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A PREPRINT

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14th January 2020

ABSTRACT

This paper introduces 'commonly knowing whether', a non-standard version of classical common knowledge which is defined on the basis of 'knowing whether', instead of classical 'knowing that'. After giving five possible definitions of this concept, we explore the logical relations among them both in the multi-agent case and in the single-agent case. We focus on one definition and treat it as a modal operator. It is found that the expressivity of this operator is incomparable with the classical common knowledge operator. Moreover, some special properties of it over binary-tree models and $\mathcal{KD}45$-models are investigated.

1 Introduction

Common knowledge, as the strongest concept among group epistemic notions, has been studied extensively in artificial intelligence [13], epistemic logic [12], epistemology [10] and philosophy of language [2]. Over Kripke semantics, common knowledge among group G that p, formulated as $C_G p$, is interpreted as: on every node that can be reached from the current node via the reflexive-transitive closure of relation R_G[2], p is satisfied. As the definition suggests, common knowledge is defined on the basis of classical knowledge, 'knowing that', to represent propositional knowledge in a group.

There also exists a large amount of 'wh-'style knowledge[3] in natural language. To describe them formally, several works on epistemic logic have been undertaken, such as the logic of 'knowing whether' [5], the logic of 'knowing how' [15] [6], and the logic of 'knowing why' [17]. Especially the notion of 'knowing whether', which means that an agent knows that p is true or knows that p is false, has been studied from distinct perspectives [7] [11] [8].

Considering that classical common knowledge is based on 'knowing that', the natural question arises of what the common knowledge based on 'knowing whether' is. There is no agreement on the definition of 'commonly knowing whether' yet and some possible definitions can currently only be expressed with an infinite language. In this paper, we suggest one of the definitions is the most plausible and deserves specific study.

In this paper, we give alternative definitions of 'commonly knowing whether' based on different intuitions and prove that they are not equivalent, following with a further study on one of them on its logical properties. In detail, Section 2 gives five different definitions of 'commonly knowing whether'. Section 3 studies how these definitions are logically related over different frames. Section 4 discusses the expressivity of the language $KwCw^5L$. Some properties of Cw^5 over two special classes of frames are investigated in Section 5. Due to the limitation on pages, we only show full proofs for these vital conclusions and proof sketches for some important results. As for other facts and lemmas, proof details are omitted.

2 Definitions of 'Commonly Knowing Whether' (Cw)

One way of approaching standard common knowledge is an infinite conjunction of all finite iterations of 'everyone knowing'. Before defining 'commonly knowing whether', two different definitions of 'everyone knowing whether'

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[2]R_G is the union of accessible relations of the agents in G.

[3]'Knowing How' is also considered as a 'wh-'style knowledge since it also describes a kind of non-propositional knowledge.

will be introduced. The following definitions of 'everyone knowing whether' and 'commonly knowing whether' are due to Yanjing Wang [16]. The group G concerned in this paper is finite. Since only one group is considered, we omit G in notations for group knowledge in the remainder of the paper. Infinite conjunctions are temporarily used to define some of the following notions since they have not been expressed in any finite language yet. And $\mathcal{K}$, $\mathcal{T}$, $\mathcal{S}5$ refer to the classes of all Kripke frames, reflexive frames and equivalence relation frames, respectively. $\mathcal{KD}45$ refers to the class of serial, transitive and Euclidean frames. G^{+}[4] refers to the set of finite sequences only consisting of agents from G, excluding the empty sequence. Let P be a fixed set of propositional variables. An infinite language is introduced to define 'commonly knowing whether'.

Definition 1 *Given P and G, the infinite language $KwEL^{\infty}$ is given as follows:*

$$\varphi := p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid Kw_i\varphi \mid E\varphi \mid C\varphi \mid \bigwedge \Phi$$

where $i \in G$, $p \in P$, and Φ is a countably infinite set of formulas.

Definition 2 $Ew^1\varphi := E\varphi \vee E\neg\varphi$

Intuitively, this definition imitates the definition of 'knowing whether', describing 'everyone knows whether φ' as 'everyone knows φ or everyone knows not φ'.

Definition 3 $Ew^2\varphi := \bigwedge_{i\in G} Kw_i\varphi$*, where the semantics of $Kw_i\varphi$ is defined over Kripke pointed models (M, s) as: $M, s \models Kw_i\varphi$ iff for any t_1 and t_2 with $s \to_i t_1$ and $s \to_i t_2$, $M, t_1 \models \varphi \Leftrightarrow M, t_2 \models \varphi$.*

This definition is inspired by classical 'everyone knows', defining 'everyone knows whether' as 'everyone in the group knows whether the proposition'.

On the basis of these definitions of 'everyone knows whether', five definitions of 'commonly knows whether' are given based on distinct interpretations.

Definition 4 $Cw^1\varphi := C\varphi \vee C\neg\varphi$

A group commonly knows whether φ is possibly interpreted as they have common knowledge that φ or they have common knowledge that $\neg\varphi$.

Definition 5 $Cw^2\varphi := CEw\varphi$

A group commonly knows whether φ plausibly means that everyone knows whether φ and it is a common knowledge of the group that everyone knows whether φ. Since there are two different definitions of 'everyone knows whether', we should separate $Cw^2\varphi$ into $Cw^{2,1}\varphi := CEw^1\varphi$ and $Cw^{2,2}\varphi := CEw^2\varphi$.

