Advances in methods to support store location and design decisions
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Appendix: Model Specification and Estimation

A.1. The Spatial Error Random Effects Hierarchical Model

Assume that observations are sorted by store and, for each store, by time and then by zip code. Let $Y_i$ (which can be $\ln(NV_i)$ or $\ln(EXP_i)$), $X_i$, and $Z_i$ denote the observations, and let $\epsilon_i$ indicate the disturbance terms that are stacked for a particular store. The length of these vectors or matrices is store specific, because the number of zip codes within each store’s trade area differs, and that number might change over time. Consequently, each vector or matrix consists of $T_i \times J_i$ observations. The full model can be written as:

$$
\begin{bmatrix}
Y_1 \\
Y_2 \\
\vdots \\
Y_i \\
\vdots \\
Y_{T_i}
\end{bmatrix} =
\begin{bmatrix}
t_1 \\
\vdots \\
t_i
\end{bmatrix}
\gamma^{NV} + 
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_i
\end{bmatrix}
\alpha + 
\begin{bmatrix}
Z_1 \\
Z_2 \\
\vdots \\
Z_i
\end{bmatrix}
\beta + 
\begin{bmatrix}
\epsilon_{11} \\
\epsilon_{21} \\
\vdots \\
\epsilon_{i1} \\
\vdots \\
\epsilon_{T_i1}
\end{bmatrix},
$$

(A.1)

Because $Var(\gamma_i^{NV}) = \phi^{2}_{NV}$ and the covariance matrix $\Phi$ of the composite disturbance term $diag(t_1, \ldots, t_i) \times \gamma_i^{NV} + \epsilon$ is block diagonal, the $i^{th}$ block diagonal is given by:
where $\Omega_{J_{i}}$ is the $(J_i \times J_i)$ covariance matrix for each cross-section of zip codes that belong to the trade area of store $i$ at time $t$. This matrix takes the form

$$
\Omega_{J_{i}} = \left( I_{J_{i}} - \lambda^{NV} W_{i} \right) \left( I_{J_{i}} - \lambda^{NV} W_{i} \right)^{\top}.
$$

The inverse of $\Phi_i$ is $\Phi_i^{-1} = \frac{\Delta_{i}^{-1}}{\sigma_{NV}^{2}} - \frac{\Delta_{i}^{-1} \Delta_{i}^{-1}}{\sigma_{NV}^{2}} \frac{1}{\phi_{NV}^{2}} \phi_{NV}^{2} \phi_{NV}^{-1} \Delta_{i}^{-1} \phi_{NV}^{2} \phi_{NV}^{-1}$ (Frees 2004).

If the total number of observations is denoted $nobs = \sum_{i=1}^{T} \sum_{t=1}^{T_{i}} J_{i}$, and $\phi^{2} = \left( \phi_{NV}^{2} \right)$, the log-likelihood function can be written as:

$$
LogL = -\frac{nobs}{2} \log \left( 2\pi \sigma_{NV}^{2} \right) - \frac{1}{2} \sum_{i=1}^{T} \log \left| \phi^{2} I_{i} + \Delta_{i} \right|
- \frac{1}{2 \sigma_{NV}^{2}} \sum_{i=1}^{T} \sum_{t=1}^{T_{i}} \tilde{e}_{i} \left( \phi^{2} I_{i} + \Delta_{i} \right)^{-1} \tilde{e}_{i},
$$

where $\tilde{e}_{i} = Y_{i} - \left[ I_{i} X_{i} Z_{i} \right] \left[ \gamma_{NV} \alpha_{h} \beta \right]$, $\gamma_{NV}$ is a scalar, $\alpha_{h}$ is a $(K \times 1)$ vector of $\alpha_{h}^{NV}$ s, and $\beta$ is a $(N \times 1)$ vector of $\beta_{n}^{NV}$ s.

