Distributed coordination and partial synchronization in complex networks

Qin, Yuzhen

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3 New Lyapunov Criteria for Discrete-Time Stochastic Systems

More recently, with the fast development of network algorithms, more and more distributed computational processes are carried out in networks of computational units. Such dynamical processes are usually modeled by stochastic discrete-time dynamical systems since they are usually under inevitable random influences or deliberately randomized to improve performance. So there is a great need to further develop the Lyapunov theory for stochastic dynamical systems, in particular in the setting of network algorithms for distributed computation. And this is exactly the aim of this chapter.

3.1 Introduction

Stability analysis for stochastic dynamical systems has always been an active research field. Early works have shown that stochastic Lyapunov functions play an important role, and to use them for discrete-time systems, a standard procedure is to show that they decrease in *expectation* at every time step [65–67, 119]. Properties of supermartingales and LaSalle's arguments are critical to establishing the related proofs. However, most of the stochastic stability results are built upon a crucial assumption, which requires that the state of a stochastic dynamical system under study is Markovian (see e.g., [64–67]), and very few of them have reported bounds for the convergence speed.

In this chapter, we aim at further developing the Lyapunov criterion for stochastic discrete-time systems in order to solve the problems we encounter in studying distributed coordination algorithms in the next chapter. Inspired by the concept of *finite-step Lyapunov functions* for deterministic systems [120–122], we propose to define a *finite-step stochastic Lyapunov function*, which decreases in expectation, not necessarily at every step, but after a finite number of steps. The associated new Lyapunov criterion not only enlarges the range of choices of candidate Lyapunov functions but also implies that the systems that can be analyzed do not need to have Markovian states. An additional advantage of using this new criterion is that we are enabled to construct conditions to guarantee exponential convergence and estimate convergence rates [102].

Outline

The remainder of this chapter is structured as follows. First, we introduce the system dynamics and formulate the problem in Section 3.2. Main results on finite-step Lyapunov functions are provided in Section 3.3. Finally, some concluding remarks appear in Section 3.4.

3.2 Problem Formulation

Consider a stochastic discrete-time system described by

$$x_{k+1} = f(x_k, y_{k+1}), \qquad k \in \mathbb{N}_0, \tag{3.1}$$

where $x_k \in \mathbb{R}^n$, and $\{y_k : k \in \mathbb{N}\}$ is a $\mathbb{R}^d$-valued stochastic process on a probability space $(\Omega, \mathcal{F}, \Pr)$. Here $\Omega = \{\omega\}$ is the sample space; $\mathcal{F}$ is a set of events which is a σ-field; y_k is a measurable function mapping Ω into the state space $\Omega_0 \subseteq \mathbb{R}^d$, and for any $\omega \in \Omega$, $\{y_k(\omega) : k \in \mathbb{N}\}$ is a realization of the stochastic process $\{y_k\}$ at ω. Let $\mathcal{F}_k = \sigma(y_1, \ldots, y_k)$ for $k \geq 1$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, so that evidently $\{\mathcal{F}_k\}, k = 1, 2, \ldots,$ is an increasing sequence of σ-fields. Following [123], we consider a constant initial condition $x_0 \in \mathbb{R}^n$ with probability one. It then can be observed that the solution to (3.1), $\{x_k\}$, is a $\mathbb{R}^n$-valued stochastic process adapted to $\mathcal{F}_k$. The randomness of y_k can be due to various reasons, e.g., stochastic disturbances or noise. Note that (3.1) becomes a stochastic switching system if $f(x, y) = g_y(x)$, where y maps Ω into the set $\Omega_0 := \{1, \ldots, p\}$, and $\{g_p(x) : \mathbb{R}^n \to \mathbb{R}^n, p \in \Omega_0\}$ is a given family of functions.

A point x^* is said to be an *equilibrium* of system (3.1) if $f(x^*, y) = x^*$ for any $y \in \Omega_0$. Without loss of generality, we assume that the origin $x = 0$ is an equilibrium. Researchers have been interested in studying the limiting behavior of the solution $\{x_k\}$, i.e., when and to where x_k converges as $k \to \infty$. Most noticeably, Kushner developed classic results on stochastic stability by employing stochastic Lyapunov functions [65–67]. We introduce some related definitions before recalling some of Kushner's results. Following [124, Sec. 1.5.6] and [125], we first define convergence and exponential convergence of a sequence of random variables.

Definition 3.1 (Convergence)**.** *A random sequence* $\{x_k \in \mathbb{R}^n\}$ *in a sample space* Ω *converges to a random variable* x almost surely *if* $\Pr\left[\omega \in \Omega : \lim_{k\to\infty} \|x_k(\omega) - x\| = 0\right] = 1$. *The convergence is said to be exponentially fast with a rate no slower than* γ^{-1} *for some* $\gamma > 1$ *independent of* ω *if* $\gamma^k \|x_k - x\|$ *almost surely converges to* y *for some finite* $y \geq 0$. *Furthermore, let* $\mathcal{D} \subset \mathbb{R}^n$ *be a set; a random sequence* $\{x_k\}$ *is said to converge to* $\mathcal{D}$ *almost surely if* $\Pr\left[\omega \in \Omega : \lim_{k\to\infty} \text{dist}(x_k(\omega), \mathcal{D}) = 0\right] = 1$, *where* $\text{dist}\,(x, \mathcal{D}) := \inf_{y\in\mathcal{D}} \|x - y\|$.