Definition 6 $Cw^3\varphi := \bigwedge_{k\geq 1}(Ew)^k\varphi$

'Commonly knowing whether φ' can also be defined as everyone knows whether φ and everyone knows whether everyone knows whether φ, etc. We should again separate $Cw^3\varphi$ into $Cw^{3,1}\varphi := \bigwedge_{k\geq 1}(Ew^1)^k\varphi$ and $Cw^{3,2}\varphi := \bigwedge_{k\geq 1}(Ew^2)^k\varphi$. As mentioned in Section 1, this is an infinitary form corresponding to intuition.

Definition 7 $Cw^4\varphi := \bigwedge_{i\in G} Cw^1 Kw_i\varphi$*, that is* $Cw^4\varphi := \bigwedge_{i\in G}(CKw_i\varphi \vee C\neg Kw_i\varphi)$

A group commonly knows whether φ possibly means for every member in the group, it is common knowledge of the group that she knows whether φ or it is common knowledge of the group that she does not know whether φ.

Definition 8 $Cw^5\varphi := \bigwedge_{s\in G^+} Kw_s\varphi$*. For instance: if $s = \langle i, j, l\rangle$, then $Kw_s\varphi = Kw_iKw_jKw_l\varphi$.*

Listing all the possible inter-'knowing whether' states among every subset of the group is an alternative way to define 'commonly knowing whether'.

[4] For any set T, T^+ contains all finite sequences only consisting of elements in T.

$$\begin{tikzcd}
 & & & Cw^5\varphi \\
Cw^1\varphi \arrow[r] & Cw^{2,1}\varphi \arrow[r] & Cw^{2,2}\varphi \arrow[r] \arrow[ur] \arrow[d] \arrow[dr] & Cw^4\varphi \\
 & & Cw^{3,2}\varphi & Cw^{3,1}\varphi
\end{tikzcd}$$

Figure 1: Logical relationships over $\mathcal{K}$

3 Implication Relations among the definitions of Cw

The above five definitions of 'commonly knowing whether' do not boil down to the same thing, especially over $\mathcal{K}$ and $\mathcal{KD}45$. In this section, we show how these definitions are related. The semantics presented in this paper uses Kripke models. A Kripke model is generally defined as $M = \langle W, R_i, R_j, \cdots, R_m, V\rangle$, where W is a non-empty set of nodes, R_i, R_j, $\cdots$, R_m are accessibility relations for the agents in group G and V is the valuation which is a function $V : P \to \mathcal{P}(W)$. If $(s,t) \in R_i$, we write $s \to_i t$; if $(s,t) \in R_i \cup R_j \cup ... \cup R_m$, we write $s \to t$; $\twoheadrightarrow$ is the reflexive-transitive closure of $\to$.

3.1 Over $\mathcal{K}$

In order to clarify the relations among definitions of 'commonly knowing whether', we first give the logical relations between Ew^1 and Ew^2.

Fact 1 *The following statements hold:*

$\mathcal{K} \models Ew^1\varphi \to Ew^2\varphi$;

$\mathcal{K} \not\models Ew^2\varphi \to Ew^1\varphi$;

$\mathcal{K} \models Ew^1\varphi \leftrightarrow Ew^1\neg\varphi$;

$\mathcal{K} \models Ew^2\varphi \leftrightarrow Ew^2\neg\varphi$.

Theorem 1 *Over $\mathcal{K}$, the logical relationships among Cw^1, Cw^2, Cw^3, Cw^4 and Cw^5 are shown in Figure 1.*[5]

Every pair of these definitions are not equivalent over $\mathcal{K}$. Conversely, the 'knowing that'-versions of Cw^2, Cw^3 and Cw^5 exactly correspond to three approaches to common knowledge [1] which are logically equivalent.

We will show the proof sketches on an interesting case where $Cw^2\varphi$ implies Cw^5 but the converse does not hold.

$\mathcal{K} \models Cw^2\varphi \to Cw^5\varphi$ Proof Sketch: prove with induction.

Consider $Cw^2\varphi$ in two cases:

The first case is $Cw^2\varphi := Cw^{2,2}\varphi$. Let (M, r) be an arbitrary pointed model, such that $M = \langle W, R_i, R_j, ..., R_m, V\rangle$ and $M, r \models Cw^2\varphi$. Then, for any node t with $r \twoheadrightarrow t$, we have $M, t \models \bigwedge_{i\in G} Kw_i\varphi$. For arbitrary $s \in G^+$, let $s = \langle i_1, i_2, ..., i_k\rangle$, where $\{i_n \mid 1 \le n \le k\} \subseteq G$. We show a stronger result: $M, r \models Kw_s\varphi$ and for any t such that $r \twoheadrightarrow t$, we have that $M, t \models Kw_s\varphi$.

By induction on the length $n = |s|$ of s:

When $n = 1$, $s = \langle i_1\rangle$, where i_1 is an arbitrary agent. By $M, r \models Cw^2\varphi$, we have $M, r \models Kw_{i_1}\varphi$. And since for any t with $r \twoheadrightarrow t$, there is $M, t \models Ew^2\varphi$. Thus there is $M, t \models Kw_{i_1}\varphi$.

