Low regularity ray tracing for wave equations using Gaussian beam

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ABSTRACT. We prove observability estimates for oscillatory Cauchy data modulo a small kernel for $n$-dimensional wave equations with space and time dependent $C^2$ and $C^{1,1}$ coefficients using Gaussian beams. We assume the domains and observability regions are in $\mathbb{R}^n$, and the GCC applies. This work generalizes previous observability estimates to higher dimensions and time dependent coefficients. The construction for the Gaussian beamlets solving $C^{1,1}$ wave equations represents an improvement and simplification over Waters (2011).

1. Introduction. Significant interest in inverse problems and imaging centers around low-regularity wave equations. Oftentimes in nature equations governing physical situations can be irregular and discontinuities are difficult to model. Ray tracing techniques used in geophysics often require $C^3$ coefficients to model the physical phenomenon. These techniques have been popular with the inverse problems community [1, 13, 14, 21, 22, 25, 26]. The goal of this article is to introduce a construction which generalizes the Gaussian beam Ansatz and is suitable for proving multidimensional observability estimates for low regularity equations. The relationship of these observability estimates to hyperbolic inverse boundary value problems is clear. One can “observe” or recover a source from following the geometric optics rays from the observation back to the source. This technique is known as ray-tracing. Our results indicate that in some cases, for practical applications imaging is still possible. The main theorems extend the observability estimates of [9] both in dimension and generality of the geometry. We use the observability criterion of Bardos-Lebeau-Rauch [3] on the rays.
Specifically, we prove in this paper observability estimates for $n$-dimensional classical wave equations with $C^{1.1}$ and $C^2$ time dependent coefficients. Whenever the coefficients are in these low regularity classes we establish observability estimates even though the typical assumption is $C^3$ coefficients. We will give a precise idea of why this is true when specifying the error estimates for the construction of the approximate solution, or Ansatz solution. This appears to be the first case in which space and time dependent coefficients are also examined in the context of observability estimates.

These estimates are proved for one-dimensional wave equations by means of a uniquely one-dimensional technique [9] called sidewise energy estimates, where the use of space and time are interchanged. We employ here a similar idea, developed in [29], that on ray paths in higher dimensions time and arc-length can be interchanged. This extension seems like a natural generalization of sidewise energy estimates.

In higher space dimensions the problem is more complex and other techniques are required. Bardos-Lebeau-Rauch [3] proved it is necessary and sufficient for observability estimates to hold for wave equation solutions if the observability region satisfies the geometric control condition.

For this article we assume $\Omega$ is a domain in $\mathbb{R}^n$, and $\Omega_0 \subset \Omega$ is the observation region. In general the work of Lebeau showed that control from $\Omega_0 \subset \Omega$ (the dual statement to the existence of the observability constants for the wave equation) also holds for the wave equation under the Geometric Control Condition (GCC):

- There exists $L = L(\Omega, \Omega_0) > 0$ such that every Hamiltonian ray path of length $L$ on $\Omega$ intersects $\Omega_0$.

In one dimension, rays can only travel in a sidewise manner, so this condition is automatic to verify. However for the higher dimensional case the development of more complex Ansatzes to trace the ray path in space-time involves $C^2$ [23] or at minimum $C^{1.1}$ [28] coefficients. The main novelty of this paper is an Ansatz which does not require the usual $C^3$ coefficients necessary for the construction of the localized tail.

For the system in question, in 1d observability estimates with a loss of derivatives continue to hold for coefficients in log-Zygmund spaces [9], but fail for $C^\alpha$ [6] with $\alpha < 1$. In higher dimensions, Burq extended the observability estimates to $C^2$ time independent coefficients and $C^3$
domains [5]. For this work, we concentrate on results in the positive direction, and use an equivalent hypothesis on the geometry of the rays as in [5]. In contrast to previous works, we assume that Cauchy data has an oscillatory phase part. This assumption could be removed by using the frame in [28], and will be reported on in future work.

2. Statement of the main theorems. We recall the definition of $C^{1,1}$ coefficients. The collection \( \{g^{jk}(x, t)\}_{j,k=1}^{n} \) with \((x, t) \in \Omega \times [0, T]\) is said to be $C^{1,1}$ if the coefficients satisfy a Lipschitz condition

\[
|g^{jk}(x, t) - g^{jk}(x', t')| \leq M(|t - t'| + |x - x'|)
\]

in $x$ and $t$, and their first derivatives in $x$ satisfy a Lipschitz condition

\[
|\nabla_x g^{jk}(x, t) - \nabla_x g^{jk}(x', t)| \leq M_0|x - x'|
\]

for some positive constants $M, M_0$ independent of $x$ and $t$. In general we assume the coefficient $g^{jk}(x, t)$ is variable over the set $\Omega \times [0, T]$, and smoothly extended outside $\Omega$ to be equal to $\delta^{jk}$. As such, in the article we only specify the regularity of the variable part of the coefficients for the theorems where it is understood we mean the extended coefficients.