Here "almost surely" is exchangeable with "with probability one", and we sometimes use the shorthand notation "a.s.". We now introduce some stability concepts for stochastic discrete-time systems analogous to those in [64] and [126] for continuous-time systems[1].

Definition 3.2. *The origin of (1) is said to be:*

1) stable in probability *if* $\lim_{x_0\to 0} \Pr\left[\sup_{k\in\mathbb{N}} \|x_k\| > \varepsilon\right] = 0$ *for any* $\varepsilon > 0$*;*

2) asymptotically stable in probability *if it is stable in probability and moreover* $\lim_{x_0\to 0} \Pr\left[\lim_{k\to\infty} \|x_k\| = 0\right] = 1$*;*

3) exponentially stable in probability *if for some* $\gamma > 1$ *independent of* ω, *it holds that* $\lim_{x_0\to 0} \Pr\left[\lim_{k\to\infty} \|\gamma^k x_k\| = 0\right] = 1$*;*

Definition 3.3. *For a set* $\mathcal{Q} \subseteq \mathbb{R}^n$ *containing the origin, the origin of (1) is said to be:*

1) locally a.s. asymptotically stable in $\mathcal{Q}$ (globally a.s. asymptotically stable, respectively) *if a) it is stable in probability, and b) starting from* $x_0 \in \mathcal{Q}$ *(*$x_0 \in \mathbb{R}^n$*, respectively) all the sample paths* x_k *stay in* $\mathcal{Q}$ *(*$\mathbb{R}^n$*, respectively) for all* $k \geq 0$ *and converge to the origin almost surely;*

2) locally a.s. exponentially stable in $\mathcal{Q}$ (globally a.s. exponentially stable, respectively) *if it is locally (globally, respectively) a.s. asymptotically stable and the convergence is exponentially fast.*

Now let us recall some Kushner's results on convergence and stability, where stochastic Lyapunov functions have been used.

Lemma 3.1 (Asymptotic Convergence and Stability [67, 127])**.** *For the stochastic discrete-time system* (3.1)*, let* $\{x_k\}$ *be a Markov process. Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuous positive definite and radially unbounded function. Define the set* $\mathcal{Q}_\lambda := \{x : 0 \leq V(x) < \lambda\}$ *for some* $\lambda > 0$, *and assume that*

$$\mathbb{E}\left[V\left(x_{k+1}\right) | x_k\right] - V\left(x_k\right) \leq -\varphi(x_k), \forall k, \tag{3.2}$$

[1]Note that 1) and 2) of Definition 3.2 follow from the definitions in [64, Chap. 5], in which an arbitrary initial time s rather than just 0 is actually considered; we define 3) following the same lines as 1) and 2). In Definition 3.3, 1) follows from the definitions in [126], and we define 2) following the same lines as 1).

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies $\varphi(x) \geq 0$ for any $x \in \mathcal{Q}_\lambda$. Then the following statements apply:

(i) for any initial condition $x_0 \in \mathcal{Q}_\lambda$, x_k converges to $\mathcal{D}_1 := \{x \in \mathcal{Q}_\lambda : \varphi(x) = 0\}$ with probability greater than or equal to $1 - V(x_0)/\lambda$ [67];

(ii) if moreover $\varphi(x)$ is positive definite on $\mathcal{Q}_\lambda$, and $h_1(\|s\|) \leq V(s) \leq h_2(\|s\|)$ for two class $\mathcal{K}$ functions h_1 and h_2, then $x = 0$ is asymptotically stable in probability [67], [127, Theorem 7.3].

Lemma 3.2 (Exponential Convergence and Stability [66, 127])**.** *For the stochastic discrete-time system* (3.1)*, let $\{x_k\}$ be a Markov process. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonnegative function. Assume that*

$$\mathbb{E}\left[V\left(x_{k+1}\right)|x_k\right] - V\left(x_k\right) \leq -\alpha V(x_k), \qquad 0 < \alpha < 1. \tag{3.3}$$

Then the following statements apply:

(i) for any given x_0, $V(x_k)$ almost surely converges to 0 exponentially fast with a rate no slower than $1 - \alpha$ [66, Th. 2, Chap. 8], [127];

(ii) if moreover V satisfies $c_1\|x\|^a \leq V(x) \leq c_2\|x\|^a$ for some $c_1, c_2, a > 0$, then $x = 0$ is globally a.s. exponentially stable [127, Theorem 7.4].

To use these two lemmas to prove asymptotic (or exponential) stability for a stochastic system, the critical step is to find a stochastic Lyapunov function such that (3.2) (respectively, (3.3)) holds. However, it is not always obvious how to construct such a stochastic Lyapunov function. We use the following simple but suggestive example to illustrate this point.

