Chapter 3

Tests of goodness-of-fit†

A popular approach to model network panel data is to embed the discrete observations of the network in a latent, continuous-time Markov process. A score-type test statistic for goodness-of-fit tests is proposed, which is useful for studying the goodness-of-fit of a wide range of models. The finite-sample behavior of the test statistic is evaluated by a Monte Carlo simulation study, and its usefulness is demonstrated by an application to empirical data.

Keywords: continuous-time Markov process, regular estimating functions, goodness-of-fit, Lagrange multiplier / Rao score test, Neyman C(α) test.

3.1 Introduction

Social network analysis (Wasserman and Faust, 1994) is concerned with links among entities. The network data considered here correspond to the directed ties among the members of a set of actors. It is common that the ties are binary, but ties may take on arbitrary values.

When modeling networks, the ties between the actors are treated as random variables. The tie variables are, however, not independent. Some well-known examples of dependencies are reciprocity (second-order dependence) and transitivity (Holland and Leinhardt, 1970, 1976), which represents third-order dependence among ties and implies clustering (“group structure”) in social networks.

Since the tie variables are dependent, statistical inference proved to be hard. (Curved) exponential random graph models (ERGMs) (Snijders, Pattison, Robins, and Handcock, 2006, Hunter and Handcock, 2006) have been used to model networks

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observed at one time point, but—in spite of recent advances in the specification and estimation of ERGMs—some theoretical and practical issues remain.

Here, the focus is on longitudinal network data. Longitudinal network data come frequently in the form of panel data. There is a large agreement in the literature (see Frank, 1991) that the most promising models for network panel data are continuous-time Markov models which assume that the network was observed at discrete time points, and that between these time points latent, continuous-time Markov processes shape the network. Holland and Leinhardt (1977) and Wasserman (1979, 1980) proposed methods for statistical inference for such Markov models, but these methods are limited to models with second-order dependence, and thus neglect fundamental third- and higher-order dependencies.

Snijders (2001) considered a family of continuous-time Markov models which allows to model third- and higher-order dependencies, and proposed the method of moments to estimate the parameter $\theta$. The probabilistic framework may be described as actor-driven, that is, the nodes are assumed to represent actors who make choices concerning the ties to other actors by either adding or deleting ties, and the choices are assumed to be based on mathematical functions, containing choice “determinants” (tendencies) such as reciprocity, transitivity, and covariate-related effects.

In the tradition of K. Popper’s conception of science, deeply rooted in contemporary social science and statistics (Healy, 1978), a natural question to ask is “whether the observed data support a given specification” (Rao, 2002, p. 9). The study of goodness-of-fit, dating back to Pearson (1900), is considered so natural that model-based inference without goodness-of-fit evaluation is almost inconceivable. However, as is argued below in some more detail, no goodness-of-fit measure has been proposed up to now which is applicable in a wide range of applications.

The present paper proposes a new goodness-of-fit test statistic that (1) has many applications; (2) does not require to estimate the parameters to be tested, which is in practice a decisive advantage as it (a) saves valuable computation time, and (b) allows to test hard-to-estimate parameters (which are not uncommon); (3) admits multi-parameter tests; and (4) has an appealing interpretation in terms of goodness-of-fit, in the sense that it compares the expected value of some function of the data—evaluated under some assumed model—to the observed value of the function; that is, it uses the observed data as an external benchmark to which model predictions are compared.

The paper is structured as follows. Model specification and estimation are sketched in section 3.2. A goodness-of-fit test statistic is proposed in section 3.3. Section 3.4
reviews interesting goodness-of-fit tests. In section 3.5, the finite-sample behavior of the test statistic is studied by Monte Carlo simulation. The usefulness of the test statistic for real-world problems is demonstrated in section 3.6 by an application to the cross-ownership network among more than 400 business firms in Slovenia (EU).

3.2 Model specification and estimation

Due to space restrictions, the discussion is restricted to some basic model specifications. Extensions to model the co-evolution of networks and other outcome variables (Snijders, Steglich, and Schweinberger, 2006) are possible.

A binary, directed relation \( \rightarrow \) (or digraph) on a finite set of nodes \( N = \{1, 2, \ldots, n\} \) is considered. The digraph is observed at discrete, ordered time points \( t_1 < t_2 < \cdots < t_M \), and the observations are represented as binary matrices \( x(t_1), x(t_2), \ldots, x(t_M) \), where element \( x_{ij}(t_m) \) of \( n \times n \) matrix \( x(t_m) \) is defined by

\[
x_{ij}(t_m) = \begin{cases} 
1 & \text{if } i \rightarrow j \text{ at time point } t_m, \\
0 & \text{otherwise}, 
\end{cases}
\]  

(3.1)

where \( i \rightarrow j \) means that node \( i \) is related to node \( j \); the diagonal elements \( x_{ii}(t_m) \) are regarded as structural zeros.

Although the model described below assumes that \( x_{ij}(t_m) \) is binary, it is possible to extend the model to the case where \( x_{ij}(t_m) \) takes on discrete, ordered values.

3.2.1 Model specification

The observed digraph \( x(t_1) \) is taken for granted, that is, the model conditions on \( x(t_1) \). It is postulated that the observed digraphs \( x(t_2), \ldots, x(t_M) \) are generated by an unobserved Markov process operating in time interval \( [t_1, t_M] \). Consider the case \( M = 2 \); the extension to the case \( M > 2 \) is straightforward due to the Markov property.