Induction hypothesis: when $n = k$, $s = \langle i_1, i_2, ..., i_k\rangle$, there is $M, r \models Kw_s\varphi$ and for any t with $r \twoheadrightarrow t$, there is $M, t \models Kw_s\varphi$.

When $n = k+1$, $s = \langle i_1, i_2, ..., i_k, i_{k+1}\rangle$. By induction hypothesis, for any t such that $r \to t$, for any $|s_k| = k$, there is $M, t \models Kw_{s_k}\varphi$. Thus, for any $i \in G$, there is $M, r \models Kw_i Kw_{s_k}\varphi$, saying, for any $|s_{k+1}| = k+1$, we have

[5] In the following figures, the directed arrows between two definitions refer to the strict implications. The logical relations are surely transitive.

$$
\begin{tikzcd}
Cw^1\varphi \arrow[r, leftrightarrow] \arrow[d] & Cw^{2,1}\varphi \arrow[r, leftrightarrow] & Cw^{2,2}\varphi \arrow[r, leftrightarrow] & Cw^{3,1}\varphi \arrow[r, leftrightarrow] & Cw^{3,2}\varphi \arrow[r, leftrightarrow] & Cw^5\varphi \\
Cw^4\varphi & & & & &
\end{tikzcd}
$$

Figure 2: Logical relationships over $\mathcal{T}$ and $\mathcal{S}5$ (multi-agent).

$M, r \models Kw_{s_{k+1}}\varphi$. Now considering any u with $r \twoheadrightarrow u$, let $u \to v$. By the definition of $\to$, $r \twoheadrightarrow v$. By induction hypothesis, we get $M, v \models Kw_{s_k}\varphi$. And since v is an arbitrary node such that $u \to v$, we have for any $i \in G$, $M, u \models Kw_i Kw_{s_k}\varphi$. That means, for any $|s_{k+1}| = k+1$, there is $M, u \models Kw_{s_{k+1}}\varphi$.

Therefore, we proved that, for arbitrary $s \in G^+$ and arbitrary t with $r \twoheadrightarrow t$, there are $M, r \models Kw_s\varphi$ and $M, t \models Kw_s\varphi$, which means $M, r \models \bigwedge_{s \in G^+} Kw_s\varphi$. Thus, we have $M, r \models Cw^5\varphi$. Above all, we obtain $\mathcal{K} \models Cw^{2,2}\varphi \to Cw^5\varphi$.

For the second case where $Cw^2\varphi := Cw^{2,1}\varphi$, since $\mathcal{K} \models Cw^{2,1}\varphi \to Cw^{2,2}\varphi$, we have $\mathcal{K} \models Cw^{2,1}\varphi \to Cw^5\varphi$. Therefore, we proved $\mathcal{K} \models Cw^2\varphi \to Cw^5\varphi$.

$\mathcal{K} \not\models Cw^5\varphi \to Cw^2\varphi$ Proof Sketch: find a counter-model.

Consider the following model M:

$$
\begin{tikzcd}
r : p \arrow[r, "i"] & t_1 : p \arrow[r, "i"] \arrow[dr, "i"] & t_2 : p \\
 & & t_3 : \neg p
\end{tikzcd}
$$

In this model, there is only one successor of r. By semantics, we have $M, r \models Cw^5 p$. And since $M, t_1 \not\models Ew^1 p$ and $M, t_1 \not\models Ew^2 p$, it follows that $M, r \not\models Cw^2 p$. Thus, $\mathcal{K} \not\models Cw^5\varphi \to Cw^2\varphi$.

3.2 Over $\mathcal{KD}45$

$\mathcal{KD}45$ is considered to be the usual class of frames for doxastic (belief) logic [9, 14]. The implication relations over $\mathcal{KD}45$ among these five definitions are the same as those over $\mathcal{K}$.

Theorem 2 *Over $\mathcal{KD}45$, the implication relations are shown as Figure 1.*

3.3 Over $\mathcal{T}$ and $\mathcal{S}5$

The model of knowledge is generally defined over $\mathcal{S}5$ [3, 14], which commits knowledge must be true via axiom T. The implication relations among five definitions over $\mathcal{T}$ or $\mathcal{S}5$ are given in Theorem 3.

Theorem 3 *Over $\mathcal{T}$ and $\mathcal{S}5$, the implication relations are shown in Figure 2.*

$Cw^1\varphi$, $Cw^2\varphi$, $Cw^3\varphi$ and $Cw^5\varphi$ boil down to the same thing once the frame is reflexive, which should be attributed to agents' agreements on the values of φ. For example, if there is $M, s \models Kw_i\varphi \wedge Kw_j\varphi$ where M is reflexive, the values of φ on all i-successors must agree with those on all j-successors since they share a common successor s. Comparatively, if M is not reflexive, the case of $M, s \models K_i\varphi \wedge K_j\neg\varphi$ is possible.

3.4 Single-agent Case

The results above concern the multi-agent case. Considering the single-agent case, observe that $\mathcal{K} \models Ew^1\varphi \leftrightarrow Ew^2\varphi$ which implies that Cw^{21} is equivalent to Cw^{22} and that Cw^{31} is equivalent to Cw^{32}. Moreover, since only one agent is involved, it is trivial to find that Cw^3 is equivalent to Cw^5.