Example 3.1 Consider a randomly switching system described by $x_k = A_{y_k} x_{k-1}$, where y_k is the switching signal taking values in a finite set $\mathcal{P} := \{1, 2, 3\}$, and

$$A_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix}.$$

The stochastic process $\{y_k\}$ is described by a Markov chain with initial distribution $v = \{v_1, v_2, v_3\}$. The transition probabilities are described by a transition matrix

$$\pi = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

whose ijth element is defined by $\pi_{ij} = \Pr[y_{k+1} = j | y_k = i]$. Since $\{y_k\}$ is not independent and identically distributed, the process $\{x_k\}$ is not Markovian. Nevertheless, we might conjecture that the origin is globally a.s. exponentially stable. In order to try to prove this, we might choose a stochastic Lyapunov function candidate $V(x) = \|x\|_\infty$, but the existing results introduced in Lemma 3.2 cannot be used since $\{x_k\}$ is not Markovian. Moreover, by calculation we can only observe that $\mathbb{E}\left[V\left(x_{k+1}\right) | x_k, y_k\right] \le V\left(x_k\right)$ for any y_k, which implies that (3.3) is not necessarily satisfied. Thus $V(x)$ is not an appropriate stochastic Lyapunov function for which Lemma 3.2 can be applied. As it turns out however, the same $V(x)$ can be used as a Lyapunov function to establish exponentially stability via the alternative criterion set out subsequently. △

It is difficult, if not impossible, to construct a stochastic Lyapunov function, especially when the state of the system is not Markovian. So it is of great interest to generalize the results in Lemmas 3.1 and 3.2 such that the range of choices of candidate Lyapunov functions can be enlarged. For deterministic systems, Aeyels et al. have introduced a new Lyapunov criterion to study asymptotic stability of continuous-time systems [120]; a similar criterion has also been obtained for discrete-time systems, and the Lyapunov functions satisfying this criterion are called *finite-step Lyapunov functions* [121, 122]. A common feature of these works is that the Lyapunov function is required to decrease along the system's solutions after a finite number of steps, but not necessarily at every step. We now use this idea to construct stochastic finite-step Lyapunov functions, a task which is much more challenging compared to the deterministic case due to the uncertainty present in stochastic systems. The tools for analysis are totally different from what are used for deterministic systems. We will exploit supermartingales [109] and their convergence property, as well as another lemma found in [66, P.192]; these concepts are introduced in the two following lemmas.

Lemma 3.3 ([109, Sec. 5.2.9])**.** *Let the sequence $\{X_k\}$ be a nonnegative* supermartingale *with respect to $\mathcal{F}_k = \sigma(X_1, \dots, X_k)$, i.e., suppose: (i) $\mathbb{E}X_n < \infty$; (ii) $X_k \in \mathcal{F}_k$ for all k; (iii) $\mathbb{E}\left(X_{k+1} | \mathcal{F}_k\right) \le X_k$. Then there exists some random X such that $X_k \xrightarrow{a.s.} X, k \to \infty$, and $\mathbb{E}X \le \mathbb{E}X_0$.*

Lemma 3.4 ([66, P.192])**.** *Let $\{X_k\}$ be a nonnegative random sequence. If $\sum_{k=0}^{\infty} \mathbb{E}X_k < \infty$, then $X_k \xrightarrow{a.s.} 0$.*

Lemma 3.4 is also called Borel-Cantelli Lemma by Kushner in his book [66]. However, it is a bit different from the standard Borel-Cantelli Lemma (see [109, Chap. 2]). We provide a proof of Lemma 3.4 following the ideas in [66], which can be found in Section 3.5.

3.3 Finite-Step Stochastic Lyapunov Criteria

In this subsection, we present some finite-step stochastic Lyapunov criteria for stability analysis of stochastic discrete-time systems, which are the main results in the chapter. In these criteria, the expectation of a Lyapunov function is not required to decrease at every time step, but is allowed to decrease after some finite steps. The relaxation enlarges the range of choices of candidate Lyapunov functions. In addition, these criteria can be used to analyze non-Markovian systems.

Theorem 3.1. *For the stochastic discrete-time system* (3.1)*, let* $V:\mathbb{R}^n\to\mathbb{R}$ *be a continuous nonnegative and radially unbounded function. Define the set* $\mathcal{Q}_\lambda := \{x : V(x) < \lambda\}$ *for some* $\lambda > 0$*, and assume that*

(a) $\mathbb{E}\left[V\left(x_{k+1}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq 0$ *for any* k *such that* $x_k \in \mathcal{Q}_\lambda$*;*

(b) there is an integer $T \geq 1$*, independent of* ω*, such that for any* k*,*

$$\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq -\varphi(x_k),$$

where $\varphi:\mathbb{R}^n\to\mathbb{R}$ *is continuous and satisfies* $\varphi(x)\geq 0$ *for any* $x\in\mathcal{Q}_\lambda$*.*

Then the following statements apply:

(i) for any initial condition $x_0\in\mathcal{Q}_\lambda$*,* x_k *converges to* $\mathcal{D}_1 := \{x\in\mathcal{Q}_\lambda : \varphi(x) = 0\}$ *with probability greater than or equal to* $1 - V(x_0)/\lambda$*;*

(ii) if moreover $\varphi(x)$ *is positive definite on* $\mathcal{Q}_\lambda$*, and* $h_1\left(\|s\|\right) \leq V(s) \leq h_2\left(\|s\|\right)$ *for two class* $\mathcal{K}$ *functions* h_1 *and* h_2*, then* $x = 0$ *is asymptotically stable in probability.*

Proof. Before proving (i) and (ii), we first show that starting from $x_0\in\mathcal{Q}_\lambda$ the sample paths $x_k(\omega)$ stay in $\mathcal{Q}_\lambda$ with probability greater than or equal to $1 - V(x_0)/\lambda$ if Assumption a) is satisfied. This has been proven in [66, p. 196] by showing that

$$\Pr\left[\sup_{k\in\mathbb{N}} V\left(x_k\right) \geq \lambda\right] \leq V\left(x_0\right)/\lambda. \tag{3.4}$$

Let $\bar{\Omega}$ be a subset of the sample space Ω such that for any $\omega\in\bar{\Omega}$, $x_k(\omega)\in\mathcal{Q}_\lambda$ for all k. Let J be the smallest $k\in\mathbb{N}$ (if it exists) such that $V(x_k)\geq\lambda$. Note that, this integer J does not exist when $x_k(\omega)$ stays in $\mathcal{Q}_\lambda$ for all k, i.e., when $\omega\in\bar{\Omega}$.