The model is specified by the generator of the Markov process, which corresponds to a \( W \times W \) matrix \( Q \) indexed by a parameter \( \theta \), where \( W = 2^{n(n-1)} \) is the number of digraphs on \( N \). The elements \( q_{\theta}(x^*, x) \) of generator \( Q \) are the rates of moving from digraph \( x^* \) to digraph \( x \). If \( x \) deviates from \( x^* \) in more than one arc variable \( X^*_{ij} \), then \( q_{\theta}(x^*, x) = 0 \) by assumption (see Snijders, 2001); in other words, it is assumed that the process moves forward by changing not more than one arc variable \( X^*_{ij} \) at the time. Let \( x^* \) be an arbitrary digraph on \( N \), and let \( x \) be the digraph that is
obtained from $x^*$ by changing one and only one specified arc variable, say $X_{ij}^*$. Since the transition from $x^*$ to $x$ involves only the ordered pair of nodes $(i,j)$, one can rewrite $q_\theta(x^*, x)$ as $q_\theta(x^*, i, j)$ and decompose $q_\theta(x^*, i, j)$ as follows:

$$q_\theta(x^*, i, j) = \lambda_i(\theta, x^*) r_i(\theta, x, j), \quad (3.2)$$

where

$$\lambda_i(\theta, x^*) = \sum_{h \neq i} q_\theta(x^*, i, h) \quad (3.3)$$

is called the rate function of node (“actor”) $i$, while

$$r_i(\theta, x, j) = \frac{q_\theta(x^*, i, j)}{\lambda_i(\theta, x^*)} \quad (3.4)$$

is the conditional probability that $i$ changes $X_{ij}^*$, given that $i$ changes some arc variable $X_{ih}^*$, $h \neq i$.

A simple specification of $\lambda_i$ is

$$\lambda_i(\theta, x^*) = \rho, \quad (3.5)$$

where $\rho$ is a parameter, while non-constant rate functions are given by

$$\lambda_i(\theta, x^*) = \rho \exp[\alpha' a_i(x^*, c_i)], \quad (3.6)$$

where $\alpha = (\alpha_k)$ is a vector valued parameter and $a_i = (a_{ik})$ is a vector valued function of node-bound covariates $c_i$ and graph-dependent statistics involving the arcs of node $i$. If there is more than one time interval ($M > 2$), then parameter $\rho$ can be made dependent on time interval $[t_{m-1}, t_m]$.

A convenient, multinomial logit parametrization of $r_i$ is given by

$$r_i(\theta, x, j) = \frac{\exp[f_i(\beta, x, j)]}{\sum_{h \neq i} \exp[f_i(\beta, x, h)]}, \quad (3.7)$$

where the real valued function

$$f_i(\beta, x, j) = \beta's_i(x, j) \quad (3.8)$$

is called the objective function, while $\beta = (\beta_k)$ is a vector valued parameter and $s_i = (s_{ik})$ is a vector valued statistic. Examples of statistics $s_{ik}$ are the number of arcs $\sum_{h=1}^n x_{ih}$, the number of transitive triplets $\sum_{h, t=1}^n x_{ih} x_{ht} x_{tl}$, and interactions of covariates with these and other statistics; see section 3.4. Such statistics can be used to define third- and higher-order dependencies.
3.2.2 Model estimation

Model estimation is concerned with estimating parameter $\theta$—corresponding to rate parameters $\rho_1, \ldots, \rho_{M-1}, \alpha$ and objective function parameter $\beta$—from the observed data $z$, consisting of the observed digraphs $x(t_1), x(t_2), \ldots, x(t_M)$ and covariates.

Since the Markov process is not observed in continuous time, the likelihood function is intractable. The parameter $\theta$ is therefore estimated by the method of moments (Pearson, 1902a,b). The moment estimate $\hat{\theta}$ is defined here as the solution of the moment equation

$$
g_n(z, \theta) = \sum_{m=1}^{M-1} \left[ E_\theta [s(X(t_{m+1})) \mid X(t_m) = x(t_m)] - s(x(t_{m+1})) \right] = 0, \tag{3.9}
$$

where $E_\theta$ denotes the expectation under $\theta$. The coordinates of the vector valued function $s = (s_k)$ in (3.9) correspond to the coordinates of parameter $\theta = (\theta_k)$. In the absence of formal methods to derive statistics with certain optimum properties (neglecting some close-to-trivial models), statistics $s_k$ are chosen heuristically so as to be sensitive to changes of the value of $\theta_k$. To estimate objective function parameter $\beta_k$, the statistic $\sum_{i,j=1}^{n} s_{ik}(x,j)$ is a natural choice; with regard to the rate parameters, see Snijders (2001).

In terms of numerical implementation, finding the moment estimate involves finding the root(s) of $g_n = g_n(z, \theta)$ as a function of $\theta$. A suitable root-finding algorithm, based on iterative, stochastic approximation methods (Robbins and Monro, 1951) and Monte Carlo simulation, is described in Snijders (2001). The variance-covariance matrix of the moment estimator $\hat{\theta}$ can be derived by the delta method (see Snijders, 2001).

3.3 Test statistic

Two papers address significance testing and / or goodness-of-fit in the considered family of models, though both of them have limitations, as is argued in section 3.3.1. A new test statistic is proposed in section 3.3.2, which has important advantages compared to the existing approaches.
3.3.1 Literature review

Snijders (1996) proposed to base significance tests on the pseudo-$t$-test statistic

$$
\frac{\hat{\theta}_k}{\text{s.e.}(\hat{\theta}_k)},
$$

where $\hat{\theta}_k$ is the moment estimator of coordinate $\theta_k$ of $\theta$ and s.e.$(\hat{\theta}_k)$ is its standard error. The distribution of the test statistic is unknown. Snijders (1996) assumed that its sampling distribution under the null hypothesis $H_0: \theta_k = 0$ is approximately the standard Gaussian distribution.

Snijders (2003) stresses the modeling of (out)degree distributions. A goodness-of-fit plot with confidence intervals is provided as a means to evaluate the fit of the model with respect to the observed (out)degree distribution. However, the goodness-of-fit study is limited to the (out)degree distribution, and while the (out)degree distribution represents a fundamental feature of the data which should be taken into account, the (out)degree distribution in itself is in most applications a nuisance. In the social sciences, the focus is on second- and third-order dependencies among arcs and covariate-related parameters. The goodness-of-fit of corresponding models cannot, however, be addressed within the Snijders (2003) framework.

3.3.2 Goodness-of-fit test statistic

As to the choice of suitable test statistics for goodness-of-fit tests, observe that the choice is constrained, in the first place, by the intractable likelihood function, the absence of saturated models, and the computational burden imposed by estimating non-trivial models. Thus the “holy trinity”, corresponding to the Wald, likelihood ratio (LR), and Lagrange multiplier / Rao score (RS) test (Rao and Poti, 1946, Rao, 1948), is not readily available.

