Preliminaries

2.1 Network Games

Non-cooperative network games have three main ingredients: the network structure, the action space, and the combined payoff function. The action space is defined for both finite games, and convex games, in which the action sets are finite discrete sets and infinite compact and convex sets, respectively.

2.1.1 Network structure, action space and payoff functions

Let $G = (V, E)$ be a graph whose node set $V = \{1, \ldots, N\}$ represents players. The edge set $E \subseteq V \times V$, represents the player interaction topology. Let $A_i$ denote the set of actions for player $i \in V$ and let $\sigma_i \in A_i$ denote the action of player $i$. The action space of the game is defined as the Cartesian product of the action sets of the players, i.e., $A = \prod_{i \in V} A_i$. An action profile of the game is an element of this set $\sigma := (\sigma_i)_{i \in V} \in A$, representing the actions chosen by all players in the network. To emphasize the $i^{th}$ element of $\sigma \in \mathbb{R}^N$, we write $\sigma = (\sigma_i, \sigma_{-i})$ where $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_N)$. Let $\pi_i : A \rightarrow \mathbb{R}$ indicate the payoff function of player $i$. The combined payoff function $\pi : A \rightarrow \mathbb{R}^N$ maps each action profile $\sigma \in A$ to a payoff vector $\pi(\sigma) = (\pi_i(\sigma))_{i \in V}$ whose elements correspond to the payoffs that the players receive for a single round of interaction. In network games, the spatial structure is incorporated into the payoff function $\pi$. Thus, the network structure
determined by the graph $\mathcal{G}$, the action space $\mathcal{A}$, and combined payoff function $\pi$ defines the network game as the triplet $\Gamma = (\mathcal{G}, \mathcal{A}, \pi)$.

2.1.2 Finite and convex games

We say $\Gamma$ is a finite game if the action set of each player is a finite discrete set such that $\mathcal{A}_i \subset \mathbb{Z}$ and $\mathcal{A} \subset \mathbb{Z}^N$. A finite game is denoted by $\Gamma_f$. We say $\Gamma$ is a convex game if the action set of each player is a non-empty, convex subset of $\mathbb{R}^m$, i.e., $\mathcal{A} \subset \mathbb{R}^m$ and $\mathcal{A} \subset \mathbb{R}^{Nm}$. A convex game is denoted by $\Gamma_c$. The convexity assumption over the action set for convex games is common in the literature of monotone games [47, 65].

2.2 Potential games

In Part I of the thesis, the theory of potential games is used. In [42], Monderer and Shapley identify several classes of games for which there exists a function that increases or decreases monotonically along the trajectory of rational decisions in a game. The most restrictive class is known as exact potential games that are defined as follows.

2.2.1 Finite games

**Definition 1** (Exact potential game). Given a finite game $\Gamma_f$, if there exists a function $P: \mathcal{A} \to \mathbb{R}$ such that for every $i \in \mathcal{V}$, for every $\sigma_i, \sigma_i' \in \mathcal{A}_i$ and every $\sigma_{-i} \in \prod_{j \in \mathcal{V} \setminus \{i\}} \mathcal{A}_j$, the following implication holds:

$$\pi_i(\sigma_i', \sigma_{-i}) - \pi_i(\sigma_i, \sigma_{-i}) = P(\sigma_i', \sigma_{-i}) - P(\sigma_i, \sigma_{-i})$$

(2.1)

then $\Gamma_f$ is an exact potential game.

Several generalizations of exact potential games exist. The following definitions provide an overview of increasingly general classes of games.

**Definition 2** (Weighted potential game). Given a finite game $\Gamma_f$, if there exists a function $P: \mathcal{A} \to \mathbb{R}$ such that for every $i \in \mathcal{V}$, for every $\sigma_i, \sigma_i' \in \mathcal{A}_i$ and every $\sigma_{-i} \in \prod_{j \in \mathcal{V} \setminus \{i\}} \mathcal{A}_j$, the following implication holds:

$$\pi_i(\sigma_i', \sigma_{-i}) - \pi_i(\sigma_i, \sigma_{-i}) = \alpha_i [P(\sigma_i', \sigma_{-i}) - P(\sigma_i, \sigma_{-i})]$$

(2.2)

then $\Gamma_f$ is a weighted potential game.
2.2. Potential games

Definition 3 (Ordinal potential game). Given a finite game \( \Gamma_f \), if there exists a function \( P : \mathcal{A} \rightarrow \mathbb{R} \) such that for every \( i \in V \), for every \( \sigma, \sigma' \in \mathcal{A}_i \) and every \( \sigma_{-i} \in \prod_{j \in V \setminus \{i\}} \mathcal{A}_j \), the following implication holds:
\[
\pi_i(\sigma', \sigma_{-i}) - \pi_i(\sigma, \sigma_{-i}) > 0 \Leftrightarrow P(\sigma', \sigma_{-i}) - P(\sigma, \sigma_{-i}) > 0,
\] (2.3)
then \( \Gamma_f \) is an ordinal potential game.