We first prove (i) by showing that the sample paths staying the $\mathcal{Q}_\lambda$ converge to $\mathcal{D}_1$ with probability one, i.e., $\Pr[x_k\to\mathcal{D}_1|\bar{\Omega}] = 1$. Towards this end, define a new function $\tilde{\varphi}(x)$ such that $\tilde{\varphi}(x) = \varphi(x)$ for $x \in \mathcal{Q}_\lambda$, and $\tilde{\varphi}(x) = 0$ for $x \notin \mathcal{Q}_\lambda$. Define another random process $\{\tilde{z}_k\}$. If J exists, when $J > T$ let

$$\begin{aligned} \tilde{z}_k &= x_k, && k < J - T, \\ \tilde{z}_k &= \epsilon, && k \geq J - T, \end{aligned}$$

where ϵ satisfies $V(\epsilon) = \tilde{\lambda} > \lambda$; when $J \leq T$, let $\tilde{z}_k = \epsilon$ for any $k \in \mathbb{N}_0$. If J does not exist, we let $\tilde{z}_k = x_k$ for all $k \in \mathbb{N}_0$. Then it is immediately clear that $\mathbb{E}\left[V\left(\tilde{z}_{k+T}\right)|\mathcal{F}_k\right] - V\left(\tilde{z}_k\right) \leq -\tilde{\varphi}(\tilde{z}_k) \leq 0$. By taking the expectation on both sides of this inequality, we obtain

$$\mathbb{E}\left[V\big(\tilde{z}_{k+T}\big)\right] - \mathbb{E}V\big(\tilde{z}_k\big) \leq -\mathbb{E}\tilde{\varphi}\big(\tilde{z}_k\big), k \in \mathbb{N}_0. \tag{3.5}$$

For any $k \in \mathbb{N}$, there is a pair $p, q \in \mathbb{N}_0$ such that $k = pT + q$. From (3.5) one obtains that

$$\mathbb{E}\left[V\big(\tilde{z}_{pT+j}\big)\right] - \mathbb{E}V\big(\tilde{z}_{(p-1)T+j}\big) \leq -\mathbb{E}\tilde{\varphi}\big(\tilde{z}_{(p-1)T+j}\big)$$

holds for all $j = 0, \ldots, q$, and

$$\mathbb{E}\left[V\big(\tilde{z}_{iT+m}\big)\right] - \mathbb{E}V\big(\tilde{z}_{(i-1)T+m}\big) \leq -\mathbb{E}\tilde{\varphi}\big(\tilde{z}_{(i-1)T+m}\big)$$

holds for all $i = 1, \ldots, p-1$ and $m = 0, \ldots, T-1$. By summing up all the left and right sides of these inequalities respectively for all the i, j and m, we have

$$\begin{aligned} &\sum_{m=0}^{T-1} \Big(\mathbb{E}\big[V(\tilde{z}_{(p-1)T+m} - \mathbb{E}V(\tilde{z}_m)\big]\Big) \\ &\qquad + \sum_{j=1}^{q} \Big(\mathbb{E}\big[V(\tilde{z}_{pT+j} - \mathbb{E}V(\tilde{z}_{(p-1)T+j})\big]\Big) \leq -\sum_{i=1}^{k-T} \mathbb{E}\tilde{\varphi}\big(\tilde{z}_i\big). \end{aligned} \tag{3.6}$$

As $V(x)$ is nonnegative for all x, from (3.5) it is easy to observe that the left side of (3.6) is greater than $-\infty$ even when $k \to \infty$ since T and q are finite numbers, which implies that $\sum_{i=0}^{\infty} \mathbb{E}\tilde{\varphi}\big(\tilde{z}_k\big) < \infty$. By Lemma 3.4, ones knows that $\tilde{\varphi}\big(\tilde{z}_k\big) \xrightarrow{a.s.} 0$ as $k \to \infty$. For $\omega \in \bar{\Omega}$, one can observe that $\tilde{\varphi}(x_k(\omega)) = \varphi(x_k(\omega))$ and $\tilde{z}_k(\omega) = x_k(\omega)$ according to the definitions of $\tilde{\varphi}$ and $\{\tilde{z}_k\}$, respectively. Therefore, $\tilde{\varphi}(\tilde{z}_k(\omega)) = \varphi(x_k(\omega))$ for all $\omega \in \bar{\Omega}$, and subsequently

$$\Pr[\varphi(x_k) \to 0|\bar{\Omega}] = \Pr[\tilde{\varphi}(\tilde{z}_k) \to 0|\bar{\Omega}] = 1.$$

From the continuity of $\varphi(x)$ it can be seen that $\Pr[x_k \to \mathcal{D}_1|\bar{\Omega}] = 1$. The proof of (i) is complete since (3.4) means that the sample paths stay in $\mathcal{Q}_\lambda$ with probability greater than or equal to $1 - V(x_0)/\lambda$.

