Distributed coordination and partial synchronization in complex networks

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2 Preliminaries

In this chapter, we introduce some theories and concepts that will be used in the remainder of this thesis.

2.1 Probability Theory

Probability Space and Random Variables

The *sample space* Ω of an experiment is the set of all possible outcomes. A collection $\mathcal{F}$ of subsets of Ω is called a σ-field if it satisfies: 1) $\emptyset \in \mathcal{F}$; 2) if $A_1, A_2, \cdots \in \mathcal{F}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$; and 3) $A \in \mathcal{F}$, then its complement $A^c \in \mathcal{F}$. A probability space is defined by a triple $(\Omega, \mathcal{F}, \Pr)$, where $\Pr : \mathcal{F} \to [0, 1]$ is a function (called a *probability measure*) that assigns probabilities to events [109].

A random variable X is a measurable function from a sample space to the set of real numbers $\mathbb{R}$, i.e., $X : \Omega \to \mathbb{R}$. We are only concerned with discrete random variables in this thesis. Thus, the subsequent concepts are all associated with discrete random variables. A vector-valued random variable Y is defined by $Y : \Omega \to \mathbb{R}^n$.

Conditional Probability and Conditional Expectation

In probability, a conditional probability measures the probability of an event A occurring given that another event B has occurred. It is usually denoted by $\Pr[A|B]$, and can be calculated by

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]},$$

assuming that $P(B) > 0$.

A conditional expectation of a random variable X is its expected value given an event has already occurred. It can be calculated in the following way

$$\mathbb{E}[X|B] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega|B].$$

Stochastic Processes

A stochastic process is an infinite collection of (vector-valued) random variables, indexed by an integer often interpreted as time, usually denoted by $\{X(k) : k \in \mathbb{N}_0\}$.

Joint Probability Distribution

Given n random variables $X_1, X_2, \ldots, X_n$, the joint probability distribution of them is

$$p_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \Pr[X_1 = x_1, \ldots, X_n = x_n].$$

2.2 Graph Theory

Graphs are used to describe network topologies. An n-node graph is defined by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of nodes, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges. A *directed graph* is a graph where all the edges are directed from one node to another. We use (i, j) to denote a directed edge from i to j; i is said to be the *source*, and j is said to be the *target*. Given $\mathcal{E}_p \in \mathcal{E}$, we let $s(\mathcal{E}_p)$ denote the source of $\mathcal{E}_p$, and $t(\mathcal{E}_p)$ the target of $\mathcal{E}_p$. A directed *path* is a sequence of edges of the form $(p_1, p_2), (p_2, p_3), \ldots, (p_{m-1}, p_m)$, where p_i are distinct nodes in $\mathcal{V}$, and $(p_j, p_{j+1}) \in \mathcal{E}$.

On the other hand, a graph, in which all the edges are undirected, is called an *undirected graph*. An undirected path is defined in the same way as the directed one, but the edges are undirected.

Directed Graph

A directed graph is said to be *strongly connected* if there is a path from every node to every other node [110]. A directed graph is said to be a *directed spanning tree* if there is exactly one node, called root, such that any other node can be reached from it via exactly one directed path. A directed graph is said to be *rooted* if it contains a directed spanning tree that contains all the nodes.

Given two directed graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ with the same node set $\mathcal{V}$, the *composition* of them, denoted by $\mathcal{G}_2 \circ \mathcal{G}_1$, is a directed graph with the node set $\mathcal{V}$ and edge set defined in such a way that (i, j) is an edge of the composition if there is a node i_1 such

that (i, i_1) is an edge in $\mathcal{G}_1$ and meanwhile (i_1, j) is an edge in $\mathcal{G}_2$. Given a sequence of graphs $\{\mathcal{G}(1), \mathcal{G}(2), \ldots, \mathcal{G}(k)\}$, a *route* over it is a sequence of vertices $i_0, i_1, \ldots, i_k$ such that (i_{j-1}, i_j) is an edge in $\mathcal{G}(j)$ for all $1 \leq j \leq k$.

Undirected Graph

An undirected graph is said to be *connected* if there is an undirected path between any pair of nodes. A *complete graph* is a graph in which each node is directly connected to all the other nodes.

Laplacian Matrices and Incidence Matrices

Let $w_{ij} > 0$, $i, j \in \mathcal{V}$, be the weight of the direct edge from i to j in the directed graph $\mathcal{G}$ (if there is no edge between them, $w_{ij} = 0$). The weighted adjacency matrix is defined by $W = [w_{ij}]_{n\times n}$. The degree matrix of this graph is given by $D = \text{ diag}(W\mathbf{1}_n)$.The Laplacian matrix of this direct graph is then defined by

$$L = D - W = \text{ diag}(W\mathbf{1}_n) - W.$$

If $\mathcal{G}$ is an undirected graph, the Laplacian matrix L is symmetric, i.e., $L^\top = L$. For an undirected graph, the second smallest eigenvalue of L, denoted by $\lambda_2(L)$, is referred to as the *algebraic connectivity* [110].