In terms of logical relationships, the five definitions share the same relations with the multi-agent case over $\mathcal{K}$, $\mathcal{KD}45$, $\mathcal{T}$ and $\mathcal{S}5$, respectively.

The following sections concentrate on Cw^5 in the single-agent case.

4 The Case of Cw^5

We focus on Cw^5 since the idea of it is inspired by the hierarchy of inter-knowledge of a group given in [13]. The language of 'commonly knowing whether' is defined in the same way as KwL[6], except introducing a new operator Cw^5. We will investigate the expressivity between Cw^5 and C in Section 4.3, so the language of classical common knowledge is also given here.

Definition 9 *Given P and G, the language $KwCw^5L$ is given as follows:*

$$\varphi := p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid Kw_i\varphi \mid Cw^5\varphi$$

where $p \in P$, $i \in G$.

Definition 10 *Given P and G, the language KCL is given as follows:*

$$\varphi := p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid C\varphi$$

where $p \in P$, $i \in G$.

4.1 Some Valid or Invalid $KwCw^5L$-formulas

In the single-agent case, we consider some validities and invalidities over $\mathcal{K}$:

Fact 2 $\mathcal{K} \models Cw^5\varphi \to Kw_iCw^5\varphi$

This valid formula illustrates that when $Cw^5\varphi$ is true at the current node, all successors of it agree the values of $Cw^5\varphi$. However, the converse formula is invalid, saying $\mathcal{K} \not\models Kw_iCw^5\varphi \to Cw^5\varphi$.

Fact 3 $\mathcal{K} \models \neg Kw_i\psi \to (Kw_iCw^5\varphi \wedge Kw_i(\psi \to Cw^5\varphi) \wedge Kw_i\varphi \to Cw^5\varphi)$

This valid formula is established in the light of Almost Definability (AD) from [4] , which is $\neg Kw_i\psi \to (K_i\varphi \leftrightarrow Kw_i\varphi \wedge Kw_i(\psi \to \varphi))$. AD shows that under the precondition $\neg Kw_i\psi$ for some ψ, the classical knowledge operator K_i can be defined in terms of Kw_i. Since $\mathcal{K} \models K_iCw^5\varphi \wedge Kw_i\varphi \to Cw^5\varphi$, we can replace $K_iCw^5\varphi$ with AD.

Fact 4 $\mathcal{K} \not\models Cw^5(\chi \to \varphi) \wedge Cw^5(\neg\chi \to \varphi) \to Cw^5\varphi$

Although $Kw_i(\chi \to \varphi) \wedge Kw_i(\neg\chi \to \varphi) \to Kw_i\varphi$ is the basic axiom of KwL according to [5], its Cw^5-version is not valid.

Fact 5 $\mathcal{K} \not\models Cw^5\varphi \to Cw^5(\varphi \to \psi) \vee Cw^5(\neg\varphi \to \chi)$

Similarly, for another basic axiom $Kw_i\varphi \to Kw_i(\varphi \to \psi) \vee Kw_i(\neg\varphi \to \chi)$ in KwL, the Cw^5-version is invalid.

Fact 6 $\mathcal{K} \not\models Cw^5\varphi \wedge Cw^5\psi \to Cw^5(\varphi \wedge \psi)$

This formula is invalid, which leads to the failure in defining Cw^5 with accessibility relations.

4.2 $KwCw^5L$ is Bisimulation Invariant

Definition 11 *Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be two Kripke models. A non-empty binary relation $Z \subseteq W \times W'$ is called bisimulation between M and M', written as $M \cong M'$, if the following conditions are satisfied:*

(i) If $wZ'w'$, then w and w' satisfy the same proposition letters.

(ii) if $wZ'w'$ and wRv, then there is a $v' \in W'$ such that vZv' and $w'R'v'$.

[6] The language for the logic of 'knowing whether' defined in [5].

(iii) If $wZ'w'$ and $w'R'v'$, then there exists $v \in W$ such that vZv' and wRv.

When Z is a bisimulation linking two states w in M and w' in M', we say that two pointed models are bisimilar and write $Z : (M, w) \cong (M', w')$. If a language L cannot distinguish any pair of bisimilar models, L is bisimulation invariant.

Theorem 4 *$KwCw^5L$ is bisimulation invariant.*

Proof 1 *By induction on formulas φ of $KwCw^5L$.*

When φ is a Boolean formula, the proof is classical.

When $\varphi = Kw_i\psi$, we prove it in three cases. For arbitrary two bisimilar models (M, r) and (N, t), we have:

- *if $M, r \models Kw_i\psi$ and for all r_n with $r \rightarrow_M r_n$, $M, r_n \models \psi$. Since $M, r \cong N, t$, for any t_n with $t \rightarrow_N t_n$, there is an r_n such that $r \rightarrow_M r_n$ and $M, r_n \cong N, t_n$. By induction hypothesis, ψ is bisimulation invariant. Thus $N, t_n \models \psi$. So $N, t \models K_i\psi$.*
- *if $M, r \models Kw_i\psi$ and for all r_n with $r \rightarrow_M r_n$, $M, r_n \models \neg\psi$, similar to above case.*
- *if $M, r \models \neg Kw_i\psi$, that means there are r_1 with $r \rightarrow_M r_1$ and r_2 with $r \rightarrow_M r_2$, such that $M, r_1 \models \psi$ and $M, r_2 \models \neg\psi$. Since $M, r \cong N, t$, there are t_1 with $t \rightarrow_N t_1$ and t_2 with $t \rightarrow_N t_2$, such that $M, r_1 \cong N, t_1$ and $M, r_2 \cong N, t_2$. By induction hypothesis, ψ is bisimulation invariant. Thus $N, t_1 \models \psi$ and $N, t_2 \models \neg\psi$. Thus $N, t \models \neg Kw_i\psi$.*

  Thus, $Kw_i\psi$ is bisimulation invariant.