Now we introduce our operator. Let $H(x, t, \partial_x, \partial_t)$ be a second-order hyperbolic operator of the form

\[
(2-1) \quad H(x, t, \partial_x, \partial_t) = -\frac{\partial^2}{\partial t^2} + \sum_{j,k=1}^{n} g^{jk}(x, t) \frac{\partial^2}{\partial x_j \partial x_k}.
\]

We assume that there exists a constant $\alpha_0$ such that

\[
(2-2) \quad \sum_{j,k=1}^{n} g^{jk}(x, t)\xi_i \xi_j \geq \alpha_0 |\xi|^2.
\]

Sometimes we will assume the variable part of the coefficients are $g^{jk}(x, t) \in C^2(\Omega \times [0, T])$, or $C^{1,1}(\Omega \times [0, T])$, and we will specify the regularity as needed.

We let

\[
(2-3) \quad S\Omega = \{(x, \omega) \in T\Omega : |\omega| = 1\}
\]

denote the sphere bundle of $\Omega$. We consider pairs \((x_0, \omega_0) \in S\Omega\) of initial conditions of the system of ODEs (3-22) for which the Hamiltonian or bicharacteristic flow (instead of the geodesic flow) satisfies the GCC.
However, it can be shown, using the coefficients $g^{jk}$ to endow $\Omega$ with a metric topology so that $(g, \Omega)$ is a manifold, that the bicharacteristic flow and the geodesic flow on the manifold are equivalent; see the Appendix of [29] for this computation.

We define a phase function $\psi_0(x)$ as
\[
\psi_0(x) = (x - x_0) \cdot \omega_0 + i|x - x_0|^2
\]
and let $\lambda \in \mathbb{R}^+$ be a large asymptotic parameter. Again, the vectors $(x_0, \omega_0) \in S\Omega$ are in the admissible set defined above.

We consider the Cauchy problem with Dirichlet boundary conditions:
\[
Hu(x, t) = 0 \quad \text{in } \Omega \times \mathbb{R}^+_t,
\]
\[
u_0 = u(x, 0) = \lambda^{n/4} f_1(x) \exp(i\lambda \psi_0) \quad \text{in } \Omega,
\]
\[
u_1 = \frac{\partial u}{\partial t}(x, 0) = \lambda^{n/4} f_2(x) \exp(i\lambda \psi_0) \quad \text{in } \Omega,
\]
\[
u(x, t) = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+_t,
\]
where each function $f_k(x)$ is in $C^3(\Omega)$, $k = 1, 2$ ($H^3$ regularity is also possible with some slightly more technical lemmas). It is well known under these conditions that this problem is well posed for some finite time $T$, with $u \in C([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega))$; see Theorem 3.3 below.

Let $\Omega_0 \subset \Omega$ be such that $\Omega_0$ satisfies the geometric control condition with respect to the Hamiltonian flow. Let $\Omega' \subset \Omega$ with $\Omega'$ a larger domain. The domain $\Omega'$ will be specified by finite speed of propagation in the proof. In particular, we will have $\text{diam}_g(\Omega') \geq T + \text{diam}_g(\Omega)$. We let the constant $C_{12}$ be such that $C_{12}$ depends on $T, \text{diam}_g(\Omega), \alpha_0$ and $\|g^{jk}(x, t)\|_{C^2(\Omega \times [0, T])}$ only. We also let $C_{22}$ be such that it depends on $\|f_1\|_{C^3(\Omega)} + \|f_2\|_{C^2(\Omega)}, \text{diam}_g'(\Omega'), \alpha_0, \|g^{jk}(x, t)\|_{C^2(\Omega \times [0, T])}$, and $T$. We have the following theorems, provided the set $\Omega_0 \subset \Omega$ satisfies the GCC, and $T > \text{diam}(\Omega)$, where the diameter is taken with respect to the Hamiltonian flow path defined by (3-22).