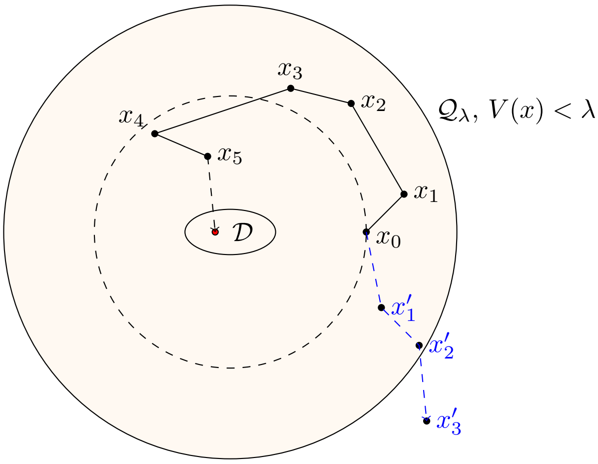

Figure 3.1: An illustration of the asymptotic behavior in $\mathcal{Q}_\lambda$.

Next, we prove (ii) in two steps. We first prove that the origin $x = 0$ is stable in probability. The inequalities $h_1(\|s\|) \leq V(s) \leq h_2(\|s\|)$ imply that $V(x) = 0$ if and only if $x = 0$. Moreover, it follows from $h_1(\|s\|) \leq V(s)$ and the inequality (3.4) that for any initial condition $x_0 \in \mathcal{Q}_\lambda$,

$$\Pr\left[\sup_{k\in\mathbb{N}} h_1(\|x_k\|) \geq \lambda_1\right] \leq \Pr\left[\sup_{k\in\mathbb{N}} V(x_k) \geq \lambda_1\right] \leq \frac{V(x_0)}{\lambda_1}$$

for any $\lambda_1 > 0$. Since h_1 is a class $\mathcal{K}$ function and thus invertible, it can be observed that

$$\Pr\left[\sup_{k\in\mathbb{N}} \|x_k\| \geq h_1^{-1}(\lambda)\right] \leq V(x_0)/\lambda \leq h_2(\|x_0\|)/\lambda.$$

Then for any $\varepsilon > 0$, it holds that

$$\lim_{x_0\to 0} \Pr\left[\sup_{k\in\mathbb{N}} \|x_k\| > \varepsilon\right] \leq \Pr\left[\sup_{k\in\mathbb{N}} \|x_k\| \geq \varepsilon\right] = 0,$$

which means that the origin is stable in probability.

Second, we show the probability that $x_k \to 0$ tends to 1 as $x_0 \to 0$. One knows that $\mathcal{D}_1 = \{0\}$ since φ is positive definite in $\mathcal{Q}_\lambda$. From (i) one knows that x_k converges to $x = 0$ with probability greater than or equal to $1 - V(x_0)/\lambda$. Since $V(x) \to 0$ as $x_0 \to 0$, it holds that $\lim_{x_0\to 0} \Pr[\lim_{k\to\infty} \|x_k\| = 0] \to 1$. The proof is complete. □

With the help of Fig. 3.1, let us provide some explanations on what have been mainly stated in Theorem 3.1. The sample paths x_k always have a possibility to leave the set $\mathcal{Q}_\lambda$, but with probability less than $V(x_0)/\lambda$ (see the blue trajectory $\{x'_k\}$). In other words, they stay in $\mathcal{Q}_\lambda$ with probability no less than $1 - V(x_0)/\lambda$. If $\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq -\varphi(x_k)$ for a finite positive integer T, all the sample paths remaining in $\mathcal{Q}_\lambda$ will converge to the set $\mathcal{D}_1$ (see the black trajectory $\{x_k\}$). If moreover, $\mathcal{D}_1$ is a singleton $\{0\}$, and $h_1\left(\|s\|\right) \leq V(s) \leq h_2\left(\|s\|\right)$ for two class $\mathcal{K}$ functions h_1 and h_2, then $x = 0$ is asymptotically stable *in probability*.

Particularly, if $\mathcal{Q}_\lambda$ is positively invariant, i.e., starting from $x_0 \in \mathcal{Q}_\lambda$ all sample paths x_k will stay in $\mathcal{Q}_\lambda$ for all $k \geq 0$, this corollary follows from Theorem 3.1 straightforwardly.

Corollary 3.1. *Assume that $\mathcal{Q}_\lambda$ is positively invariant along the system* (3.1)*, and there hold that*

(a) $\mathbb{E}\left[V\left(x_{k+1}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq 0$ *for any k such that $x_k \in \mathcal{Q}_\lambda$;*

(b) there is an integer $T \geq 1$, independent of ω, such that for any k,

$$\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq -\varphi(x_k),$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies $\varphi(x) \geq 0$ for any $x \in \mathcal{Q}_\lambda$.

Then the following statements apply:

(i) for any initial condition $x_0 \in \mathcal{Q}_\lambda$, x_k converges to $\mathcal{D}_1$ with probability one;

(ii) if moreover $\varphi(x)$ is positive definite on $\mathcal{Q}_\lambda$, and $h_1\left(\|s\|\right) \leq V(s) \leq h_2\left(\|s\|\right)$ for two class $\mathcal{K}$ functions h_1 and h_2, then $x = 0$ is locally a.s. asymptotically stable in $\mathcal{Q}_\lambda$. Furthermore, if $\mathcal{Q}_\lambda = \mathbb{R}^n$, then $x = 0$ is globally a.s. asymptotically stable.