Definition 4 (Generalized ordinal potential game). Given a finite game \( \Gamma_f \), if there exists a function \( P : \mathcal{A} \rightarrow \mathbb{R} \) such that for every \( i \in V \), for every \( \sigma, \sigma' \in \mathcal{A}_i \) and every \( \sigma_{-i} \in \prod_{j \in V \setminus \{i\}} \mathcal{A}_j \), the following implication holds:
\[
\pi_i(\sigma', \sigma_{-i}) - \pi_i(\sigma, \sigma_{-i}) > 0 \Rightarrow P(\sigma', \sigma_{-i}) - P(\sigma, \sigma_{-i}) > 0,
\] (2.4)
then \( \Gamma_f \) is a generalized ordinal potential game.

Potential games (and their generalizations) with finite action sets have an important property called the Finite Improvement Property (FIP) that is formalized as follows.

Definition 5 (Finite Improvement Property [42, Sec. 2]). Let \( \gamma = (\sigma(0), \sigma(1), \ldots) \) denote an action profile sequence for \( \Gamma \). If for every \( t \geq 1 \) there exists a unique player, say \( i_t \in V \) such that \( \sigma(t) = (\sigma_{i_t}(t), \sigma_{-i_t}(t-1)) \) for some \( \sigma_{i_t}(t) \neq \sigma_{i_t}(t-1) \), then \( \gamma \) is called a path in the action profile. If additionally it holds that for each consecutive action profile in a path \( \gamma \) the payoff of the unique deviator \( i_t \) is strictly increasing, that is
\[
\forall t \geq 1 : \pi_{i_t}(\sigma(t)) > \pi_{i_t}(\sigma(t-1)),
\]
then \( \gamma \) is called an improvement path. \( \Gamma \) has the Finite Improvement Property (FIP) if every improvement path is finite.

Lemma 1 (Finite Improvement paths in potential games [42, Sec. 2]). \( \Gamma_f \) has the FIP if and only if \( \Gamma_f \) has a generalized ordinal potential function.

2.2.2 Infinite games

In finite potential games, the action space is finite and in turn, the potential function is bounded. Naturally, these properties do not hold when the number of actions is infinite. In the following, we shortly introduce concepts from the theory of infinite potential games, i.e. potential games with an infinite number of actions. We will focus on convex games, that unless the action sets are singleton sets, can also be characterized as infinite games. We begin with the infinite game counterpart of improvement paths, commonly referred to as approximate improvement paths.
2. Preliminaries

**Definition 6** ($\epsilon$-improvement paths). Let $\gamma$ denote a sequence in the action profile of $\Gamma_c$ and let $\epsilon > 0$ be an arbitrarily small positive real. When for every $t \geq 1$ there exists a unique player, say $i_t \in V$, such that

$$\sigma(t) = (\sigma_{i_t}(t), \sigma_{-i_t}(t-1))$$

for some $\sigma_{i_t}(t) \neq \sigma_{i_t}(t-1)$, then $\gamma$ is called a path in the action profile. When additionally it holds that for each consecutive action profile in a path $\gamma$ the payoff of the unique deviator $i_t$ is strictly increasing, i.e., \( \forall t \geq 1 \)

$$\text{if } i = i_t \text{ : } \pi_i(\sigma(t)) > \pi_i(\sigma(t - 1)) + \epsilon,$$

then $\gamma$ is called an $\epsilon$–improvement path with respect to $\Gamma_c$.

The FIP for infinite games is known as the Approximate Finite Improvement Property (AFIP).

**Definition 7** (Approximate Finite Improvement Property, [42]). $\Gamma_c$ has the AFIP if for every $\epsilon > 0$, every $\epsilon$–improvement path is finite.

Finite approximate improvement paths are naturally connected to the concept of an approximate Nash Equilibrium (NE), that is defined as follows.

**Definition 8** ($\epsilon$-Nash equilibrium). The action profile $\sigma \in A$ is an $\epsilon$–NE for $\Gamma_c$, if for all $i \in V$, $\sigma_i \in \sigma$ is such that

$$\pi_i(\sigma_i, \sigma_{-i}) > \pi_i(\sigma'_i, \sigma_{-i}) + \epsilon, \quad \forall \sigma'_i \in A_i,$$

for some $\epsilon > 0$.

To characterize the class of infinite games that have the AFIP, it is necessary to introduce the concept of a bounded game.

**Definition 9** (Bounded Game). $\Gamma_c$ is a bounded game if for all $\sigma \in A$ there exists $M \in \mathbb{R}$ such that $\forall i \in V$ it holds that $|\pi_i(\sigma)| \leq M$.

Bounded games thus have bounded payoff functions on the action space of the infinite game. For weighted and exact potential games, this implies that also the potential function is bounded. This leads to the following Lemma.

**Lemma 2** (Lemma 4.2, [42]). Every bounded $w$–potential game has the AFIP.
Part I

RATIONALITY AND SOCIAL INFLUENCE IN NETWORK GAMES