**Theorem 2.1.** Assume that $g^{jk}(x, t)$ are $C^2(\Omega \times [0, T])$. Then for every $\epsilon > 0$ there exist nonzero constants $C_{12}$ and $C_{22}$ as above such that for all $\lambda$ sufficiently large,
\[
\|u_0\|_{L^2(\Omega)} + \|u_1\|_{H^{-1}(\Omega)} \leq C_{12}\|u\|_{L^2(\Omega_0 \times [0, T])} + \frac{C_{22}}{\lambda^{1/3}} + C_{22}C_2(\epsilon),
\]
where $C_2(\epsilon)$ is a positive function of $\epsilon$ such that $C_2(\epsilon) \to 0$ as $\epsilon \to 0$. 
Corollary 2.2. As $\epsilon \to 0$ and $\lambda \to \infty$, this gives a perfect observability estimate:

\begin{equation}
\|u_0\|_{L^2(\Omega)} + \|u_1\|_{H^{-1}(\Omega)} \leq C_{12}\|u\|_{L^2(\Omega_0 \times [0,T])}.
\end{equation}

We let the constant $C_{11}$ be such that $C_{11}$ depends on $T$, $\text{diam}_g(\Omega)$, $\alpha_0$, and $\|g^{jk}(x,t)\|_{C^{1,1}(\Omega \times [0,T])}$ only. We also let $C_{21}$ be such that it depends on $\|f_1\|_{C^3(\Omega)} + \|f_2\|_{C^2(\Omega)}$, $\text{diam}_g'(\Omega')$, $\alpha_0$, $\|g^{jk}(x,t)\|_{C^{1,1}(\Omega \times [0,T])}$ and $T$.

Theorem 2.3. Assume that $g^{jk}(x,t)$ are $C^{1,1}(\Omega \times [0,T])$. Then there exist constants $C_{11}$ and $C_{21}$ such that for all $\lambda$ sufficiently large,

\begin{equation}
\|u_0\|_{L^2(\Omega)} + \|u_1\|_{H^{-1}(\Omega)} \leq C_{11}\|u\|_{L^2(\Omega_0 \times [0,T])} + \frac{C_{21}}{\lambda^{1/3}} + C_{21}C_1(M_0),
\end{equation}

where $C_1(M_0)$ is a positive constant depending on $M_0$ such that $C_1(M_0) \to 0$ as $M_0 \to 0$.

Corollary 2.4. As $M_0 \to 0$, and $\lambda \to \infty$, this gives a perfect observability estimate:

\begin{equation}
\|u_0\|_{L^2(\Omega)} + \|u_1\|_{H^{-1}(\Omega)} \leq C_{11}\|u\|_{L^2(\Omega_0 \times [0,T])}.
\end{equation}

Even though the second theorem requires less regularity, we show both theorems for completeness because the computations will indicate that the higher the regularity the sharper the estimates. Moreover, if the modulus of continuity for the derivative of the metric is too large in the case of $C^{1,1}$ coefficients, meaningful error estimates are not possible. In the second corollary, the equation is close to $\partial_t^2 u - \Delta u = 0$, for which the ODEs defining the Hamiltonian flow can be solved in an exact explicit way. In the limit, the oscillatory phase functions then match the classical wave-packets popularized in Córdoba-Fefferman [7].

3. Ansatz construction. If the reader is expecting a typical Ansatz to the wave equation, that will not be found here as $C^3$ coefficients are necessary for easy control over the error terms. The low regularity of the coefficients increases the complexity of the construction. We must, as such, modify the existing construction for Gaussian beams to low regularity coefficients. Gaussian beams are asymptotically valid high frequency solutions to hyperbolic differential equations which
are concentrated on a single physical curve in the domain. One can extend them to dispersive equations such as the Schrödinger equation. We present a systematic construction of Gaussian beams for the wave equation, but modified to handle the extremely low regularity coefficients. Gaussian beams have a long history, starting with Keller [15] and Maslov and Fedoriuk [19]. Related complex phase Fourier integral operators were introduced by Hörmander [11]. We follow loosely for our parametrix construction the one found in [8, Section 64], which is for real phase Fourier integral operators. The major differences to [8] are that we have to be more careful about the treatment of our error terms, and we continue his construction to include the Gaussian beam tails which are the complex part of the phase.

We let $H_0$ be the principal symbol of the operator (2-1). One can factor the symbol as

\[(3-1)\quad H_0(x, t, \xi, \tau) = (\tau - \lambda_1(x, t, \xi))(\tau - \lambda_2(x, t, \xi)),\]

where the roots are

\[(3-2)\quad \lambda_1 = \sqrt{\sum_{j,k=1}^{n} g^{jk}(x, t) \xi_j \xi_k}, \quad \lambda_2 = -\sqrt{\sum_{j,k=1}^{n} g^{jk}(x, t) \xi_j \xi_k}.

We now proceed to construct a parametrix to (2-5).

**Theorem 3.1.** Assume $g^{jk}(x, t)$ are $C^2(\Omega \times [0, T])$. Then for every $\epsilon > 0$ and $\lambda = \lambda(\epsilon)$ sufficiently large, there is an approximate solution $U_\lambda(x, t)$ to the problem (2-5) such that

\[(3-3)\quad \|u - U_\lambda(x, t)\|_{L^2(\Omega \times [0, T])} \leq C_{22} C_2(\epsilon),\]

where $C_{22}$ depends on $\|f_1\|_{C^3(\Omega)} + \|f_2\|_{C^2(\Omega)}$, diam$_g'(\Omega')$, $T$, $\alpha_0$ and $\|g^{jk}(x, t)\|_{C^2(\Omega \times [0, T])}$. In particular, we have that $C_2(\epsilon) \to 0$ as $\epsilon \to 0$.