Theorem 3.1 and Corollary 3.1 provide some Lyapunov criteria for asymptotic stability and convergence of stochastic discrete-time systems. The next theorem provides a new criterion for exponential convergence and stability of stochastic systems, relaxing the conditions required by Lemma 3.2.

Theorem 3.2. *Suppose that the following conditions are satisfied*

(a) $\mathbb{E}\left[V\left(x_{k+1}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq 0$ *for any k such that $x_k \in \mathcal{Q}_\lambda$;*

(b) there is an integer $T \geq 1$, independent of ω, such that for any k,

$$\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq -\alpha V(x_k), \qquad 0 < \alpha < 1, \tag{3.7}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies $\varphi(x) \geq 0$ for any $x \in \mathcal{Q}_\lambda$.

Then, the following statements apply:

(i) for any given $x_0 \in \mathcal{Q}_\lambda$, $V(x_k)$ converges to 0 exponentially at a rate no slower than $(1-\alpha)^{1/T}$, and x_k converges to $\mathcal{D}_2 := \{x \in \mathcal{Q}_\lambda : V(x) = 0\}$, with probability greater than or equal to $1 - V(x_0)/\lambda$;

(ii) if moreover V satisfies that $c_1\|x\|^a \leq V(x) \leq c_2\|x\|^a$ for some $c_1, c_2, a > 0$, then $x = 0$ is exponentially stable in probability.

Proof. We first prove (i). From the proof of Theorem 3.1, we know that the sample paths x_k stay in $\mathcal{Q}_\lambda$ with probability greater than or equal to $1 - V(x_0)/\lambda$ for any initial condition $x_0 \in \mathcal{Q}_\lambda$ if the assumption a) is satisfied. We next show that for any sample path that always stays in $\mathcal{Q}_\lambda$, $V(x_k)$ converges to 0 exponentially fast. Towards this end, we define a random process $\{\hat{z}_k\}$. Let J be as defined in the proof of Theorem 3.1. If J exists, when $J > T$, let

$$\begin{aligned} \hat{z}_k &= x_k, & k < J - T, \\ \hat{z}_k &= \varepsilon, & k \geq J - T, \end{aligned}$$

where ε satisfies $V(\varepsilon) = 0$, when $J \leq T$, let $\hat{z}_k = \varepsilon$ for any $k \in \mathbb{N}_0$; if J does not exist, we let $\hat{z}_k = x_k$ for all $k \in \mathbb{N}_0$.

If the inequality (3.7) is satisfied, one has $\mathbb{E}\left[V\left(\hat{z}_{k+T}\right)|\mathcal{F}_k\right] - V\left(\hat{z}_k\right) \leq -\alpha V(\hat{z}_k)$. Using this inequality, we next show that $V\left(\hat{z}_{k+T}\right)$ converges to 0 exponentially. To this end, define a subsequence

$$Y_m^{(r)} := V(\hat{z}_{mT+r}), \qquad m \in \mathbb{N}_0,$$

for each $0 \leq r \leq T-1$. Let $\mathcal{G}_m^{(r)} := \sigma(Y_0^{(r)}, Y_1^{(r)}, \ldots, Y_m^{(r)})$, and one knows that $\mathcal{G}_m^{(r)}$ is determined if we know $\mathcal{F}_{mT+r}$. It then follows from the inequality (3.7) that for any r, $\mathbb{E}[Y_{m+1}^{(r)}|\mathcal{G}_m^{(r)}] - Y_m^{(r)} \leq -\alpha Y_m^{(r)}$. We observe from this inequality that

$$\mathbb{E}\left[(1-\alpha)^{-(m+1)}Y_{m+1}^{(r)}|\mathcal{G}_m^{(r)}\right] - (1-\alpha)^{-m}Y_m^{(r)} \leq 0.$$

This means that $(1-\alpha)^{-m}Y_m$ is a supermartingale, and thus there is a finite random number $\bar{Y}^{(r)}$ such that $(1-\alpha)^{-m}Y_m^{(r)} \xrightarrow{a.s.} \bar{Y}^{(r)}$ for any r. Let $\gamma = \sqrt[T]{1/(1-\alpha)}$, and then by the definition of $Y_m^{(r)}$ we have

$$\gamma^{mT}V(\hat{z}_{mT+r}) \xrightarrow{a.s.} \bar{Y}^{(r)}.$$

Straightforwardly, it follows that $\gamma^{mT+r}V(\hat{z}_{mT+r}) \xrightarrow{a.s.} \gamma^r\bar{Y}^{(r)}$. Let $k = mT + r, \bar{Y} = \max_r\{\gamma^r\bar{Y}^{(r)}\}$, then it almost surely holds that $\lim_{k\to\infty}\gamma^k V(\hat{z}_k) \leq \bar{Y}$. From Definition 3.1, one concludes that $V(\hat{z}_k)$ almost surely converges to 0 exponentially no slower than $\gamma^{-1} = (1-\alpha)^{1/T}$. From the definition of $\hat{z}_k$, we know that $V(\hat{z}_k(\omega)) = V(x_k(\omega))$ for all $\omega \in \bar{\Omega}$, with $\bar{\Omega}$ defined in the proof of Theorem 3.1. Consequently, it holds that

$$\Pr\big[\lim_{k\to\infty} \gamma^k V(x_k) \le \bar{Y} | \bar{\Omega}\big] = \Pr\big[\lim_{k\to\infty} \gamma^k V(\hat{z}_k) \le \bar{Y} | \bar{\Omega}\big] = 1. \quad (3.8)$$

The proof of (i) is complete since the sample paths stay in $\mathcal{Q}_\lambda$ with probability greater than or equal to $1 - V(x_0)/\lambda$.