For a directed graph with edge set $\mathcal{E} = \{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$, its incidence matrix is an $n \times m$ matrix, denoted by $B = [b_{ij}]_{n\times m}$, whose elements are defined by

$$b_{ip} = \begin{cases} 1, & \text{if } s(\mathcal{E}_p) = i; \\ -1, & \text{if } t(\mathcal{E}_p) = i; \\ 0, & \text{otherwise.} \end{cases}$$

For an undirected graph, its incidence matrix and Laplacian matrix satisfy the equality $L = B\mathcal{W}B^\top$, where $\mathcal{W} \in \mathbb{R}^{m\times m}$ is a diagonal matrix whose elements represent the weights of the edges. We let B_c denote the incidence matrix of a complete graph.

2.3 Stochastic Matrices

A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is said to be (row) *stochastic* if $a_{ij} \geq 0$ for any i, j, and it satisfies

$$\sum_{j=1}^{n} a_{ij} = 1, \qquad \forall i = 1, 2, \ldots, n.$$

A stochatic matrix A is said to be *irreducible* if for any pair (i, j), there exists an $m \in \mathbb{N}$ such that $A_{ij}^m > 0$. On the other hand, it is said to be *reducible* if it is not irreducible [71]. A stochastic matrix A is *indecomposable* and *aperiodic* (SIA) if

$$Q = \lim_{k \to \infty} A^k$$

exists and all the rows of Q are identical [68].

A stochastic matrix $A \in \mathbb{R}^{n \times n}$ is said to be: 1) *scrambling* if no two rows are orthogonal; 2) *Markov* if it has a column with all positive elements [71]. If two stochastic matrices A_1 and A_2 have zero elements in the same positions, we say these two matrices are of the same type, denoted by $A_1 \sim A_2$.

Given a stochastic matrix $A \in \mathbb{R}^{n \times n}$, we can associate it with a directed, and weighted graph $\mathcal{G}_A = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} := \{1, \ldots, n\}$ is the set of vertices, and $\mathcal{E}$ is the set of edges. A directed edge $\mathcal{E}_{ij} = (i, j)$ is in the set of $\mathcal{E}$ if $a_{ji} > 0$, and then its weight is a_{ji}.

Part I

Stochastic Distributed Coordination Algorithms:

Stochastic Lyapunov Methods

Overview of Part I

The past few decades have witnessed the fast development of network computational algorithms, in which computational processes are carried out in coupled computational units. The distributed coordination algorithms [111] are a typical type of network algorithms. Units in a network compute individually, but communicate and coordinate locally. They repeatedly update their states (computed results) to the weighted average of their neighbors', seeking for coordination. This type of algorithms are widely applied to many research topics, including distributed optimization [25,26], distributed control of networked robots [112], distributed linear equation solving [29,30,113,114], and opinion dynamics modeling [6,32,115,116].

When applying distributed coordination algorithms, one cannot ignore the fact that the computational processes are usually under inevitable random influences, resulting from random changes of network structures [36,37,117,118], stochastic communication delays [38–40], and random asynchronous updating events [41,42]. Moreover, some randomness may also be introduced deliberately to improve the global performance in a network [44,45]. Traditional methods for stability analysis of deterministic systems cannot be directly applied due to the presence of random uncertainty in the system dynamics. Instead, the stochastic Lyapunov theory serves as a powerful tool for the analysis of such stochastic systems. Different from deterministic Lyapunov theory, one needs to evaluate the expectation of a constructed Lyapunov function. For example, if the expectation of a Lyapunov candidate decreases at every time step along the solution to a stochastic discrete-time system, the stability of this system can be shown [65,66]. However, it is sometimes quite difficult to construct a Lyapunov function using the existing stochastic Lyapunov theory, especially when the systems are influenced by non-Markovian random processes.

The purpose of this part of the thesis is to further develop Lyapunov criteria for stochastic discrete-time systems, and use them to study stochastic distributed coordination algorithms. In Chapter 3, we establish some finite-step stochastic Lyapunov criteria, which enlarge the range of choices of applicable Lyapunov functions for stochastic stability analysis. In Chapter 4 , we show how these new criteria can be applied to the analysis of some stochastic distributed coordination algorithms.