For the lower regularity coefficients we have a second, slightly weaker characterization of the errors in the Ansatz.

**Theorem 3.2.** Assume $g^{jk}(x, t)$ are $C^{1,1}(\Omega \times [0, T])$. For every $\lambda$ sufficiently large, there exists a solution $U_\lambda(x, t)$ to the problem (2-5) such that

\[(3-4)\quad \|u - U_\lambda(x, t)\|_{L^2(\Omega \times [0, T])} \leq C_{21} C_1(M_0) + \frac{C_{21}}{\sqrt{\lambda}},\]
where $C^{21}$ depends on $\|f_1\|_{C^3(\Omega)} + \|f_2\|_{C^2(\Omega)}$, $\text{diam}_{g'}(\Omega')$, $T$, $\alpha_0$ and $\|g^{jk}(x,t)\|_{C^{1,1}(\Omega \times [0,T])}$. Recall that $M_0$ is the modulus of continuity for $\nabla_x g^{jk}(x,t)$. In particular, we have that $C_1(M_0) \to 0$ as $M_0 \to 0$.

We call the factor $M_0$ the relative error in the problem. It is not possible to obtain the second set of estimates without the relative error.

We concentrate on the $C^2$ coefficient case first. We look for a solution to (2-5) of the form

$$U_\lambda(x,t) = \sum_{j=1}^{2} \sum_{k=0}^{2} \lambda^{n/4} a_{jk}(x,t,\eta) \exp(i\psi_j(x,t,\eta)),$$

where $a_{jk}(x,t,\eta) \in C^3(\Omega \times [0,T])$, and $\eta = \lambda \omega_0$, with $\lambda \in \mathbb{R}^+$ and $\omega_0 \in \mathbb{R}^n$ such that $|\omega_0| = 1$. We let $v_j(x,t,\eta) \in C^3(\Omega \times [0,T])$ be the correction terms in the equation, so we have

$$u(x,t) = U_\lambda(x,t) + \lambda^{n/4} \sum_{j=1}^{2} v_j(x,t,\eta) \exp(i\psi_j(x,t,\eta)).$$

Substituting (3-5) into (2-5) and grouping the powers of $\eta$, we obtain the following set of equations:

$$H_0(x,t,\psi_{jx},\psi_{jt}) = 0, \quad j = 1, 2,$$

$$\frac{\partial H_0(x,t,\psi_{jx},\psi_{jt})}{\partial \xi_0} \frac{\partial a_{j0}}{\partial t} + \sum_{k=1}^{n} \frac{\partial H_0(x,t,\psi_{jx},\psi_{jt})}{\partial \xi_k} \frac{\partial a_{j0}(x,t,\eta)}{\partial x_k} + \left(H_0(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}) \psi_j\right) a_{j0} = 0,$$

$$L_j a_{jk} = -iH(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}) a_{j,k-1}, \quad k \geq 1, j = 1, 2.$$

Here $L_j$ is the operator on the left-hand side of the second equation in (3-7), that is,

$$L_j = \frac{\partial H_0(x,t,\psi_{jx},\psi_{jt})}{\partial \xi_0} \frac{\partial}{\partial t}$$

$$+ \sum_{k=1}^{n} \frac{\partial H_0(x,t,\psi_{jx},\psi_{jt})}{\partial \xi_k} \frac{\partial}{\partial x_k} + \left(H_0(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}) \psi_j\right).$$

We need to be able to solve these equations to some order along a centralized curve. To this end, it helps if we use Cauchy data for (2-5) with Gaussian packets rather than nonoscillatory $L^2$ functions.
In order to find the Cauchy conditions we look at the following initial conditions on \( a_{jk}, j = 1, 2 \). We obtain

\[
\sum_{j=1}^{2} \sum_{k=0}^{2} a_{jk}(x, 0, \eta) = f_1(x),
\]

\[
i \psi_1 t(x, 0, \eta) \sum_{k=0}^{2} a_{1k}(x, 0, \eta) + i \psi_2 t(x, 0, \eta) \sum_{k=0}^{2} a_{2k}(x, 0, \eta)
\]

\[
+ \sum_{j=1}^{2} \sum_{k=0}^{2} \frac{\partial a_{jk}(x, 0, \eta)}{\partial t} = f_2(x).
\]