Although the “holy trinity” is not available, it can be instructive to consider some of its features. An important practical concern is computation time. As the LR test requires the estimation of both the restricted and the unrestricted model, it is inferior to the Wald and RS test in terms of computation time. The RS test is most appealing, since only the restricted model must be estimated, while the Wald test requires the more computation-intensive estimation of the unrestricted model. Besides, it is not uncommon to encounter convergence problems in high-dimensional parameter spaces. In fact, it may be argued that, in the considered family of models (given the absence of saturated models), forward model selection, combined with the
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RS test, is preferable to backward model selection, because one can start with simple models and is not required to estimate additional parameters in the first place.

In the method of moments framework, the RS test is not available, but a RS-type test can be obtained by generalizing the $C(\alpha)$ test (Neyman, 1959) and the RS test by replacing the Fisher score function by regular estimating functions along the lines of Basawa (1985, 1991). The classical $C(\alpha)$ test is designed to test parametric hypotheses in the presence of nuisance parameters, where the nuisance parameters are replaced by consistent estimates under the null hypothesis. If maximum likelihood estimates (under the null hypothesis) are used to estimate the nuisance parameter, the classical $C(\alpha)$ test reduces to the RS test. The RS test encompasses the classical goodness-of-fit test of Pearson (1900) as a special case (see Rao, 2002). Interrelations among the “holy trinity” test statistics and the $C(\alpha)$ test statistic were studied by, among others, Godfrey (1988, pp. 27—28), Hall and Mathiason (1990), Basawa (1991), and Bera and Bilias (2001a,b).

Partition $\theta' = (\theta'_1, \theta'_2)$, where $\theta_1$ is the (vector valued) nuisance parameter and $\theta_2$ the (vector valued) parameter of primary interest. In the classical Neyman and Pearson tradition, goodness-of-fit can be studied by specifying hypotheses regarding the postulated family of probability distributions $\{P_\theta, \theta \in \Theta\}$, for instance, the null hypothesis

$$H_0 : \theta_2 = \theta_{20},$$

(3.11)

tested against

$$H_1 : \theta_2 \neq \theta_{20},$$

(3.12)

where $\theta_{20}$ is some specified value (commonly, $\theta_{20} = 0$), and $\theta_1$ is unspecified; that is, both $H_0$ and $H_1$ are composite.

A test statistic for testing such composite hypotheses is proposed below, based on the estimating function $g_n$ along the lines of Basawa (1985, 1991). The estimating function and some additional notation is introduced first.

**The estimating function**

For convenience, it is assumed that $M = 2$; the extension to the case $M > 2$ is immediate.

The parameter $\theta$ is estimated by the solution of the estimating equation $g_n = g_n(z, \theta) = 0$, where $g_n$ is the estimating function satisfying some mild smoothness conditions (see Godambe, 1960). Examples of estimating functions include the Fisher
score function, moment functions, or more generally, some function involving parameter \( \theta \) and the data. Here, the estimating function \( g_n \) is defined in (3.9).

Observe that \( g_n \) as defined in (3.9) is an unbiased estimating function in the sense that

\[
E_\theta[g_n(Z, \theta) \mid X(t_1) = x(t_1)] = 0
\]  

(3.13)

holds for all \( n \) and all \( \theta \).

Let \( \Sigma_n \) be the \( L \times L \) variance-covariance matrix of \( g_n \), and denote the limit of \( w_n \Sigma_n \) as \( n \rightarrow \infty \) by \( \Sigma \), where \( w_n \) are appropriate norming constants.

The function \( s \) defined in section 3.2.2 does not depend on \( \theta \), so that the definition of \( g_n \) implies that the derivative of \( g_n \) with respect to \( \theta \) is deterministic and given by

\[
\Delta_n(\theta) = \frac{\partial g_n(z, \theta)}{\partial \theta'} = \frac{\partial}{\partial \theta'} E_\theta[s(X(t_2)) \mid X(t_1) = x(t_1)].
\]  

(3.14)

Denote the limit of \( w_n \Delta_n \) as \( n \rightarrow \infty \) by \( \Delta \), where \( g_n \) is of order \( L \times 1 \) and \( \theta' \) is of order \( 1 \times L \), so that \( \Delta \) is of order \( L \times L \).

Last, partition \( g_n \), \( \Sigma \), and \( \Delta \) in accordance with \( \theta' = (\theta'_1, \theta'_2) \):

\[
\begin{align*}
g_n(z, \theta) &= \left( \begin{array}{c} g_{1n}(z, \theta) \\ g_{2n}(z, \theta) \end{array} \right), \\
\Sigma &= \left( \begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{array} \right), \\
\Delta(\theta) &= \left( \begin{array}{cc} \Delta_{11}(\theta) & \Delta_{12}(\theta) \\ \Delta_{21}(\theta) & \Delta_{22}(\theta) \end{array} \right).\end{align*}
\]  

(3.15)

The test statistic

As was mentioned above, the test statistic is based on estimating function \( g_n \). Since the statistics contained in \( g_n \) are commonly based on sums of weakly dependent random variables, it is plausible that

\[
w_n^{1/2} g_n(Z, \theta) \xrightarrow{d} N_L(0, \Sigma) \quad \text{as} \quad n \rightarrow \infty
\]  

(3.16)

for the norming constants \( w_n \) introduced above, where \( \xrightarrow{d} \) denotes convergence in distribution, \( N_L \) refers to the \( L \)-variate Gaussian distribution, and \( \Sigma \) is non-singular.

In the case of the Independent Arcs model (Snijders and Van Duijn, 1997), where the arc variables \( X_{ij}(t) \) follow independent Markov processes, the asymptotic normality can be proved directly from the Central Limit Theorem. However, in the general case where the arc variables \( X_{ij}(t) \) are dependent and the Markov process is non-stationary, a rigorous proof of (3.16) is beyond the scope of the present paper (see
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The approximate normality is supported by some limited simulation studies (not further reported here); in the remainder of the section, it is assumed that (3.16) holds.