Next, we prove (ii). If the inequalities $c_1\|x\|^a \le V(x) \le c_2\|x\|^a$ are satisfied, and then we know that $V(x) = 0$ if and only if $x = 0$. Moreover, it follows from (3.8) that for all the sample paths that stay in $\mathcal{Q}_\lambda$ it holds that $c_1\gamma^k\|x\|^a \le \gamma^k V(x_k) \le \bar{Y}$ since $c_1\|x_k\|^a \le V(x)$. Hence,

$$\|x_k(\omega)\| \le \left(\bar{V}/c_1\right)^{1/a} \gamma^{-k/a}, \qquad \forall \omega \in \bar{\Omega},$$

and one can check that this inequality holds with probability greater than or equal to $1 - V(x_0)/\lambda$. If $x_0 \to 0$, we know that $1 - V(x_0)/\lambda \to 1$, which completes the proof. □

If $\mathcal{Q}_\lambda$ is positively invariant, the following corollary follows straightforwardly.

Corollary 3.2. *Suppose that $\mathcal{Q}_\lambda$ is positively invariant along the system* (3.1)*, and the following conditions are satisfied*

a) $\mathbb{E}\left[V\left(x_{k+1}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \le 0$ *for any k such that $x_k \in \mathcal{Q}_\lambda$;*

b) *there is an integer $T \ge 1$, independent of ω, such that for any k,*

$$\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \le -\alpha V(x_k), \qquad 0 < \alpha < 1, \quad (3.9)$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies $\varphi(x) \ge 0$ for any $x \in \mathcal{Q}_\lambda$.

Then, the following statements apply:

(i) for any given $x_0 \in \mathcal{Q}_\lambda$, $V(x_k)$ converges to 0 exponentially no slower than $(1-\alpha)^{1/T}$ with probability one;

(ii) if moreover V satisfies that $c_1\|x\|^a \le V(x) \le c_2\|x\|^a$ for some $c_1, c_2, a > 0$, then $x = 0$ is locally a.s. exponentially stable in $\mathcal{Q}_\lambda$. Furthermore, if $\mathcal{Q}_\lambda = \mathbb{R}^n$, then $x = 0$ is globally a.s. exponentially stable.

The following corollary, which can be proven following the same lines as Theorems 3.1 and 3.2, shares some similarities to LaSalle's theorem for deterministic systems. It is worth mentioning that the function V here does not have to be radially unbounded.

Corollary 3.3. *Let $\mathbb{D} \subset \mathbb{R}^n$ be a compact set that is positively invariant along the system* (3.1)*. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonnegative function, and $\bar{\mathcal{Q}}_\lambda := \{x \in \mathbb{D} : V(x) < \lambda\}$ for some $\lambda > 0$. Assume that $\mathbb{E}\left[V\left(x_{k+1}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq 0$ for all k such that $x_k \in \bar{\mathcal{Q}}_\lambda$, then*

(i) if there is an integer $T \geq 1$, independent of ω, such that for any $k \in \mathbb{N}_0$, $\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq -\varphi(x_k)$, where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies $\varphi(x) \geq 0$ for any $x \in \bar{\mathcal{Q}}_\lambda$, then for any initial condition $x_0 \in \bar{\mathcal{Q}}_\lambda$, x_k converges to $\bar{\mathcal{D}}_1 := \{x \in \bar{\mathcal{Q}}_\lambda : \varphi(x) = 0\}$ with probability greater than or equal to $1 - V(x_0)/\lambda$;

(ii) if the inequality in a) is strengthened to $\mathbb{E}\left[V\left(x_{k+T}\right)|\mathcal{F}_k\right] - V\left(x_k\right) \leq -\alpha V(x_k)$ for some $0 < \alpha < 1$, then for any given $x_0 \in \bar{\mathcal{Q}}_\lambda$, $V(x_k)$ converges to 0 exponentially at a rate no slower than $(1-\alpha)^{1/T}$, and x_k converges to $\bar{\mathcal{D}}_2 := \{x \in \bar{\mathcal{Q}}_\lambda : V(x) = 0\}$, with probability greater than or equal to $1 - V(x_0)/\lambda$;

(iii) if $\bar{\mathcal{Q}}_\lambda$ is positively invariant w.r.t the system (3.1)*, then all the convergence in both (i) and (ii) takes place* almost surely*.*

Continuation of Example 3.1 Now let us look back at Example 1 and still choose $V(x) = \|x\|_\infty$ as a stochastic Lyapunov function candidate. It is easy to see that $V(x)$ is a nonnegative supermartingale. To show the stochastic convergence, let $T = 2$ and one can calculate the conditional expectations

$$\begin{aligned}
&\mathbb{E}\left[V\left(x_{k+T}\right)|\, x_k, y_k = 1\right] - V\left(x_k\right) \\
&= 0.5\left\| \begin{array}{c} 0.2x_k^1 \\ 0.8x_k^2 \end{array} \right\|_\infty + 0.5\left\| \begin{array}{c} 0.2x_k^1 \\ 0.6x_k^2 \end{array} \right\|_\infty - \left\| \begin{array}{c} x_k^1 \\ x_k^2 \end{array} \right\|_\infty \\
&\leq -0.3V\left(x_k\right), \forall x_k \in \mathbb{R}^2.
\end{aligned}$$