Because \( \psi_j(x, 0, \eta) = \lambda_j(x, 0, \eta) \) for \( j = 1, 2 \) we can find \( a_{10}(x, 0, \eta) \) and \( a_{20}(x, 0, \eta) \) as the unique solution of the algebraic \( 2 \times 2 \) system

\[
a_{10}(x, 0, \eta) + a_{20}(x, 0, \eta) = f_1(x),
\]

\[
i \lambda_1(x, 0, \eta) a_{10} + i \lambda_2(x, 0, \eta) a_{20} = f_2(x)
\]

as the determinant of this system is \( \lambda_2 - \lambda_1 \neq 0 \), for all \( x, t, \eta \neq 0 \). We determine \( a_{jk}(x, 0, \eta) \) from the equations

\[
a_{1k}(x, 0, \eta) + a_{2k}(x, 0, \eta) = 0,
\]

\[
i \lambda_1(x, 0, \eta) a_{1k} + i \lambda_2(x, 0, \eta) a_{2k}
\]

\[
= - \frac{\partial a_{1,k-1}(x, 0, \eta)}{\partial t} - \frac{\partial a_{2,k-1}(x, 0, \eta)}{\partial t}, \quad 1 \leq k \leq 2.
\]

We see that for some \( t_j(x, t, \eta) \in L^2(\Omega' \times [0, T]) \), \( j = 1, 2 \),

\[
HU_{\lambda}(x, t) = \sum_{j=1}^{2} \lambda^{n/4} t_j(x, t, \eta) \exp(i \psi_j).
\]

If we set \( u - U_{\lambda}(x, t) = w(x, t, \eta) \), then it follows that

\[
Hw(x, t, \eta) = - \sum_{j=1}^{2} \lambda^{n/4} t_j(x, t, \eta) \exp(i \psi_j),
\]

\[
w(x, 0, \eta) = 0, \quad \frac{\partial w(x, 0, \eta)}{\partial t} = - \lambda^{n/4} t_j(x, 0, \eta) \exp(i \psi_j).
\]

Hence we can relabel \( w(x, t, \eta) = w_{\lambda}(x, t) \). In order to bound the errors, we recall the following somewhat classical energy estimate. Let \( u \) be a solution to the system
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\[ Hu = f(x,t) \quad \text{in } \Omega \times [0,T], \]
\[ u(x,0) = u_0, \quad \partial_t u(x,0) = u_1 \quad \text{in } \Omega, \]
\[ u(x,t) = 0 \quad \text{on } \partial\Omega \times [0,T], \]

with \( f(x,t) \in L^2(\Omega \times [0,T]), \) \( u_0 \in H^1(\Omega) \) and \( u_1 \in L^2(\Omega). \) The following holds for the above Cauchy problem.

**Theorem 3.3.** There exists \( C \) depending on \( \alpha_0 \) and \( \| g^{jk} \|_{C^{1,1}(\Omega \times [0,T])}, \) and \( \tilde{A}_1 \) depending on \( \| g^{jk} \|_{C^{1,1}(\Omega \times [0,T])} \) such that
\[
\| u \|_{C([0,T];H^1(\Omega)) \cap C^1([0,T];L^2(\Omega))} 
\leq C(\| u_0 \|_{H^1(\Omega)} + \| u_1 \|_{L^2(\Omega)} + \| f(x,t) \|_{L^2(\Omega \times [0,T])}) \exp(\tilde{A}_1 T).
\]

This particular variant is an exercise in integration by parts but is also found in the literature; see Liones and Magenes [18]. In order to obtain the desired result, one needs to gain precision over \( t_j \) for \( j = 1, 2 \) in order to use the energy estimate above. These estimates are where we use the regularity hypothesis to build our Gaussian beam. We chose the phase functions \( \psi_j \) to be real-valued along the curve \( x = x(t), \) and we fix the initial value of the phase function as
\[
\psi(0, x, \eta) = (i|\eta||x-x_0|^2/2 + (x-x_0) \cdot \omega_0)|\eta|.
\]
We assume \( \psi \) is positive homogeneous of degree one in \( \| \eta \|, \) so we can write
\[
\psi(x,0, \eta) = |\eta|\tilde{\psi}(0, x, \omega_0) = \lambda\tilde{\psi}(x,0, \omega_0) = \lambda\psi_0.
\]
We now work with \( \tilde{\psi} \) and drop the tilde everywhere where it is understood.

We can write the phase function in a Taylor series. Note that for the rest of this paper all of the \( o \) and \( \mathcal{O} \) terms are considered uniform over compact sets.