Let \( \theta'_0 = (\theta'_1, \theta'_2) \) be the parameter under \( H_0 : \theta_2 = \theta_{20} \). To eliminate the impact of the estimated nuisance parameter \( \theta_1 \) on the test, Neyman’s (1959) orthogonalization method can be exploited, as suggested by Basawa (1985, 1991). Let

\[
b_n(z, \theta_0) = g_{2n}(z, \theta_0) - \Gamma(\theta_0) g_{1n}(z, \theta_0),
\]

where \( \Gamma = \Delta_{21} \Delta_{11}^{-1} \) and \( \Delta_{11} \) is non-singular. Since both \( g_{2n} \) and \( \Gamma g_{1n} \) have zero expectation by (3.13) and are asymptotically Gaussian distributed by (3.16), one obtains

\[
w_n^{1/2} b_n(Z, \theta_0) \xrightarrow{d} N_R(0, \Xi) \quad \text{as} \ n \to \infty,
\]

where the variance-covariance matrix \( \Xi \) is given by

\[
\Xi = \Sigma_{22} - (\Sigma'_{12} \Gamma(\theta_0) + \Gamma(\theta_0) \Sigma_{12}) + \Gamma(\theta_0) \Sigma_{11} \Gamma(\theta_0)',
\]

and \( R \) is the number of coordinates of \( \theta_2 \). Thus, under \( H_0 \),

\[
w_n b_n(Z, \theta_0)' \Xi^{-1} b_n(Z, \theta_0) \xrightarrow{d} \chi^2_R \quad \text{as} \ n \to \infty
\]

is asymptotically central chi-square distributed with \( R \) degrees of freedom.

The entities \( \Delta \) and \( \Sigma \) can be replaced by the constant matrices \( w_n \Delta_n \) and \( w_n \Sigma_n \), respectively; it is evident that these plug-ins do not change the asymptotic distribution of (3.20). The unknown nuisance parameter \( \theta_1 \) can be replaced by a consistent estimator under \( H_0 \) without changing the asymptotic distribution of (3.20). The parameter \( \theta_1 \) can be estimated by the \( \hat{\theta}_1 \) that solves \( g_n = g_n(z, \theta_0) = 0 \) (see section 3.2.2), where \( \theta'_0 = (\theta'_1, \theta'_{20}) \) and \( \theta_{20} \) is the value of \( \theta_2 \) under \( H_0 \). It is plausible that \( \hat{\theta}_1 \) is a consistent estimator of \( \theta_1 \), although a rigorous proof is beyond the scope of the present paper (see section 3.7). Denote by \( C_n \) the test statistic (3.20) obtained by plugging in \( \hat{\theta}_1 \), \( \Delta_n \), and \( \Sigma_n \) for \( \theta_1 \), \( \Delta \), and \( \Sigma \), respectively.

The entities \( g_n \), \( \Delta_n \), and \( \Sigma_n \) are not available in closed form, but can be estimated by Monte Carlo methods. Crude Monte Carlo estimation (Hammersley and Handscomb, 1964) of the expectations of which \( g_n \) is composed and of \( \Sigma_n \) is straightforward. Monte Carlo estimators of \( \Delta_n \) can be found in Schweinberger and Snijders (2007). These Monte Carlo estimators are simulation-consistent in the sense that the Monte Carlo estimators converge in probability to the desired quantities as the
number of Monte Carlo simulations increases without bound. Therefore, the Monte Carlo estimator of test statistic $C_n$, which is obtained by plugging in the Monte Carlo estimators of $g_n$, $\Delta_n$, and $\Sigma_n$, is simulation-consistent.

Remarks and extensions

Observe that, to test restrictions on parameter vector $\theta_2$, $\theta_2$ needs not be estimated, as is generally the case for RS tests.

If $\theta_2$ is a scalar, the test statistic (3.20) can be used both in its quadratic form, as presented above, and in its corresponding linear form

$$-w_n^{1/2} \frac{b_n(Z, \theta_0)}{\Xi^{1/2}},$$

(3.21)

where $b_n$ and $\Xi$ are scalars. The linear form is convenient when one-sided one-parameter tests are desired. The minus sign in (3.21) facilitates the interpretation in the sense that, if $s_2$ denotes the statistic corresponding to parameter $\theta_2$ and its conditional expectations are non-decreasing functions of $\theta_2$, then, by the definition of $g_n$ in (3.9), $\theta_2 - \theta_{20} > 0$ is associated with positive values of (3.21). By (3.18), the asymptotic distribution of (3.21) under $H_0 : \theta_2 = \theta_{20}$ is standard Gaussian.

Furthermore, tests with $R > 1$ degrees of freedom can be complemented with one degree of freedom tests, testing the restrictions one by one; two-sided one-parameter tests can be based on (3.20), while one-sided one-parameter tests can be based on (3.21).

Note that test statistic (3.20) has an appealing interpretation in terms of goodness-of-fit. Let $M = 2$ and observe that the test statistic is based on

$$g_{2n}(z, \theta) = E_{\theta} [s_2(X(t_2)) \mid X(t_1) = x(t_1)] - s_2(x(t_2)),$$

(3.22)

where $s_2$ is the part of the statistics vector $s$ which corresponds to $\theta_2$. In other words, the test statistic is based on the “distance” between the expected value of the function $s$ of the data—evaluated under the model restricted by $H_0$—and the observed value of $s$. The argument extends to the case $M > 2$.