According to Elhorst and Zeilstra (2007), the maximum likelihood estimators of the response parameters $\gamma$, $\alpha$, and $\beta$ (provided that $X$ and $Z$ do not include a lagged dependent or any endogenous explanatory variables) are equal to the generalized least squares (GLS) estimator:
\[
\begin{bmatrix}
\gamma \\
\alpha \\
\beta_{GLS}
\end{bmatrix} = \left[ \sum_{i=1}^{I} \begin{bmatrix} t_i & X_i & Z_i \end{bmatrix} \Phi_i^{-1} \begin{bmatrix} t_i & X_i & Z_i \end{bmatrix} \right]^{-1} \\
\times \left[ \sum_{i=1}^{I} \begin{bmatrix} t_i & X_i & Z_i \end{bmatrix} \Phi_i^{-1} Y_i \right]
\]

\[
= \left[ \sum_{t=1}^{T} \sum_{i=1}^{I} S_{it}^* S_{it}^* - \sum_{t=1}^{T} \sum_{i=1}^{I} S_{it}^* t_{it}^* \left[ \frac{1}{\phi^2} + \sum_{i=1}^{I} \sum_{t=1}^{T} t_{it}^* t_{it}^* \right]^{-1} t_{it}^* S_{it}^* \right]^{-1} \\
\times \left[ \sum_{t=1}^{T} \sum_{i=1}^{I} S_{it}^* Y_{it}^* - \sum_{t=1}^{T} \sum_{i=1}^{I} S_{it}^* t_{it}^* \left[ \frac{1}{\phi^2} + \sum_{i=1}^{I} \sum_{t=1}^{T} t_{it}^* t_{it}^* \right]^{-1} t_{it}^* Y_{it}^* \right],
\]

(A.4)

where \( S_{it}^* = \begin{bmatrix} t_{it}^* & X_{it}^* & Z_{it}^* \end{bmatrix} \), and the superscript * denotes the transformation \( S_{it}^* = \begin{bmatrix} t_{it}^* & X_{it}^* & Z_{it}^* \end{bmatrix} \), applied to the variables \( t_{it}, X_{it}, \) and \( Z_{it} \). In addition,

\[
\sigma_{NV}^2 = \gamma_{nobs} \sum_{i=1}^{I} \sum_{t=1}^{T} \left( \phi^2 t_{it}^* + \Delta_i \right)^{-1} \tilde{e}_{it}.
\]

In contrast, there is no closed-form solution for \( \lambda_{NV} \) and \( \phi^2 \). Therefore, we develop an iterative, two-step estimation procedure, in which the two sets of parameters are estimated alternately until convergence; \( \gamma_{NV}, \alpha, \beta, \) and \( \sigma_{NV}^2 \), given \( \lambda_{NV} \) and \( \phi^2 \), can be estimated using the GLS estimator in Equation A.4, whereas \( \lambda_{NV} \) and \( \phi^2 \), given \( \gamma_{NV}, \alpha, \beta, \) and \( \sigma_{NV}^2 \), can be estimated by maximizing the log-likelihood in Equation A.3. A Matlab routine of this estimation procedure can be downloaded from the Web site http://www.regroningen.nl/elhorst/.

A.2. The Spatial Error Random Effects Model

Assume that the observations are sorted by time and then, for each time period, by zip code. Let \( Y_t \) \( (= \logit(\text{PR}_t)) \) and \( U_t = \begin{bmatrix} t_t & X_t & Z_t \end{bmatrix} \) denote observations stacked
within a particular time period ($Y_t$ and $U_t$ consist of $J$ observations). Further assume that $\theta = \begin{bmatrix} \gamma^{PR} & \alpha' & \beta' \end{bmatrix}$, where $\alpha$ is a $(K \times 1)$ vector of $\alpha_k^{PR}$'s, and $\beta$ is a $(N \times 1)$ vector of $\beta_n^{PR}$'s. The model then can be written as:

$$
\begin{bmatrix}
Y_1 \\
Y_2 \\
\vdots \\
Y_T
\end{bmatrix} = 
\begin{bmatrix}
U_1 \\
U_2 \\
\vdots \\
U_T
\end{bmatrix} + \theta + \delta, \quad \delta = (t_T \otimes I_J)\nu + (I_T \otimes B^{-1})\epsilon,
$$

(A.5)

where $t_T$ is a $(T \times 1)$ vector of unit elements, $\nu = (v_1^{PR}, \ldots, v_J^{PR})$, $\epsilon$ is a $(TJ \times 1)$ vector of disturbance terms $\epsilon_j^{PR}$, and $B = (I_J - \lambda^{PR} W)$.

