Chapter 1

Introduction

The study of ultrashort pulses is of fundamental importance for two significant fields: for telecommunication and for the study of rapidly varying phenomena. For optical telecommunication one wants to send short light pulses, that are still short when they arrive (at the other side of the ocean). For studying rapidly varying processes like photosynthesis, with a time scale of some femtoseconds, one can probe them with light pulses that are also some femtoseconds long. Currently the world record is held by the group of Prof. Wiersma in Groningen, with a pulse of $4.6 \text{ fs (FWHM)}$. Both for optical telecommunication and for the study of rapidly varying phenomena, one has to know what happens to the pulse as it propagates.

When one wants to propagate short light pulses over long distances along fibres, group velocity dispersion leads to pulse spreading. However, this can be counteracted by intensity dependence of the refractive index. When both are present, the envelope of the pulse is governed by the nonlinear Schrödinger equation (NLS). This equation has soliton solutions, that is, it has stable solutions that will propagate undeformed, and that can pass through each other without deformation.

The soliton ‘industry’ started in 1967 when Gardner et al. [13] discovered the inverse scattering method for solving the Korteweg-de Vries equation. Subsequently Zakharov and Shabat [53] extended the method and solved the Nonlinear Schrödinger equation (NLS) with it.

In 1973, Hasegawa and Tappert found, that pulses in nonlinear dispersive fibres could be described by the NLS, both in the case of anomalous dispersion [21] and in the case of normal dispersion [22], which meant that the fibres should be able to support soliton pulses. This was experimentally verified (in the case of anomalous dispersion) by Mollenauer, Stolen and Gordon in 1980 [38].

The NLS excellently describes pulses longer than a picosecond. For shorter pulses, one has to take the finite response times of atomic and molecular processes into account, that lead to higher order terms. Most important of these is the stimulated Raman scattering (SRS) term, which leads to a down shift in the pulse frequency, as discovered by Mitschke and Mollenauer [34] and explained by Gordon [14].
Usually, the extended NLS, including these higher order terms, is derived making *ad hoc* assumptions (often, the second order derivative of the amplitude with respect to the propagation coordinate, \( A_{zz} \), is neglected, like in [2, 30]; however, careful analysis shows that the first order contribution to \( A_{zz} \) is not negligible, but cancels with another neglected term). But it is possible to derive the extended NLS in a consistent manner, by using the method of multiple scales, in optics literature also known as the slowly varying envelope (or amplitude) approximation or as the rotating wave approximation. Kodama and Hasegawa [27, 28] were the first to give a proper derivation with the higher order terms included.

The rest of this chapter gives some general assumptions that will be made, some notational conventions, and a short introduction about the nonlinear Schrödinger equation.

Chapter 2 gives the physical background, mostly on the nonlinear polarisation. For the ultrashort pulses under consideration, the time dependence of the nonlinear polarisation must explicitly be taken into account. However, one can still treat the nonlinear polarisation in the Born-Oppenheimer approximation: the motion of the nuclei is independent of the motion of the electron. This greatly simplifies the expression for the nonlinear polarisation.

Chapter 2 results in a wave equation that the electric field has to satisfy. The method of multiple scales that is used to solve this equation in later chapters is explained in chapter 3.

Following (and extending) an approach of Newell and Moloney [43], in chapter 4 an ansatz for the pulse envelope is chosen in a ‘clever’ way, that particularly suits the linear mode of the waveguide. In this approach, functions of the frequency are not used at one fixed centre frequency \( \omega_1 \), but they are expanded like \( f(\omega) \rightarrow f(\omega_1) + i\varepsilon f'(\omega_1)\partial_T + \cdots \). In later chapters, this expansion is applied to terms nonlinear in the pulse amplitude as well. It is shown that this leads to an automatic cancellation of some of the arising perturbation terms.

In chapter 5 the derivation of the propagation equation for the envelope of the pulse is presented, taking the nonlinear polarisation and its time dependence into account, resulting in a partial differential equation. It is shown that this equation preserves the energy of a pulse.

However, it turns out that for pulses of the order of ten femtoseconds, the term corresponding to the stimulated Raman scattering cannot be properly incorporated in a single partial differential equation for the envelope of the pulse. In chapter 6 an integrodifferential equation for the envelope is derived, in which the SRS term has to be taken into account as a convolution product. This is the same equation as the one that Blow and Wood [6] and Mamyshev and Chernikov [30] found. They both derived the equation in the Fourier domain, but there the problem is that one cannot properly separate the transverse dependence of the electric field from the longitudinal dependence. Also they don’t make exactly clear what approximations have been made, or they impose unnecessary conditions. In particular, the condition that the spectral width of the pulse is less than one third of its centre frequency [6] is not needed.