When the test indicates that there is empirical evidence against $H_0 : \theta_2 = \theta_{20}$, it may be desired to estimate $\theta_2$. If $g_n$ is differentiable at $\hat{\theta}_0$, then, by definition (Magnus and Neudecker, 1988, p. 82),

$$g_n(z, \theta) = g_n(z, \hat{\theta}_0) + \Delta_n(\hat{\theta}_0) (\theta - \hat{\theta}_0) + h_{\hat{\theta}_0}(\theta - \hat{\theta}_0),$$

(3.23)

where the Jacobian matrix $\Delta_n$ is given by (3.14), and

$$\lim_{(\theta - \hat{\theta}_0) \to 0} \frac{h_{\hat{\theta}_0}(\theta - \hat{\theta}_0)}{||\theta - \hat{\theta}_0||} = 0.$$

(3.24)
Thus, solving $g_n = g_n(z, \theta) = 0$ is asymptotically the same as solving
\[ g_n(z, \hat{\theta}_0) + \Delta_n(\hat{\theta}_0) (\theta - \hat{\theta}_0) = 0, \] (3.25)
giving rise to the one-step estimator
\[ \theta^* = \hat{\theta}_0 - \Delta_n(\hat{\theta}_0)^{-1} g_n(z, \hat{\theta}_0), \] (3.26)
where $\Delta_n$ is non-singular. In general, one-step estimators can be useful as approximations of estimators which are hard to obtain (see Lehmann, 1999). Here, the one-step estimator $\theta^*$ is an approximation of the unrestricted moment estimator $\hat{\theta}$, and is useful because all ingredients required to evaluate $\theta^*$ are available once the ingredients of (3.20) are available, while the estimation of $\hat{\theta}$ requires an additional, time-consuming estimation run (see section 3.2.2). Note that $\theta^* = (\theta^*_1, \theta^*_2)$ is an estimator of both $\theta_1$ and $\theta_2$, and in practice both $\theta^*_1$ and $\theta^*_2$ are interesting, because in network models these estimators can be considerably correlated.

### 3.4 Model misspecifications

Two classes of model misspecifications are briefly considered. First, the model is misspecified in the sense that the true parameter $\theta$ contains other (additional) coordinates than the specified model. Section 3.4.1 deals with such cases. Second, the model is misspecified in the sense that homogeneity assumptions regarding $\theta$ do not hold. Such homogeneity assumptions are convenient for statistical and computational reasons. Two important cases are the assumption that

1. $\beta^{(1)} = \beta^{(2)} = \cdots = \beta^{(M-1)}$, where the superscript refers to the period, provided $M \geq 3$.
2. $\beta^{(1)} = \beta^{(2)} = \cdots = \beta^{(n)}$, where the superscript refers to the node.

Testing such homogeneity assumptions is considered in sections 3.4.2 and 3.4.3.

#### 3.4.1 Misspecifications I

The simple case is considered where the specified objective function $f_i$ by mistake excludes effects inherent to the true $f_i$.

Besides second- and third-order dependencies, covariates may have an impact on the digraph evolution. As an example, adolescents may favor friendship ties to adolescents of the same gender. Let $c$ be some
(1) node-bound covariate \( c_i \), containing information about node \( i \): (a) demographic covariate; (b) behavioral or other covariate (such as smoking).

(2) dyadic covariate \( c_{ij} \), containing information about the ordered pair of nodes \((i, j)\): a frequently used form of dyadic covariates is “similarity” of nodes \( i \) and \( j \) with respect to node-bound covariates.

The objective function can be specified as

\[
 f_i(\beta, x) = \sum_{k=1}^{K} \beta_k s_{ik}(x, j) + \beta_{K+1} s_{iK+1}(x, c, j). \tag{3.27}
\]

Some possible specifications of statistic \( s_{iK+1} \) are

1. node-bound covariate: \( \sum_{h=1}^{n} x_{ih} c_h \) or \( c_i \sum_{h=1}^{n} x_{ih} \).
2. dyadic covariate: \( \sum_{h=1}^{n} x_{ih} c_{ih} \).
3. interaction node-bound covariate \( \times \) reciprocated arcs: \( \sum_{h=1}^{n} x_{ih} x_{hi} c_h \).

Numerous other interactions between covariates and digraph statistics are conceivable. The covariate \( c \) may, or may not, depend on time. If \( c \) depends on time, it is—in the presence of panel data—convenient to assume that it is constant in time interval \([t_m, t_{m+1}]\), so that the value of \( c(t_m) \) observed at time point \( t_m \) can be used to evaluate the corresponding statistic in time interval \([t_m, t_{m+1}]\).

A natural approach is to test the model restricted by

\[
 H_0 : \beta_{K+1} = 0 \tag{3.28}
\]

against the unrestricted model

\[
 H_1 : \beta_{K+1} \neq 0. \tag{3.29}
\]

### 3.4.2 Misspecifications II: homogeneous periods

An important assumption underlying the Snijders (2001) family of models is that \( \beta \) is constant across time intervals \([t_m, t_{m+1}]\), \( m = 1, \ldots, M - 1 \). To test whether the data-generating mechanism is indeed constant across time intervals, the model can be specified as

\[
 f_i(\beta, x) = \sum_{k=1}^{K} \beta_k^{(m)} s_{ik}(x, j), \tag{3.30}
\]
where parameter $\beta_k^{(m)}$ depends on time interval $[t_m, t_{m+1}]$. Observe that an equivalent formulation is obtained by including, for the $M - 1$ periods, $M - 2$ period-dependent dummy variables interacting with statistic $s_{ik}$ (assuming that the main effect of $s_{ik}$ is included), where the dummies are equivalent to node- and time-dependent covariates $c^{(m)}_i$.

To test whether coordinate $\beta_k$ of $\beta$ is constant across time intervals, the model restricted by

$$H_0 : \beta_k^{(1)} = \beta_k^{(2)} = \cdots = \beta_k^{(M-1)}$$

(3.31)

can be tested against the unrestricted model

$$H_1 : \beta_k^{(a)} \neq \beta_k^{(b)} \text{ for some } a \neq b,$$

(3.32)

where $a$ and $b$ indicate periods.

### 3.4.3 Misspecifications II: homogeneous nodes

Another assumption of the Snijders (2001) family of models is that $\beta$ is constant across nodes $i$. Models with node-dependent parameters can be specified, which is equivalent to using $n - 1$ node-dependent dummy variables $c_i$ interacting with statistic $s_{ik}$ (assuming that the main effect of $s_{ik}$ is included). Tests can be carried out as in section 3.4.2.

### 3.5 Monte Carlo study

The present section studies the finite-sample behavior of the proposed test statistic (3.20) and the pseudo-$t$-test (3.10). For convenience, the two tests will be referred to as “score test” and “$t$-test”, respectively, although the tests are not equivalent to the tests which are known in the statistical literature under these names.