When $\varphi = Cw^5\psi$, assume two bisimular models (M, r) and (N, t), such that $M, r \models Cw^5\psi$ and $N, t \models \neg Cw^5\psi$. That means there exists a sequence of agents s, such that $M, r \models Kw_s\psi$ and $N, t \models \neg Kw_s\psi$. Let $s = \langle i_1, i_2, ..., i_n\rangle$. So $M, r \models Kw_{i_1}\gamma$ and $N, t \models \neg Kw_{i_1}\gamma$, where $\gamma = Kw_{\langle i_2, i_3, ..., i_n\rangle}\psi$. However, we have proved that for any formula of the form $Kw_{i_1}\gamma$, they are bisimulation invariant. Thus, if $M, r \models Kw_{i_1}\gamma$, there must be $N, t \models Kw_{i_1}\gamma$. Contradiction.

Therefore, we proved that $KwCw^5L$ is bisimulation invariant.

4.3 $KwCw^5L$ and KCL

Although Cw^5 is formed merely with Kw_i, which can be defined by classical operator K_i, surprisingly, Cw^5 cannot be defined with K_i and C.

Theorem 5 *Over $\mathcal{K}$, $KwCw^5L$ is not weaker than KCL in expressivity.*

We prove Theorem 5 by defining two classes of models, which no KCL-formula can distinguish while there exists one $KwCw^5L$-formula that can distinguish them.

Definition 12 *For every $n \geq 1$, define two sets of possible worlds T_n and Z_n with induction:*

- *$t_{00} \in T_n$; $z_0 \in Z_n$*
- *If $t_i \in T_n$, then $t_{i0} \in T_n$ and $t_{i1} \in T_n$; if $z_i \in Z_n$, then $z_{i0} \in Z_n$ and $z_{i1} \in Z_n$*
- *For every $t_j \in T_n$, $|j| \leq n + 2$; for every $z_j \in Z_n$, $|j| \leq n + 1$*
- *Besides t_i defined above, T_n has no more possible worlds; besides z_i defined above, Z_n has no more possible worlds.*

Then define the class of models $M = \{M_n = \langle W_n, R_n, V_n\rangle \mid n \in \mathbb{N}\}$ as follows: for every $n \geq 1$,

- *$W_n = \{r\} \cup T_n \cup \{t_0\}$*
- *$R_n = \{\langle t_i, t_{i0}\rangle, \langle t_i, t_{i1}\rangle \mid t_i \in T_n, t_{i0} \in T_n\} \cup \{\langle r, t_0\rangle, \langle t_0, t_{00}\rangle\}$*
- *$V_n(p) = W_n - \{t_{0i}\}$, where $|i| = n + 1$, $i \in \{0\}^+$.*

Define the class of models $N = \{N_n = \langle W'_n, R'_n, V'_n\rangle \mid n \in \mathbb{N}\}$ with M: for every $n \geq 1$:

- $W'_n = W_n \cup Z_n$
- $R'_n = R_n \cup \{\langle z_i, z_{i0}\rangle, \langle z_i, z_{i1}\rangle \mid z_i \in Z_n, z_{i0} \in Z_n\}$
- $V'_n(p) = V_n(p) \cup Z_n - \{z_{0i}\}$, *where* $|i| = n$, $i \in \{0\}^+$

The first model M_1 and N_1 in M and N are shown as Figure 3.

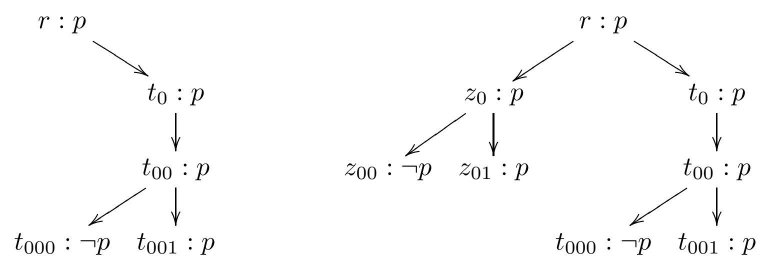

Figure 3: M_1(left) and N_1(right)

The model N_n is constructed by adding a new subtree rooted with z_0 to the root r and just make p unsatisfied on the leaf node whose index only consists of 0.

We will prove that no KCL-formula can distinguish M and N with the CL-game[7]. If there is a winning strategy for duplicator in n-round games, (M_n, r) and (N_n, r) agree on all KCL-formulas whose modal depth is n.