If instead of working in the Fourier domain one derives an amplitude equation strictly
in the time domain, like Kodama and Hasegawa do [18,19,27,28], one has to asymptotically expand the nonlinear polarisation, so cannot treat the stimulated Raman scattering properly then. Therefore this thesis in effect combines the time domain and the Fourier domain. By working mainly in the time domain, one can keep track of the approximations made, and one can exactly separate the longitudinal part from the transverse part. By changing to the Fourier domain sometimes one can treat the stimulated Raman scattering properly, and one can simplify some expressions. Using this mixed approach this thesis will more clearly identify the approximations made implicitly in previous derivations.

An alternative would be to take the SRS term into account as a separate differential equation, leaving a system of differential equations, e.g. [1,17]. However, then for the stimulated Raman scattering one is limited to a single Lorentzian line.

It is shown that the equation derived in chapter 6 does not conserve the pulse energy, but it conserves the number of photons instead. This is the proper conservation law when stimulated Raman scattering is present, as is shown in section 2.5.

Chapter 7 compares the two equations found, by numerically integrating the 1-soliton solution. Also the frequency decrease caused by the stimulated Raman scattering is investigated, and the effect of higher order linear polarisation.

Chapter 8 discusses the limits of the validity of the equations found. A lower limit on the pulse length is necessary, or the asymptotic expansion will break down. For the equation derived in chapter 5 to be valid, it turns out that the pulse length must be much larger than 10 fs; for the validity of the equation derived in chapter 6, the pulse can be a factor 10 shorter.

General books on solitons in fibres are [2,18,49]. A general book on shorter pulses (femtosecond) is [3].

A part of this thesis has been published before as [44].

### 1.1 Assumptions

A completely general rigorous analysis of the propagation of a pulse through a slab or fibre is infeasible, so one has to make some assumptions.

First of all, the polarisation is supposed to be local. That is, the polarisation is allowed to depend on the past of the electric field, but it may only depend on the electric field at the same position.

Secondly, the material is supposed to be isotropic. Also, the fibre or waveguide should be single-mode and polarisation-preserving. Without this assumption, mixing of modes could occur and one would have to consider coupled equations.

The nonlinear polarisation will be treated in the Born-Oppenheimer approximation, i.e. the motion of the nuclei and the motion of the electrons will be treated separately [8]. This simplifies the expression for the nonlinear susceptibility.

Since the advent of Er\textsuperscript{3+} doped fibre amplifiers in 1989 [32,39,40], for optical telecommunication the wavelength of choice is 1.55 μm. Glass is transparent at that wavelength, and an erbium doped fibre amplifier pumped at 0.98 μm or 1.48 μm can amplify light at
that wavelength very efficiently. So when a definite figure for the wavelength is needed, e.g. for the numerical integration, the wavelength will be taken to be 1.55 μm.

The ultrashort pulses made in the laboratory of Prof. Wiersma have a centre wavelength of 800 nm. Indeed a short piece of glass fibre is used as the nonlinear element, but at that wavelength the dispersion is normal; to get net anomalous dispersion after a roundtrip, gratings or prisms are used. The equation found in this thesis, equation (6.39), is still valid for the propagation in the fibre in that case, if one changes the sign of the dispersion term $a_\tau$.

But apart from the choice of the wavelength, damping and amplification is neglected. Therefore subjects such as the ‘average’ or ‘guiding-center’ solitons [4, 20, 35], Gordon Haus jitter [15] or bandwidth limited amplification [5, 36] are not touched upon.

In the final dimensionless form of the envelope equations, the linear dispersion is supposed to be anomalous. This is the case that leads to bright solitons (were it not for the higher order terms).

1.2 Notational conventions

Appendix E contains a list of symbols used, notational conventions for products, derivatives, etc., and definitions of the various direct and inverse Fourier transforms.

To keep in line with the most common convention, the Fourier transform with respect to $t$ is defined with an exponent $+i\omega t$, and the Fourier transform with respect to $z$ is defined with an exponent $-ikz$. Fourier transform with respect to $t$ is also denoted with a hat accent, $\hat{f}(\omega) = \mathcal{F}_t[f(\omega)]$, and its inverse also with the check accent, $g(t) = \mathcal{F}_t^{-1}[g](t)$. In chapter 7, the Fourier transform of functions of $\tau$ is expressed in terms of the frequency $\nu$, instead of in terms of the circular frequency $\omega$.

The convolution product of two functions of time, $f(t)$ and $g(t)$, is denoted by $f * g$. It is defined by $(f * g)(t) = \int_{-\infty}^{\infty} f(t - t_1)g(t_1) \, dt_1$.