Data are generated by simulating the Markov process with $n = 30$ (“small data set”) and $n = 60$ nodes (“moderate data set”) in time interval $[0, 2]$, starting with a real-world digraph at time point $t_1 = 0$ and “observing” the random digraph at time points $t_2 = 1$ and $t_3 = 2$. The rate function $\lambda_i$ is constant and equal to $\rho_m$ for period $m = 1, 2$, and the objective function is given by

$$f_i(\beta, x) = \sum_{k=1}^{K} \beta_k s_{ik}(x, j).$$

(3.33)
The choice of statistics $s_{ik}$ is based on their importance in empirical social science research; the chosen statistics are:

$$s_{i1}(x,j) = \sum_{h=1}^{n} x_{ih}: \text{the number of arcs},$$

$$s_{i2}(x,j) = \sum_{h=1}^{n} x_{ih}x_{hi}: \text{the number of reciprocated arcs},$$

$$s_{i3}(x,j) = \sum_{h,l=1}^{n} x_{ih}x_{hl}x_{il}: \text{the number of transitive triplets},$$

$$s_{i4}(x,j) = \sum_{h=1}^{n} (1 - x_{ih}) \max_{l} x_{il}x_{lh}: \text{the number of indirect connections},$$

$$s_{i5}(x,j) = \sum_{h=1}^{n} x_{ih}c_{ih}: \text{interaction of arcs and dyadic covariate } c_{ih},$$

$$s_{i6}(x,j) = \sum_{h=1}^{n} x_{ih}d_{h}: \text{interaction of arcs and node-bound covariate } d_{h}.$$

The values of the dyadic covariate $c_{ih}$ are generated by independent draws from the Poisson(1) distribution; the values of the node-bound covariate $d_{h}$ are generated by independent draws from the Bernoulli(1/2) distribution.

Two main purposes of goodness-of-fit testing can be distinguished, and hence the simulation study consists of two main parts: (1) testing parameters which capture structural features of the data such as third-order dependence among arcs (“clustering”), and (2) testing the impact of covariates on the digraph evolution. The basic data-generating model corresponds to parameters $\rho_1 = \rho_2 = 4$, $\beta_1 = -1$ (corresponding to $s_{i1}$), and $\beta_2 = 1$ (corresponding to $s_{i2}$), which are common to all models used for generating data. The first part of the simulation study tests hypotheses involving the parameters $\beta_3$ and $\beta_4$, corresponding to $s_{i3}$ and $s_{i4}$, respectively; both parameters capture third-order dependence among arcs. The second part of the simulation study tests hypotheses involving the parameters $\beta_5$ and $\beta_6$, corresponding to $s_{i5}$ and $s_{i6}$, respectively. All tests are two-sided.

Part I: testing effects capturing third-order dependence The basic model is $P_\theta, \theta = (\rho_1, \rho_2, \beta_1, \beta_2, \beta_3, \beta_4)' = (4, 4, -1, 1, \beta_3, \beta_4)';$ three values of $\beta_3$ (0, .2, and .4) and $\beta_4$ (0, .3, and .6) are considered including all combinations, giving 9 models. The values of $\beta_3$ and $\beta_4$ are primarily chosen to study the power of the test against local alternatives. For each model, 500 data sets are generated; the reason for limiting the number of data sets to 500 is the computing time required to estimate models. For each model and each data set, the score test is evaluated in one estimation run (where the 4 unrestricted parameters are estimated) and the $t$-test in another...
Figure 3.1: Monte Carlo results, model $P_\theta$, $\theta = (\rho_1, \rho_2, \beta_1, \beta_2, \beta_3, \beta_4)' = (4, 4, -1, 1, 0, 0)'$, $n = 30$ and $n = 60$: distribution of test statistics

Note: the curves represent the expected distributions under $H_0$; $c$ and $t$ refer to score test (3.20) and $t$-test (3.10), respectively.

estimation run (where all 6 parameters are estimated); the parameters are estimated by the conditional method of moments of Snijders (2001).

Figure 3.1 (p. 53) shows histograms of the distributions of the score test statistic for testing $H_0 : \beta_3 = \beta_4 = 0$, $H_0 : \beta_3 = 0$, and $H_0 : \beta_4 = 0$, and histograms of the distributions of the $t$-test statistic for testing $H_0 : \beta_3 = 0$ and $H_0 : \beta_4 = 0$, where the true, data-generating model is $P_\theta$, $\theta = (\rho_1, \rho_2, \beta_1, \beta_2, \beta_3, \beta_4)' = (4, 4, -1, 1, 0, 0)'$. For the score tests, the distributions agree very well with the chi-square distributions expected under $H_0$, but the $t$-tests appear to be slightly conservative.

Table 3.1 (p. 54) shows the empirical rejection probabilities for the mentioned hypotheses and the 9 data-generating models using the nominal significance level .05; space restrictions do not allow to show the results for other nominal significance levels (which do not alter the conclusions). Note that if $H_0$ is true and the test is of size .05, then the binomial distribution of the number of rejections implies that the empirical rejection rate should be roughly between .03 and .07.

The two-parameter score test behaves reasonable under $H_0 : \beta_3 = \beta_4 = 0$ and seems to have (for all practical purposes) sufficient power to detect departures from $H_0$. 
Table 3.1: Monte Carlo results, model $P_\theta$, $\theta = (\rho_1, \rho_2, \beta_1, \beta_2, \beta_3, \beta_4)' = (4, 4, -1, 1, \beta_3, \beta_4)'$, $n = 30$ and $n = 60$: empirical rejection probabilities for tests with nominal significance level .05

<table>
<thead>
<tr>
<th>true model: $P_\theta, \beta_3 = 0, \beta_4 = 0$:</th>
<th>score test: $H_0: \beta_3 = \beta_4 = 0$</th>
<th>score test: $\beta_3 = 0$</th>
<th>score test: $\beta_4 = 0$</th>
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Figure 3.2: Monte Carlo results, model $P_{\theta}$, $\theta = (\rho_1, \rho_2, \beta_1, \beta_2, \beta_3, \beta_5, \beta_6)' = (4, 4, -1, 1, 2, 0, 0)'$, $n = 30$ and $n = 60$: distribution of test statistics

Note: the curves represent the expected distributions under $H_0$: $c$ and $t$ refer to score test (3.20) and $t$-test (3.10), respectively.