**Lemma 3.4** (from [29]). Given initial conditions (3-17), for \( \beta = 1, 2 \) we have
\[
\psi_1^\beta(x,t) = (x-x(t)) \cdot \omega(t),
\]
\[
\psi_2^\beta(t,x) = M_{ij}(t)(x-x(t))^i(x-x(t))^j,
\]
where $M(t)$ is a matrix such that $\text{Im} M(t)$ is positive definite, and

$$
\psi(t, x) = \psi_1^2(x, t) + \psi_2^2(x, t)
$$

solves the eikonal (first equation in (3-7)) to order $o(|x - x(t)|^2)$, provided $g^{jk}(x, t) \in C^2(\Omega \times [0, T])$.

**Proof.** The first equation in (3-7) can be recast in the form

$$
\psi_{1t} = h(x, t, \psi_{1x}).
$$

We suppress the subscript 1 where it is understood, with the same estimates holding for the subscript 2 but with $\lambda_2$ as the root of (3-7). Working backwards, if we differentiate the equation (3-19) with respect to $x$ we obtain the relations

$$
\psi_{tx} - h_{pi}(x, t, \psi_x)\psi_{xi}x_j = h_{xj}(x, t, \psi_x),
\psi_{tt} - h_{pi}(x, t, \psi_x)\psi_{xt} = 0,
$$

(3-20) \begin{align*}
\psi_{txj}x_k - h_{pi}(x, t, \psi_x)\psi_{xjx_k} &= h_{xjx_k}(x, t, \psi_x) + h_{xjp}(x, t, \psi_x)\psi_{xix_j}x_k \\
&\quad + h_{xjp}(x, t, \psi_x)\psi_{xix_j}x_k + h_{pim}(x, t, \psi_x)\psi_{xix_j}\psi_{xix_m}x_k.
\end{align*}

To simplify this set of relations we consider the matrices $A$, $B$, and $C$ which are defined with entries as follows:

$$
A^j_i = \{h_{xi}(x(t), t, \omega(t))\},
B^j_i = \{h_{xi}(x(t), t, \omega(t))\},
C^j_i = \{h_{pi}(x(t), t, \omega(t))\}.
$$

If we set $\nabla_x \psi(t, x(t)) = \omega(t)$ we know that the phase must satisfy (3-20) along the path $\{(t, x(t)) : 0 \leq t \leq T\}$. The equations then become

$$
\begin{cases}
\frac{dx(t)}{dt} = -h_p(x(t), t, \omega(t)), \\
\frac{d\omega(t)}{dt} = h_x(x(t), t, \omega(t)), & (x(0), \omega(0)) = (x_0, \omega_0) \in S\Omega,
\frac{d\psi}{dt}(x(t), t) = 0,
\end{cases}
$$

(3-22)

$$
\frac{dM}{dt} = A + BM + MB^t + MCM.
$$

(3-23)

The set (3-22) is well posed for some finite time $T$ by the Picard existence theorem [27, Theorem 3]. The last equation (3-23) is a matrix Riccati
equation associated to (3-22). It is a nonlinear equation which is not always well posed. From the equation \( \psi_t(x(t), t) = 0 \) in (3-22), and the initial condition, this implies \( \psi_1(t) = \omega(t) \) as claimed. The crucial choice is therefore the Hessian \( M(t) \), which is associated to the second-order terms in \( (x - x(t)) \). We chose the initial condition \( M(0) = iI \). We also associate the matrices \( Y(t) \) and \( N(t) \) to the Hessian \( M(t) \). Now we let \( Y(t) \) and \( N(t) \) satisfy the system

\[
\begin{align*}
\frac{dY}{dt} &= -B^t Y - CN, \\
\frac{dN}{dt} &= AY + BN,
\end{align*}
\]

(3-24) \( (Y(0), N(0)) = (I, iI) \).

We claim whenever \( (Y(t), N(t)) \) is a solution to (3-24), then \( Y(t) \) is invertible, and the solution \( M(t) = N(t)Y^{-1}(t) \) to (3-23) exists for all bounded time intervals, if and only if \( \text{Im} M(t) \) is positive definite. With the given initial conditions (3-17), this is equivalent to the claim that we can find a phase function such that \( \text{Im} M(t) \) is positive definite. The computation is now standard in the literature and we will not include it here; for details, see [29, Theorem 1] or [23].

Now we proceed to find the coefficients \( a_j(x, t) \) of the beam. The linear operator \( L \) is the transport operator which acts on functions \( a(x, t) \) by

\[
L a = 2\psi_t a_t - 2g^{kl} \psi_{x_k} a_{x_l} + (H\psi)a.
\]

It is natural to consider \( a_j(x, t) \) as a sum of homogeneous polynomials with respect to \( x - x(t) \) as well, so we Taylor expand

\[
a_j(x, t) = \sum_{0 \leq l} a_{j,l}(t)(x - x(t))^l.
\]

(3-25)