Theorem 6 *For arbitrary $n \in \mathbb{N}$, duplicator has a winning strategy in the CL-game on (M_n, r) and (N_n, r) in n rounds.*

Proof 2 *We describe Duplicator's winning strategy case by case. Starting with the initial state (r, r), we mainly concerns the case where spoiler does a K-move. Otherwise, duplicator can move to a isomorphic sub-model such that there must be a winning strategy in following rounds. Thus, the cases below exhaust all possibilities.*

- *The initial state is (r, r):*
  - *If spoiler does a K-move or a C-move on M_n reaching t_i, then duplicator does a K-move or a C-move on N_n to reach the corresponding t_i. Since $(M_n, t_i) \cong (N_n, t_i)$, there is a winning strategy after this move.*
  - *If spoiler does a K-move or a C-move on N_n reaching t_i, then duplicator does a K-move or a C-move on M_n to reach the corresponding t_i. Since $(M_n, t_i) \cong (N_n, t_i)$, there is a winning strategy after this move.*
  - *If spoiler does a K-move on N_n reaching z_0, duplicator moves to t_0.*
  - *If spoiler does a C-move on N_n reaching an arbitrary node $z_{i(i\neq 0)}$ in Z_n, then duplicator does a C-move to reach t_{0i}. Since $(M_n, t_{0i}) \cong (N_n, z_i)$, there is a winning strategy after this move.*
- *The current state is (z_0, t_0):*
  - *If spoiler does a K-move reaching z_{00} or z_{01}, then duplicator moves on M_n to reach t_{00}.*
  - *If spoiler does a K-move reaching t_{00} on M_n, then duplicator moves to z_{00} on N_n.*
  - *If spoiler does a C-move reaching $z_{i(i\neq 0)}$, then duplicator moves to t_{0i}. Since $(M_n, t_{0i}) \cong (N_n, z_i)$, there is a winning strategy after this move.*
  - *If spoiler does a C-move reaching t_{00} on M_n, then duplicator moves to z_{00} on N_n.*
  - *If spoiler does a C-move reaching $t_{0i(i\neq 0)}$[8], then duplicator moves to z_i on N_n. Since $(M_n, t_{0i}) \cong (N_n, z_i)$, there is a winning strategy after this move.*
- *The current state is (z_i, t_i) and $i \neq 0$: this means before the game gets to this state, both players have only done K-moves. In the current state, there have been at most $(n-1)$ rounds. Thus, $i \leq (n-1)$ and players can do next round as follows:*

[7]The definition of the CL-game is given in Appendix.
[8]The notation t_{0i} is correct since the index for every node in T_n begins with 0.

- *If spoiler does a K-move reaching z_{i0} or z_{i1}, then duplicator does a K-move to reach t_{i0} where $M_n, t_{i0} \models p$ since $|i0| = |i1| = (i+1) \leq n$ and there are $N_n, z_{i0} \models p$ and $N_n, z_{i1} \models p$.*
- *If spoiler does a K-move or a C-move reaching t_{i0} or t_{i1}, then duplicator does a K-move or a C-move to reach z_{i0} where $M_n, z_{i0} \models p$ since $|i0| = |i1| = (i+1) \leq n$ and there are $N_n, t_{i0} \models p$ and $N_n, t_{i1} \models p$.*
- *If spoiler does a C-move reaching $z_{j(|j|>|i|)}$, then duplicator does a C-move to reach t_{0j}. Since $(M_n, t_{0j}) \cong (N_n, z_j)$, there is a winning strategy after this move.*
- *If spoiler does a C-move reaching $t_{0i(i \neq 0)}$, duplicator moves to z_i on N_n. Since $(M_n, t_{0i}) \cong (N_n, z_i)$, there is a winning strategy after this move.*

Therefore, for arbitrary $n \in \mathbb{N}$, there is a winning strategy for duplicator in the n-round CL-game over (M_n, r) and (N_n, r).

The above proof shows the nonexistence of some KCL-formula which can distinguish M and N since for any KCL-formula, its modal depth is a natural number n which results in that it cannot distinguish M_n and N_n. Comparatively, there is a $KwCw^5L$-formula, Kw_iCw^5p such that $M_n, r \models Kw_iCw^5p$ and $N_n, r \models \neg Kw_iCw^5p$ for every $n \in \mathbb{N}$. Moreover, consider the following two pointed models, M and M':

$$r : p \xrightarrow{\quad i \quad} t : p \qquad\qquad r' : p \xrightarrow{\quad i \quad} t' : \neg p$$

Figure 4: M(left) and M'(right)

We can find a KCL-formula to distinguish them, where $M, r \models Kp$ and $M', r' \models \neg Kp$. But there is no any $KwCw^5L$-formula can distinguish M and M'. It follows that KCL is not weakly expressive than $KwCw^5L$.

Therefore, together with Theorem 5, the following theorem is obtained:

Theorem 7 *Language KCL and $KwCw^5L$ are incomparable with respect to expressivity.*

5 $KwCw^5L$ over Special Frames

Because of the invalidity of the formula $(Cw^5\varphi \wedge Cw^5\psi) \to Cw^5(\varphi \wedge \psi)$, the operator Cw^5 is not normal, in the sense that it cannot be defined with accessibility relations. However, an interesting observation over binary-tree models can be proved.

