{s}_{ij}) - \rho_j(\bar{s}_{ij}') \right) \right]
    = \alpha_i \sum_{j=1}^n w_j \left( \rho_j(\bar{s}_{j(N_j)}) - \rho_j(\bar{s}_{j(N_j)'}) \right).
\end{equation}

The last equality in Eq. (5.10) follows from the fact that when the unique deviator is not a member of some closed neighborhood $N_h$, then $\rho_h(s_{j(N_j)}) - \rho_h(s_{j(N_j)'}) = 0$. This implies that
\begin{equation}
    \bar{P} = \sum_{j \in V} w_j \rho_j(\bar{s}_{j(N_j)}),
\end{equation}
with weights $\alpha_i$ is a weighted potential function and thus $\Xi$ is a strategically differentiated weighted potential game. The convergence of the differentiated myopic best response then follows from the argument used in traditional potential games. Clearly, since $\bar{S}$ is finite, $\bar{P}$ is bounded. Moreover it is increasing along the trajectory generated by asynchronous myopic best responses of non-differentiators and asynchronous differentiated myopic best responses of differentiators. This implies convergence of the differentiated myopic best response action update dynamics to a differentiated pure Nash equilibrium.

The proof of Theorem 7 can be easily adjusted to show that the same statement holds for strategically differentiated pairwise games on networks with $w_{ij} = w_{ji}$ for all $(i, j) \in \mathcal{E}$. The following corollary of Theorem 7 applies to the class of exact potential games in which $\alpha_i = 1$ for all $i \in V$.

**Corollary 4.** If the local groupwise interactions of $\Gamma$ are potential games, then $(\Xi, \beta)$ converges to a strategically differentiated Nash equilibrium.

**Remark 7.** Theorem 7 and its corollary hold because there always exists a weighted potential function for payoffs that are a linear combination of local payoffs obtained from either potential games or weighted potential games with the same action sets and fixed weight vectors [106]. Hence, conditions on the local game interactions extend to the entire network game and its strategically differentiated version. This linear combination property does not hold for ordinal potential games [42]. Thus, assuming that the entire network game $\Gamma$ is an ordinal potential game may not be sufficient for convergence of its strategically differentiated version $(\Xi, \beta)$. Up to now, we have not been able to find conditions for ordinal potential games that ensure that their differentiated versions share their fundamental convergence properties.
5.4 The free-rider problem with strategic differentiation

In many social situations, individual members of a group can benefit from the efforts of other group members. When the individuals tend to be selfish the possibility to profit from others naturally results in trying to balance out one's efforts and rewards. In economics, the free-rider problem describes a situation in which a good or service becomes under-provided or even depleted as a result of selfish individuals profiting from a good without contributing to it. Here, we seek to determine how strategic differentiation can result in more desirable outcomes in which contributions to a good are preserved in the long run. The problem of finding a pure Nash equilibrium in a finite potential game is PLS-complete [107]. Therefore, we investigate the effect of strategic differentiation on the equilibrium action profiles of multiplayer games on networks via simulation. Unless stated otherwise, all simulations are conducted on a threshold public goods game which is a non-linear version of the public goods game described in Example 6. The non-linearity in the payoff function is created by requiring a minimum number of cooperators in order for the players to obtain a benefit from the local interaction. For all \( i \in V \), let \( \tau_i \) denote this threshold value of cooperators in the local game played by the players in \( N_i \). The payoffs in the local game of the threshold public goods game are given by,

\[
\forall j \in N_i : u_{ji}(\sigma) = \frac{r \sum_{j \in N_i} (\sigma_j c_{ji})}{|N_i| + 1} \theta_i(\sigma, \tau_i) - c_{ji} \sigma_j,
\]

with \( \theta_i(\sigma, \tau_i) \) defined by

\[
\theta_i(\sigma, \tau_i) = \begin{cases} 
1, & \text{if } \sum_{j \in N_i} \sigma_j \geq \tau_i, \\
0, & \text{otherwise}.
\end{cases}
\]

This model is well established in the fields of economics, sociology and evolutionary biology and captures the free-rider problem because defectors can benefit from contributions of cooperators [96,108,109], and it is known that the cooperator thresholds in the model can promote moderate levels of cooperation at an equilibrium [110]. In the absence of thresholds (i.e. \( \tau_i = 1 \) for all \( i \in V \)), it can be shown that the best response set for a player is solely determined by the degree distribution of the network and the public goods multiplier \( r \), and thus, is static. The addition of thresholds thus allows for richer and more complex decision-making dynamics under best responses. In all simulations we start with 50% cooperators which are randomly assigned to the nodes on the network. The local contributions are determined by a players degree: for all \( i \in V, j \in N_i, c_{ij} = \frac{1}{|N_i|+1} \). Hence, the total contributions that a player can make is
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\[ \sum_{j \in \bar{N}_i} c_{ij} = 1. \] This corresponds to a set-up known as fixed costs per individual [96]. The total number of contributions in an equilibrium action profile \( \hat{s} \) is determined as,

\[ 0 \leq \sum_{i \in D} \sum_{j \in N_i} c_{ij} s_{ij} + \sum_{h \in F} \sum_{t \in N_h} c_h \sigma_h \leq n. \]