Table 3.1 indicates that the one-parameter score test has considerably more power than the $t$-test, in particular if $n = 30$ and the departure from $H_0$ is small.

Concerning the one-parameter score test, note that the two-parameter test and the two corresponding one-parameter tests are computed from the same estimation run, where $\theta$ was estimated under $H_0 : \beta_3 = \beta_4 = 0$. Therefore, the one-parameter score tests in table 3.1 do not control for the other parameter ($\beta_3$ or $\beta_4$). Note that, in principle, it is straightforward to carry out one-parameter score tests of $\beta_3$ and $\beta_4$ where the other parameter is controlled for, but that would require two additional estimation runs.

Concerning the $t$-test, it is notable that the power of the $t$-test for testing $H_0 : \beta_3 = 0$ is the smaller the larger the departure of $\beta_4$ from 0.

Part II: testing covariate-related effects The basic model is $P_{\theta}$, $\theta = (\rho_1, \rho_2, \beta_1, \beta_2, \beta_3, \beta_5, \beta_6)' = (4, 4, -1, 1, 2, \beta_5, \beta_6)'$, which includes transitivity parameter $\beta_3$; three values of $\beta_5$ (0, .1, and .2) and $\beta_6$ (0, .2, and .4) are considered including all combinations, giving 9 models.

Figure 3.2 (p. 55) shows histograms of the distributions of the score test statistic
Table 3.2: Monte Carlo results, model $P_\theta$, $\theta = (\rho_1, \rho_2, \beta_1, \beta_2, \beta_3, \beta_5, \beta_6)' = (4, 4, -1, 1, 2, \beta_5, \beta_6)'$, $n = 30$ and $n = 60$: empirical rejection probabilities for tests with nominal significance level .05

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<td>.984</td>
<td>.994</td>
<td>.990</td>
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</tbody>
</table>
3.6. APPLICATION

for testing $H_0 : \beta_5 = \beta_6 = 0$, $H_0 : \beta_5 = 0$, and $H_0 : \beta_6 = 0$, and histograms of the distributions of the $t$-test statistic for testing $H_0 : \beta_5 = 0$ and $H_0 : \beta_6 = 0$, where the true, data-generating model is $P_\theta, \theta = (\rho_1, \rho_2, \beta_1, \beta_2, \beta_3, \beta_5, \beta_6)' = (4, 4, -1, 1, 2, 0, 0)'$. For the score tests, the distributions agree quite well with the chi-square distributions expected under $H_0$, although the two-parameter score test appears to be slightly conservative.

Table 3.2 (p. 56) shows the empirical rejection probabilities for the mentioned hypotheses and the 9 data-generating models using the nominal significance level .05. Table 3.2 indicates that the one-parameter score test has less power than the $t$-test, in particular if $n = 30$ and the departure from $H_0$ is small.

3.6 Application

Pahor (2003) studied the directed cross-ownerships among 413 business firms in Slovenia (EU) observed at 5 time points, where directed cross-ownership $A \rightarrow B$ means that firm $A$ holds stock market shares of firm $B$. The present paper re-analyses the data in the light of the new goodness-of-fit test statistic. Once again, it is convenient to refer to the two tests as “score test” and “$t$-test”, respectively, and all tests are two-sided, unless stated otherwise.

The baseline model considered here corresponds to the rate function

$$
\lambda_i(\theta, x^*) = \rho_m \exp \left[ \alpha \left( 1 + \sum_{h=1}^{n} x_{ih}^{*} \right)^{-1} \right]
$$

(3.34)

for time interval $[t_m, t_{m+1}]$. The objective function is given by

$$
f_i(\beta, x) = \sum_{k=1}^{K} \beta_k s_{ik}(x, j),
$$

(3.35)

where

$s_{i1}(x, j) = \sum_{h=1}^{n} x_{ih}$: the number of arcs,

$s_{i2}(x, j) = \sum_{h=1}^{n} x_{ih} x_{hi}$: the number of reciprocated arcs,

$s_{i3}(x, j) = \sum_{h=1}^{n} x_{ih} d_{h1}$: interaction of arcs and covariate $d_{h1}$,

$s_{i4}(x, j) = \sum_{h=1}^{n} x_{ih} d_{h2}$: interaction of arcs and covariate $d_{h2}$,
where \( d_{h1} \) indicates whether or not firm \( h \) was quoted on the stock exchange, while \( d_{h2} \) refers to the size of firm \( h \). The parameter \( \theta \) of the baseline model \( P_{\theta} \) corresponds to \( \theta = (\alpha', \beta_1, \beta_2, \beta_3, \beta_4)' \), where \( \alpha = (\rho_1, \rho_2, \rho_3, \rho_4)' \).

Pahor (2003) suspected that the data may exhibit third-order dependence, leading to the transitivity parameter \( \beta_5 \) and the indirect connections parameter \( \beta_6 \) as candidates to be tested, which correspond to statistics \( \sum_{h,l=1}^{n} x_{ih} x_{hl} x_{il} \) and \( \sum_{h=1}^{n} (1 - x_{ih}) \max x_{il} x_{lh}, \) respectively. According to table 3.3 (p. 58), the two-parameter score test of \( H_0 : \beta_5 = \beta_6 = 0 \) clearly indicates that parameters capturing third-order dependence are required to improve the goodness-of-fit of the model, and the one-parameter score tests suggest to include \( \beta_5 \) but not \( \beta_6 \). A one-sided test of \( H_0 : \beta_5 = 0 \) can be carried out by using the linear form of the score test (see (3.21)), giving 7.86: using the asymptotic standard Gaussian distribution of the linear form under \( H_0 : \beta_5 = 0 \), it seems that \( \beta_5 \) is positive, which is supported by the one-step estimate of \( \beta_5 \) (see (3.26)) given by .996. As a result, \( \beta_5 \) is henceforth included in the model, while \( \beta_6 \) is not.