Definition 13 *(M, r) is a binary-tree model with root r if (M, r) is a tree model with root r and for any node t in M, t has precisely two successors.*

5.1 $KwCw^5L$ over Binary Trees

Theorem 8 *Consider the single-agent case. If $M, r \models Cw^5\varphi$ where (M, r) is a binary tree with root r, then on every layer of (M, r), the number of the nodes where φ is satisfied is even.*

In order to prove Theorem 8, we need to prove a stronger theorem:

Theorem 9 *For an arbitrary formula φ, if M is a binary-tree model, then $M, v_m \models Kw_i^n\varphi$ $(1 \leq n)$ iff the number of the φ-satisfied nodes on the $(|m| + n)th$ layer that v_m can reach via relation $\twoheadrightarrow$ is even.*

Proof sketch By induction on n:

- When $n = 1$, it is trivial.
- (Induction Hypothesis) When $n = k$, it holds.
- When $n = k + 1$, suppose $M, v_m \models Kw_i^{k+1}\varphi$, which is equivalent to $M, v_m \models Kw_i^k Kw_i\varphi$. Then we can use induction hypothesis. Suppose $M, v_m \not\models Kw_i^{k+1}\varphi$, which means on one successor, $Kw_i^k\varphi$ is satisfied, and on other one successor, $Kw_i^k\varphi$ is not satisfied. Then we can use induction hypothesis.

Proof details are shown in Appendix.

Remark 1 *Theorem 8 can be extended into a more general conclusion considering the multi-agent case: on any G-binary-tree model*[9] (M, r) *where* r *is the root,* $M, r \models Cw^5\varphi$ *iff for any sequence of agents* s *in* G*, on every layer of the subtree (of* (M, r)*) generated with* s[10]*, the number of the* φ*-satisfied nodes is even.*

5.2 $KwCw^5L$ over $\mathcal{KD}45$

By conclusions drawn in Section 3.2, $Cw^5\varphi$ is not equivalent to $Cw^1\varphi$ over $\mathcal{KD}45$, where Cw^5 still preserves its non-triviality.

Single-agent Case When the group concerned is a singleton, there is nestings reducing for operator Kw_i over $\mathcal{KD}45$. The proof idea of Theorem 10 is similar to the classical case proved in [12] where the nestings of operator K can also be reduced over $\mathcal{S}5$.

Theorem 10 *Over* $\mathcal{KD}45$*, every* $KwL_{(1)}$*-formula*[11] *is equivalent to some formula without nestings of the modal operator* Kw_i*.*

Corollary 1 *In the single-agent case,* $\mathcal{KD}45 \models Cw^5\varphi \leftrightarrow Kw_i\varphi$

Multi-agent Case However, when more than one agent are involved, we cannot reduce nestings. Fortunately, there are still some interesting findings.

Theorem 11 *For any sequence* $s = \langle i_1, i_2, \cdots, i_n \rangle$ $(n \geq 2)$ *of agents, if there exists* m *such that* $1 \leq m < n$ *and* $i_m = i_{m+1}$*, then* $\mathcal{KD}45 \models Kw_s\varphi$ *for any formula* φ*.*

The proof of Theorem 11 is given in Appendix.

Corollary 2 *Over* $\mathcal{KD}45$*,* $Cw^5\varphi = \bigwedge_{s \in G^+} Kw_s\varphi$ *where* $s = \langle i_1, i_2, \cdots, i_n \rangle$ *is a sequence of agents and there is no* m $(1 \leq m < n)$ *such that* $i_m = i_{m+1}$*.*

6 Conclusions

In this paper, we provide some initial results on definitions of 'commonly knowing whether', their relationships and properties, with special focus on Cw^5. We also show that over $\mathcal{K}$, the languages KCL and $KwCw^5L$ are incomparable with respect to expressivity.

There is much more expected work to be done in terms of 'commonly knowing whether'. For instance, giving a Kripke semantics of Cw^5, axiomatizing $KwCw^5L$, proving the completeness of that axiomatization with respect to models of $KwCw^5L$, studying other definitions of 'commonly knowing whether' (Cw^2 is also a good choice), etc.

Acknowledgement

The authors are greatly indebted to Yanjing Wang for many insightful discussions on the topics of this work and helpful comments on earlier versions of the paper. Jie Fan acknowledges the support of the project 17CZX053 of National Social Science Fundation of China. Xingchi Su was financially supported by Chinese Scholarship Council (CSC) and we wish to thank CSC for its fundings.

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[9] A G-binary-tree model is a tree model where every node exactly has two R_i-successors for every $i \in G$.

[10] A subtree (of some tree model (M, r)) generated with a sequence of agents s is a subtree rooted with r which only consists of all s-paths starting with r in (M, r).

[11] $KwL_{(1)}$ is the language of KwL with only one agent.