Anselin (1988) and Baltagi (2005) show that the log-likelihood function of this model is:

$$
\text{LogL} = -\frac{JT}{2} \log \left( 2\pi \sigma_{PR}^2 \right) - \frac{1}{2} \log \left| T \kappa^2 I_J + (B' B)^{-1} \right| \\
+ (T - 1) \log |B| - \frac{1}{2 \kappa^2} e' \left( \frac{1}{T} t_T' t_T \otimes \left( T \kappa^2 I_J + (B' B)^{-1} \right) \right)^{-1} e \\
- \frac{1}{2 \sigma_{PR}^2} e' \left( I_T - \frac{1}{T} t_T' t_T \right) \otimes (B' B) e,
$$

(A.6)

where $e = (e_1, \ldots, e_T)'$, $e_t = Y_t - U_t \theta$, and $\kappa^2 = (\sigma_v^2 / \sigma_{PR}^2)$.

Elhorst (2003) also shows that if the determinants of the matrices $T \kappa^2 I_J + (B' B)^{-1}$ and $B$ are expressed as a function of the characteristic roots of $W$, denoted by $\omega_j$ ($j = 1, \ldots, J$), the log-likelihood function can be rewritten as:

$$
\text{LogL} = -\frac{JT}{2} \log \left( 2\pi \sigma_{PR}^2 \right) - \frac{1}{2} \sum_{j=1}^{J} \log \left( 1 + T \kappa^2 \left( 1 - \lambda^{PR} \omega_j \right)^2 \right) \\
+ T \sum_{j=1}^{J} \log \left( 1 - \lambda^{PR} \omega_j \right) - \frac{1}{2 \sigma_{PR}^2} \sum_{t=1}^{T} \tilde{e}_t' \tilde{e}_t,
$$

(A.7)
where \( \tilde{e}_i = Y_i^* - U_i^* \theta, \)

\[
Y_i^* = P \bar{Y} + B \left( Y_i - \bar{Y} \right) = BY_i + (P - B) \bar{Y} = \left( I_J - \lambda^{PR} W \right) Y_i \\
+ \left( P - \left( I_J - \lambda^{PR} W \right) \right) \bar{Y}, \\
U_i^* = \left( I_J - \lambda^{PR} W \right) U_i + \left( P - \left( I_J - \lambda^{PR} W \right) \right) \bar{U}, \text{ and}
\]

\( P \) is such that \( P'P = \left( T \kappa^2 I_J + (B' B)^{-1} \right)^{-1}. \)

Consequently, the GLS estimator of the response parameters \( \theta \) and \( \sigma_{PR}^2 \) can be computed as:

\[
\theta = \left( u^* u^* \right)^{-1} \left( u^* y^* \right) \quad \text{and} \quad \sigma_{PR}^2 = \frac{\sum_{i=1}^{T} e_i^* \tilde{e}_i}{JT}, \tag{A.8}
\]

where \( u^* = \begin{bmatrix} U_1^* \\ \vdots \\ U_T^* \end{bmatrix} \), and \( y^* = \begin{bmatrix} Y_1^* \\ \vdots \\ Y_T^* \end{bmatrix} \).

Just as in the previous case, there is no closed-form solution for \( \lambda^{PR} \) and \( \kappa^2 \), so we need an iterative two-step estimation procedure in which the two sets of parameters get estimated alternately until convergence occurs. A Matlab routine of this estimation procedure can be downloaded from the Web site http://www.regroningen.nl/elhorst/.