Pahor (2003) argued that firms tend to hold shares of other firms close to them with respect to region \( (c_{ih1}) \) and industry branch \( (c_{ih2}) \), and to other firms having the same owner \( (c_{ih3}) \). Three parameters are added, \( \beta_7, \beta_8, \) and \( \beta_9, \) corresponding to statistics \( \sum_{h=1}^{n} x_{ih} c_{ihl}, l = 1, 2, 3, \) respectively. The three-parameter score test of \( H_0 : \beta_7 = \beta_8 = \beta_9 = 0 \) and the three corresponding one-parameter tests, shown in table 3.3, suggest to add all three covariates to the model, which is done.

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The parameter vector \( \eta \) is given by \( \eta = (\alpha', \beta_1, \beta_2, \beta_3, \beta_4)' \).
To save space, the nuisance parameters $\alpha, \alpha, \beta_2, \beta_3,$ and $\beta_4$ are omitted. The one-step estimate $\theta^*$ is based on the moment estimate of $\theta$ under the model restricted by $H_0: \beta_{10} = \beta_{11} = \beta_{12} = 0$; the moment estimate $\hat{\theta}$ is the unrestricted moment estimate of $\theta$.

It is interesting to test whether the values of the parameters are constant across time intervals. A basic parameter for which such homogeneity tests frequently make sense is the “outdegree” parameter $\beta_1$, corresponding to statistic $s_{11}$. A homogeneity test for $\beta_1$ is conducted by testing $H_0: \beta_{10} = \beta_{11} = \beta_{12} = 0$, where $\beta_{10}, \beta_{11},$ and $\beta_{12}$ correspond to statistics $e_i^{(m)} s_{i1}, m = 2, 3, 4$, respectively, where $e_i^{(m)}$ is a period-dependent dummy variable with value 1 in time interval $[t_m, t_{m+1}]$ for all $i$ and 0 otherwise. According to table 3.3, the three-parameter score test of $H_0: \beta_{10} = \beta_{11} = \beta_{12} = 0$ indicates that there is empirical evidence against $H_0$, and the three corresponding one-parameter tests suggest that it is sensible to add $\beta_{10}, \beta_{11},$ and $\beta_{12}$ to the model. The one-sided one-parameter score test statistics (see (3.21)) are $-3.22,$ $-2.15,$ and $3.72$, respectively, suggesting that the values of $\beta_{10}$ and $\beta_{11}$ are negative while $\beta_{12}$ is positive. Table 3.4 (p. 59) gives the one-step estimate (see (3.26)) of the parameter $\theta = (\alpha', \alpha, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_7, \beta_8, \beta_9, \beta_{10}, \beta_{11}, \beta_{12})'$ based on the moment estimate of $\theta$ under the model restricted by $H_0: \beta_{10} = \beta_{11} = \beta_{12} = 0$, and shows in addition the unrestricted moment estimate of $\theta$, including standard errors and $t$-tests. The one-step estimates roughly agree with the unrestricted moment estimates. The $t$-tests by and large agree with the score tests (regarding $\beta_5, \beta_7, \beta_8, \beta_9,$ and $\beta_{10}$), but slightly disagree about $\beta_{11}$ and clearly disagree about $\beta_{12}$. 

<table>
<thead>
<tr>
<th>Parameter</th>
<th>One-step Estimate</th>
<th>Moment Estimate</th>
<th>s.e.($\hat{\theta}$)</th>
<th>$t$-Test</th>
<th>p-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>-2.65</td>
<td>-2.67</td>
<td>.007</td>
<td>-27.54</td>
<td>&lt; .0001</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>.81</td>
<td>.81</td>
<td>.102</td>
<td>7.91</td>
<td>&lt; .0001</td>
</tr>
<tr>
<td>$\beta_7$</td>
<td>.57</td>
<td>.58</td>
<td>.076</td>
<td>7.60</td>
<td>&lt; .0001</td>
</tr>
<tr>
<td>$\beta_8$</td>
<td>1.01</td>
<td>1.01</td>
<td>.106</td>
<td>9.57</td>
<td>&lt; .0001</td>
</tr>
<tr>
<td>$\beta_9$</td>
<td>.89</td>
<td>.89</td>
<td>.258</td>
<td>3.45</td>
<td>.0006</td>
</tr>
<tr>
<td>$\beta_{10}$</td>
<td>-.31</td>
<td>-.29</td>
<td>.134</td>
<td>-2.12</td>
<td>.0337</td>
</tr>
<tr>
<td>$\beta_{11}$</td>
<td>-.31</td>
<td>-.28</td>
<td>.158</td>
<td>-1.79</td>
<td>.0735</td>
</tr>
<tr>
<td>$\beta_{12}$</td>
<td>.12</td>
<td>.13</td>
<td>.117</td>
<td>1.14</td>
<td>.2544</td>
</tr>
</tbody>
</table>
3.7 Discussion

A new goodness-of-fit test statistic was proposed, which has many applications, and, in contrast to the pseudo-\(t\)-test, does not require the estimation of the parameters to be tested (saving computation time and avoiding convergence problems), admits multi-parameter tests, and has an appealing interpretation in terms of goodness-of-fit.

The Monte Carlo study indicated that the finite-sample distribution of the goodness-of-fit test statistic under the considered null hypotheses matches the expected distributions fairly well.

As to the unproven assertion (3.16), observe that the statistical modeling of social networks is in its infancy, and that asymptotic properties of estimators and test statistics are largely unknown; see, for instance, Hunter and Handcock (2006). In addition, asymptotics—despite of being an indispensable tool in mathematical statistics and useful guides in applied statistics—strictly speaking make no statement about samples of given finite size (De Bruijn, 1981, p. 2, Hampel, 1997). In combination with the Monte Carlo study, the approach in the present paper can be recommended for practical use, though care must be taken.

The proposed test statistic is implemented in the Windows-based computer program Siena embedded in the program collection StOCNET, which can be downloaded free of charge from http://stat.gamma.rug.nl/stocnet.