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A Proof of Theorem 9

Proof 3 *Given a binary tree (M, v_0), where v_0 is the root, we firstly define the index of M as follows: if there are nodes v_m, t, r in M and $v_m \to_i t$, $v_m \to_i r$, then define the index of t as v_{m0} and the index of r as v_{m1}.*

Let v_m be an arbitrary node in M. Do induction on n:

- *When $n = 1$,*
  - *Assume $M, v_m \models Kw_i\varphi$. Since M is a binary tree, there must be two nodes, v_{m0} and v_{m1} such that $v_m \to_i v_{m0}$ and $v_m \to_i v_{m1}$. Since $M, v_m \models Kw_i^n\varphi$, we have ($M, v_{m0} \models \varphi$ and $M, v_{m1} \models \varphi$) or ($M, v_{m0} \models \neg\varphi$ or $M, v_{m1} \models \neg\varphi$). Thus, on the $(|m|+1)$th layer, the number of nodes where φ is satisfied is 2 or 0, both of which are even.*
  - *Assume the number of the nodes on the $(|m|+1)$th layer that v_m can reach is even. That means there are only two possible cases: ($M, v_{m0} \models \varphi$ and $M, v_{m1} \models \varphi$) or ($M, v_{m0} \models \neg\varphi$ or $M, v_{m1} \models \neg\varphi$). Thus, we have $M, v_m \models Kw_i\varphi$.*
- *Induction hypothesis: when $n = k$, $M, v_m \models Kw_i^k\varphi(1 \leq n) \Leftrightarrow$ the number of the φ-satisfied nodes on the $(|m|+k)th$ layer that v_m can reach via relation $\twoheadrightarrow$ is even.*
- *When $n = k+1$,*
  - *Assume $M, v_m \models Kw_i^{k+1}\varphi$, which is equivalent to $M, v_m \models Kw_i^k Kw_i\varphi$.*
    Let T be the set of nodes exactly consisting of all $Kw_i\varphi$-satisfied nodes on the $(|m|+k)$th layer that v_m can reach via relation $\twoheadrightarrow$. By induction hypothesis, $|T|$ is even. let $|T| = 2a$. Thus, among all the successors of T, the number of φ-satisfied nodes is $2x + 0y$, where $x + y = 2a$. $2x + 0y$ is surely an even number. Let S be a set of nodes only consisting of $\neg Kw_i\varphi$-satisfied nodes on the $(|m|+k)$th layer that v_m can reach via relation $\twoheadrightarrow$. Since M is a binary tree, let $|S| = 2b$. For every node in S has only one φ-satisfied successor, among all the successors of S, the number of φ-satisfied nodes is $2b$. Thus, the number of φ-satisfied nodes on the $(|m|+k+1)$th layer is $2x + 2b = 2(x+b)$ which must be even.
  - *Assume $M, v_m \not\models Kw_i^{k+1}\varphi$, which means $M, v_{m0} \models Kw_i^k\varphi$ and $M, v_{m1} \models \neg Kw_i^k\varphi$. By induction hypothesis, the number of the φ-satisfied nodes on the $(|m|+k+1)th$ layer that v_{m0} can reach via relation $\twoheadrightarrow$ is even. And the number of the φ-satisfied nodes on the $(|m|+k+1)th$ layer that v_{m1} can reach via relation $\twoheadrightarrow$ is odd. That means that the φ-satisfied nodes on the $(|m|+k+1)$th layer that v_m can reach via relation $\twoheadrightarrow$ is an even number plus an odd number, which equals to an odd number.*

B The definition of CL-game

The Definition of CL-Game A CL-game is a game with two players, duplicator and spoiler, playing on a Kripke-model. Given two Kripke models $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$, from an arbitrary node w in W and an arbitrary node w' in W', play games in n rounds between duplicator and spoiler as following rules:

- When $n = 0$, if the sets of satisfied formulas on node w and w' are the same, then duplicator wins; otherwise, spoiler wins.
- When $n \neq 0$,
  - K-move: If spoiler starting from node w does K-move to node x which can be reached by R, then duplicator starting from w' does K-move to a node y in W' with the same set of satisfied propositional variables as x. If spoiler starts from w', then duplicator starts from w with similar way to move.
  - C-move: If spoiler starting from node w does C-move to node x which can be reached by $\twoheadrightarrow$, then duplicator starting from w' does C-move to a node y in W' with same set of satisfied propositional variables to x. If spoiler starts from w', then duplicator starts from w with similar way to move.

  In the game, for arbitrary $x \in W$ and $y \in W'$, we say (x, y) or (y, x) is a state of CL-game.

C Proof of Theorem 11

A Corollary For arbitrary formula φ, $Kw_iKw_i\varphi$ is valid over any $\mathcal{KD}45$, saying $\mathcal{KD}45 \models Kw_iKw_i\varphi$.

This is a corollary of the results in [5].

The Proof of Theorem 11 Let $s = \langle i_1, i_2, \cdots, i_n \rangle$ where there is an $m \in N$ such that $1 \leq m < n$ and $i_m = i_{m+1}$. By the corollary above, $\mathcal{KD}45 \models Kw_{\langle i_m, i_{m+1}, \cdots, i_n \rangle}\varphi$, which means $\mathcal{KD}45 \models Kw_{\langle i_m, i_{m+1}, \cdots, i_n \rangle}\varphi \leftrightarrow \top$ for an arbitrary formula φ. Since $\mathcal{KD}45 \models Kw_{\langle i_1, i_2, \cdots, i_{m-1} \rangle}\top$, we have $\mathcal{KD}45 \models Kw_{\langle i_1, i_2, \cdots, i_n \rangle}\varphi$.