A prime will always denote a (total) derivative with respect to the frequency $\omega$, never any other derivative or a real or imaginary part; so $f' = df/d\omega$ etc. A dot over a symbol denotes a derivative with respect to time.

The complex conjugate of a function is per definition given by $f^*(z) = [f(z)]^*$.

S.I. units are used throughout.

1.3 The Nonlinear Schrödinger equation

In a material the speed of light depends on the wavelength. That is why a prism can separate colours. Now if a light pulse propagates through a fibre, it cannot be purely monochromatic: if it would be, it would have to last forever. The spectral bandwidth is at least $\pi/2$ over the pulse duration. So a finite pulse contains (slightly) different colours, that move at (slightly) different speeds. The speeds won’t be that much different, but if the pulse propagates long enough, over thousands of kilometers, it will still add up to many wavelengths. So the pulse broadens.
In the case of so called normal dispersion, i.e. if \( k''(\omega) > 0 \), lower frequencies move faster; in the case of anomalous dispersion, i.e. if \( k''(\omega) < 0 \), higher frequencies move faster. A way to see this, is to look at the group velocity \( v_g = 1/k' = 1/(k' + (\omega - \omega_1)k'') \). For higher frequencies, \( v_g \) is higher (in the case of anomalous dispersion).

Therefore, because of the group velocity dispersion (GVD), different frequencies will spread, and the pulse will broaden: for normal dispersion, the lower frequencies will move faster, and for anomalous dispersion, the higher frequencies will move faster.

But there is another effect. If the electric field becomes large, the refractive index will become intensity dependent, \( n = n_0 + n_2 |E|^2 \). This means that the phase velocity becomes intensity dependent. The nonlinear refractive index \( n_2 \) is positive, so if the intensity is higher, the velocity becomes smaller.

For a pulse this means that the peak will move slowly, and the tails will move fast. Consequently the leading tail moves away from the peak, and the trailing tail runs into the peak. So if one looks at the zero transitions of the electric field, for instance, in the leading tail, they move apart, so there the wavelength increases, and thus the frequency decreases. And in the trailing tail, the zero transitions become closer to one another, so the wavelength decreases, and the frequency increases. In other words, the lower frequencies move faster.
Now in the case of normal dispersion, both the GVD and the nonlinearity cause the lower frequencies to move faster, so the pulse will broaden irrevocably. But in the case of anomalous dispersion, the GVD will cause the higher frequencies to move faster, and the nonlinearity cause the lower frequencies to move faster, so these effects can balance. This is shown symbolically in figure 1.1.

With both effects combined the envelope of the pulse is described by the Nonlinear Schrödinger equation:

$$ia\zeta + \frac{i}{2}a_{\tau \tau} + |a|^2 a = 0.$$  \hspace{1cm} (1.1)

This is a very interesting equation, because it is integrable. This means among others that it has an infinity of conservation laws, that it can be solved with inverse scattering, and that it has soliton solutions — very stable localised solutions, with particle-like properties. For instance, if 2 solitons collide, they can pass through each other undeformed, undergoing only a delay.

The most stable solution is the 1-soliton:

$$a(\zeta, \tau) = \Lambda \sech \Lambda (\tau + \Omega \zeta)e^{-i\Omega \tau + \frac{i}{2}(\Lambda^2 - \zeta^2)\zeta}$$  \hspace{1cm} (1.2)

of which the simplest case, with \(\Lambda = 1\) and \(\Omega = 0\), is

$$a(\zeta, \tau) = \sech \tau e^{\frac{i}{2}\zeta}.$$  \hspace{1cm} (1.3)

Figure 1.2 shows how it propagates: nothing significant happens. Only the phase evolves, but one cannot see that in the figure. The plots on the left-hand side show the amplitude, the plots on right-hand side show its Fourier transform. The time or frequency runs to the right, and plots for different values of the propagation distance \(\zeta\) are shown superimposed in plots (a) and (c), and running to the back in plots (b) and (d).

If the amplitude is not completely right, the pulse will deform to an exact soliton shape whilst shedding off radiation, as figure 1.3 shows. There the initial condition is \(a(0, \tau) = 1.1 \sech \tau\); its energy is equal to the energy of a soliton with \(\Lambda = 1.21\), so it oscillates in the \(\zeta\)-direction around a 1-soliton with \(\Lambda\) slightly smaller than 1.21.
Another interesting solution is the 2-soliton, for which $a(0, \tau) = 2 \text{ sech} \, \tau$: it oscillates in the $\zeta$-direction with a period $\pi/2$, the so-called soliton period, but is undeformed otherwise. See Figure 1.4.

For more information on soliton and integrable equations, see introductory texts such as [7, 9, 10].