The natural number \( l \) depends on the regularity of the coefficients and the regularity of the \( g^{jk} \). For our purposes, we only use \( l = 0 \), which suffices for this article, even though sometimes we have \( C^2 \) coefficients. From this identity we can match up terms in our Taylor series expansion. Using (3-22), we obtain a differential equation for \( a_{j,l}(t) \):

\[
\frac{d}{dt} a_{j,l}(t) + r(t)a_{j,l}(t) = F_{j,l}(t).
\]

(3-26)

The right-hand side is a homogenous polynomial of order \( l \) in \( x - x(t) \) which depends on \( a_{j,l}(t) \) and \( \psi_j \). The factor \( r(t) \) comes from computing
$H\psi$ along the curves (3-22). We obtain ordinary differential equations defining $a_{j,l}(t)$ as

$$\frac{d}{dt} a_{j,l}(t) + \left( \frac{d}{dt} \sigma(t) \right) a_{j,l}(t) = F_{j,l}(t)$$

for some $\sigma(t)$ such that $r(t) = \frac{d\sigma(t)}{dt}$. Solutions to these equations are given by

$$a_{j,l}(t) = \sigma(t) \left( a_{j,l}(0) + \int_0^t \sigma^{-1}(s) F_{j,l}(s) \, ds \right). \tag{3-27}$$

A lengthy computation in [8, Section 64.3] results in

$$\sigma(t) = \frac{1}{\sqrt{|\partial x(t)/\partial x_0|}} \exp \left( \int_0^t F(x(s),s) \, ds \right) \tag{3-28}$$

with

$$F(x,t) = -\frac{1}{\psi_t(x,t)} \sum_{p,k=1}^n g^{pk}_{x_k}(x,t) \psi_{x_p} - \frac{1}{4\psi_t^2} \sum_{p,k=1}^n g^{pk}_{t}(x,t) \psi_{x_p} \psi_{x_k}. \tag{3-29}$$

When finding $\sigma(t)$, evaluating at $x = x(t)$, we note that on null bicharacteristics $\nabla_x \psi = \omega(t)$ and $\psi_t = \lambda_i(t,x(t))$ for $i = 1, 2$.

**Lemma 3.5.** For $\sigma(t)$ as in (3-28) above, we can write

$$a_0(t,x) = \sigma(t)a(0,x) + O(|x - x(t)|).$$

**Proof.** We can compute the first few terms given by (3-27). For the case of $C^2$ coefficients the error terms above are correct and make sense. Since we know that $F_{0,0} = 0$, we compute

$$a_{0,0}(t) = a_{0,0}(0)\sigma(t), \tag{3-30}$$

completing the proof. \qed

We now continue with the proof of the main theorem by bounding the error term $w_\lambda(x,t)$. Assuming that $H(x,t,\xi,\tau) = H(\infty, t, \xi, \tau)$ for $|x| > R$, then $w_\lambda(x,t) \equiv 0$ for all $|x| > R$, by finite speed of propagation. Assuming $\text{diam}(\Omega') > R$, $w_\lambda(x,t)$ satisfies the hypothesis of Theorem 3.3.
Inserting $U_\lambda(x,t)$ into the wave equation we now consider the sets

$$A_\lambda = \{ x \in \Omega' : |x-x(t)| \leq \sqrt{B} \lambda^{-1/2}, \ 0 < t \leq T \},$$
$$A^c_\lambda = \{ x \in \Omega' : |x-x(t)| > \sqrt{B} \lambda^{-1/2}, \ 0 < t \leq T \},$$

where $B$ is a positive constant. We will determine the optimal $B$ shortly.

From Taylor’s theorem, assuming $g^{jk}(x,t)$ has two derivatives, the phase can be expanded to second-order around $|x-x(t)|$ with error which is $o(|x-x(t)|^2)$. The usual error is $O(|x-x(t)|^3)$ for a two term expansion of a $C^3$ phase. The higher the regularity, the better the error estimates in Taylor’s theorem, and the more accurate the Ansatz construction.

Applying these definitions we can prove the following lemma.

**Lemma 3.6.** In the set $A_\lambda$, each $t_j$, $j = 1, 2$, can be bounded as

\begin{equation}
(3-31) \quad t_j = O(\epsilon \lambda^2 |x-x(t)|^2) + O(\epsilon \lambda |x-x(t)|),
\end{equation}

and in $A^c_\lambda$,

\begin{equation}
(3-32) \quad t_j = O(\lambda^2 |x-x(t)|^2) + O(\lambda |x-x(t)|).
\end{equation}