5.4.1 Differentiated Best Response

In this section, we investigate the effect of differentiators on the existence of cooperation in a differentiated Nash equilibrium. For every player the threshold is equal to two, i.e., \( \tau_i = 2 \) for all \( i \in V \). The considered network has size 50 and was formed by a preferential attachment process and thus has high degree heterogeneity. When initially there are no differentiators in the network, the total contributions in the Nash equilibrium is close to zero. Because \( r = 2.4 < \bar{d} := \frac{\sum_{i \in V} |N_i|}{N} \), this is consistent with a rule for the emergence of cooperation in games on networks proposed in [40]. When differentiators are added to the network the level of cooperation in equilibrium changes significantly. For low values of \( r \), within the two-hop neighborhood of differentiators located at high degree nodes, cooperation starts to exist in the differentiated Nash equilibrium (Fig. 5.2). However, when the differentiators are placed on low degree nodes (i.e., \( |N_i| \leq \bar{d} \)) the total number of contributions in equilibrium tends to be lower than in the Nash equilibrium without differentiators. The same qualitative effects of strategic differentiation in a set-up known as fixed cost per game, in which the total contribution that a player can make is equal to their degree in the network. This illustrates that cooperation can be promoted if individuals with a large social network (i.e. hubs) can differentiate their actions. An explanation is that, in traditional network games, cooperating hubs can be taken advantage of by many players and therefore tend to defect when they cannot differentiate. Indeed, when the group size of a multiplayer game becomes larger the emergence of cooperation becomes more difficult [111]. When players located at hubs can differentiate however, they can cooperate against cooperators and defect against defectors, thereby promoting the emergence of cooperation in their neighborhood. On the other hand, when low degree players can differentiate, network reciprocity [8] becomes less effective because cooperators surrounded by other cooperators can start to free-ride in their separate local games. When the network has a narrow degree distribution as in small-world networks or regular networks, the effect of differentiators on the emergence of cooperation in the equilibrium action profile is not as pronounced and more differentiators are needed to make a significant change to the equilibrium action profile.
Figure 5.2: Each plot is obtained by averaging 10 trajectories generated by the differentiated myopic best response dynamics for the same initial condition with differentiators placed on high degree nodes. In both cases $n = 50$ and the network is generated by a preferential attachment process [92].

5.4.2 Differentiated Imitation

We have seen that the existence of differentiators and social influence in network games can promote the emergence of cooperation at an equilibrium action profile of a non-cooperative game on a network in which players seek to optimize their payoffs by playing best responses. Now we assume the players update their actions according to an imitation process in which each differentiator updates his/her action in a local game by imitating an action of the best performing players in that local game. The players who do not differentiate, update their actions by imitating one of their best performing neighbors [112]. For these imitation based dynamics, the effect of differentiators located on high degree nodes in the network is remarkable. For a neutral benefit to cost ratio ($r = 1$), increasing the number of differentiators tends to increase the level of cooperation in the equilibrium. When there are only four differentiators located at high degree nodes, almost half of the players cooperate at the equilibrium action profile. This behavior was consistent for different activation sequences. Such levels of cooperation cannot be seen without strategic differentiation. As in the differentiated best response dynamics, when differentiators employ cooperation against groups of cooperators and defection against groups of defectors, network reciprocity allows clusters of cooperators to emerge around differentiators in the equilibrium action.
Final Remarks

We have shown how network games can be extended to include a subset of players that can employ different actions against different opponents. When the local games in the network admit a weighted potential function convergence of the strategically differentiated version under myopic best response dynamics is guaranteed. For both imitation and best response like dynamics, the topology of the network, the existence, and location of differentiators in the networks can crucially alter the action profile at an equilibrium of groupwise games. When differentiators are plentiful the equilibrium action profile becomes less sensitive to changes in the values of the payoff parameters. The convergence results in this framework can be combined with those of Chapter 3 and 4. The combination of relative best responses, rational imitation and strategic differentiation allows us to study the behavior of many classic games from a novel perspective.

This chapter concludes the first part of the thesis. We have mainly focused on network games and structural solutions to social dilemmas via network reciprocity that allows for the emergence of cooperation via a spatial or social structure. Part II of the thesis focuses on direct reciprocity in repeated games. In this setting there is no network structure, in stead, cooperation can evolve through the expectation of repeated interactions with a fixed group of players. In the following, we will investigate how such probabilistic decision-making processes can be studied by their
average behavior and show how strategic individuals can exert significant control in the long-run outcomes of repeated games.
Part II

STRATEGIC PLAY AND CONTROL IN REPEATED GAMES