When $y_k = 2, 3$, it analogously holds that

$$\mathbb{E}[V\left(x_{k+T}\right)|\, x_k, y_k] - V\left(x_k\right) \leq -0.3V(x_k), \forall x_k \in \mathbb{R}^2.$$

From these three inequalities one can observe that starting from any initial condition x_0, $\mathbb{E}V(x)$ decreases at an exponential speed after every two steps before it reaches 0. By Corollary 3.2, one knows that origin is globally a.s. exponentially stable, consistent with our conjecture. △

Kushner and other researchers have used more restricted conditions to construct Lyapunov functions than those appearing in our results to analyze asymptotic or exponential stability of random processes [66, 67, 119]. It is required that $\mathbb{E}[V(x_k)]$ decreases strictly at every step, until $V(x_k)$ reaches a limit value. However, in our result, this requirement is relaxed. In addition, Kushner's results rely on the assumption that the underlying random process is Markovian, but we work with more general random processes.

3.4 Concluding Remarks

Many distributed coordination algorithms are stochastic since they are often under inevitable random influences, or randomness is deliberately introduced into them to improve global performance. Stochastic Lyapunov theory is often needed to study them. However, it is not always easy to construct a stochastic Lyapunov function using the existing criteria. In this chapter, we have further developed a tool, termed finite-step stochastic Lyapunov criteria, using which one can study the convergence and stability of a stochastic discrete-time system together with its convergence rate. Unlike what is required in the existing Lyapunov criteria [65–67, 119], the constructed Lyapunov function does not have to decrease after every time step. Instead, decreasing after some finite time steps is sufficient to guarantee the asymptotic or exponential convergence and stability of a system, which makes the construction of a Lyapunov function easier. In addition, the states of a system under study do not have to be Markovian. The tool we developed in this chapter plays a very important role in studying some stochastic coordination algorithms, which we will discuss in more detail in the next chapter.

3.5 Appendix: Proof of Lemma 3.4

We first provide two lemmas and a definition that will be used in the proof.

Lemma 3.5 (Borel-Cantelli lemma [109, Chap. 2]). *Let $\{A_k\}$ be a sequence of events in some probability space. If the sum of the probabilities of the A_k is finite*

$$\sum_{k=1}^{\infty} \Pr\left[A_k\right] < \infty,$$

then the probability that infinitely many of them occur is 0, that is,

$$\Pr\left(\limsup_{k\to\infty} A_k\right) = 0.$$

Lemma 3.6 (Markov's inequality [109, Chap. 1]). *If X is a nonnegative random variable and $a > 0$, then $\Pr[X \geq a] \leq \mathbb{E}X/a$.*

Definition 3.4. *We say the nonnegative sequence X_k converges to* 0 almost surely *if*

$$\Pr\left[\liminf_{k\to\infty} X_k < \epsilon\right] = 1, \forall \epsilon > 0.$$

Note that this definition is equivalent to the almost sure convergence defined in Definition 3.1 in Section 3.2. We are now ready to provide the proof of Lemma 3.4.

Proof of Lemma 3.4. We complete the proof in two steps. First, we show

$$\sum_{k=1}^{\infty} \Pr\left[X_k \geq \epsilon\right] < \infty, \forall \epsilon \qquad \Rightarrow \qquad X_k \xrightarrow{a.s.} 0.$$

Second, we prove

$$\sum_{k=1}^{\infty} \mathbb{E}[X_k] < \infty \qquad \Rightarrow \qquad \sum_{k=1}^{\infty} \Pr\left[X_k \geq \epsilon\right] < \infty, \forall \epsilon.$$

Let us start with the first step. From the Borel-Cantelli lemma, one knows that if $\sum_{k=1}^{\infty} \Pr\left[X_k \geq \epsilon\right] < \infty$ for all $\epsilon > 0$, then

$$\Pr\left[\limsup_{k\to\infty}(X_k \geq \epsilon)\right] = 0, \forall \epsilon.$$

Let A^c denote the complementary of the event A. Using the property that

$$\left(\limsup_{k\to\infty}(X_k \geq \epsilon)\right)^c = \liminf_{k\to\infty}(X_k < \epsilon),$$

we have

$$\Pr\left[\liminf_{k\to\infty}(X_k < \epsilon)\right] = 1 - \Pr\left[\limsup_{k\to\infty}(X_k \geq \epsilon)\right] = 1, \forall \epsilon > 0.$$

Then one can say that $X_k \xrightarrow{a.s.} 0$.

We finally use the Markov's inequality to show the second step. Using the lemma, we know that $\mathbb{E}X_k \geq \epsilon \Pr[X_k \geq \epsilon]$ for any $\epsilon > 0$. Then there holds that

$$\epsilon \sum_{n=1}^{\infty} \Pr[X_k \geq \epsilon] \leq \sum_{k=1}^{\infty} \mathbb{E}[X_k] < \infty,$$

which implies that $\sum_{n=1}^{\infty} \Pr[X_k \geq \epsilon] < \infty$ for any $\epsilon > 0$. The proof is complete.

□