**Proof.** The first in the set of order terms of (3-31) depends on the $C^2$ norm of $g^{jk}$ and the $L^2(\Omega \times (0,T))$ norm of $a(x,t)$, from solving (3-7) up to $o(|x-x(t)|)$, using the Peano form of the remainder in Taylor’s theorem. The second in the set of order terms in (3-31) depends on the $C^2$ norm of $g^{jk}$ and the $H^3(\Omega \times (0,T))$ norm of $a(x,t)$, from solving (3-8) up to $o(|x-x(t)|)$. In other words, to obtain the first set of order terms, we have that for all $\epsilon > 0$ there exists a $\lambda(\epsilon)$ sufficiently large so that for the remainder, say $r_2(x,t)$, we have that

\begin{equation}
(3-33) \quad |r_2(x,t)| \leq \epsilon |x-x(t)|^2
\end{equation}

whenever $|x-x(t)| \leq \sqrt{B} \lambda^{-1/2}$ provided that $\lambda \geq \lambda(\epsilon)$. Away from this region, in $A^c_\lambda$ it is possible to approximate the remainder, but only to $O(|x-x(t)|^2)$, whence the equality (3-32). This computation is similar for the transport equation, and therefore the lemma is proved. \hfill \Box

**Proof of Theorem 3.1.** We define

\begin{equation}
(3-34) \quad \int_{-\infty}^{b} \exp(-x^2) \, dx = \text{erfc}(b), \quad \int_{0}^{b} \exp(-x^2) \, dx = \text{erf}(b).
\end{equation}
We know that the exponential function admits the asymptotics

\[
\text{erfc}(b) = \frac{\exp(-b^2)}{2b} + O\left(\frac{\exp(-b^2)}{b^3}\right)
\]

from Example 4 on page 255 of [4], whenever \( b \) is sufficiently large.

We now proceed to estimate

\[
\lambda^{n/2} t_j \exp(i\lambda \psi_j)
\]

in \( L^2(\Omega') \) norm. Using the integrals (A-7) and the fact

\[
|x|^2 = x_1^2 + \cdots + x_n^2
\]

for \( x = (x_1, x_2, \ldots, x_n) \), we know

\[
\int_{A_{\lambda}} \epsilon |x - x(t)|^2 \lambda^2 \exp(-\lambda |x - x(t)|^2) \, dx \\
+ \epsilon \lambda |x - x(t)| \exp(-\lambda |x - x(t)|^2) \, dx \\
\leq C\lambda \left( B\epsilon + \frac{\sqrt{B}\epsilon}{\sqrt{\lambda}} \right) \text{erf}(\sqrt{B}),
\]

with \( C \) independent of \( \lambda \) and \( \epsilon \). We also have that

\[
\int_{A_{\lambda}^\delta} |x - x(t)|^2 \lambda^2 \exp(-\lambda |x - x(t)|^2) \\
+ \lambda |x - x(t)| \exp(-\lambda |x - x(t)|^2) \, dx \\
\leq C\lambda \text{erfc}(\sqrt{B}),
\]

with \( C \) independent of \( \lambda \) and \( \epsilon \) depending on \( \text{diam}(\Omega') \).

Combining (3-38) and (3-37), after making the change of variables

\[
\sqrt{\text{Im} \, M(t)} \, |x - x(t)| \rightarrow |x - x(t)|,
\]

one sees that for \( B, \lambda > 1 \),

\[
\| \lambda^{n/4} t_j \exp(i\lambda \psi_j) \|_{L^2(\Omega')} \leq C\lambda \left( B\epsilon + \text{erfc}(\sqrt{B}) \right)
\]

with \( C \) independent of \( \lambda \) and \( \epsilon \) depending on \( \text{diam}(\Omega') \) and \( \text{Im} \, M(t) \).

Using the energy estimates in Theorem 3.3 and the asymptotic (3-35), one obtains

\[
\| w_\lambda(x, t) \|_{\dot{H}^1_0(\Omega' \times (0, T))} \leq \int_0^T \sum_{j=1}^2 \| \lambda^{n/4} t_j \exp(i\psi_j) \|_{L^2(\Omega')} \, dt \\
\leq C_{22}\lambda \left( B\epsilon + \exp(-B) \right),
\]
from which it follows from the form of the remainder that

\[(3-41) \quad \|w_\lambda(x, t)\|_{L^2(\Omega' \times [0, T])} \leq C_{22}(B\epsilon + \exp(-B)).\]

The constant \(B\) is arbitrary, so we can take the minimum in \(B\) of the right-hand side (\(\sim \log(\epsilon^{-1})\)) to reach the desired conclusion. \(\square\)

For the case of \(g^{jk}(x, t) \in C^{1,1}(\Omega \times [0, T])\), instead of the Gaussian beam tail, we no longer have the ability to construct the Hessian matrix. We use as our initial data

\[(3-42) \quad (x-x_0) \cdot \omega_0 + i|x-x_0|^2\]
as before, but we propagate it as

\[(3-43) \quad (x-x(t)) \cdot \omega(t) + i|\omega(t)||x-x(t)|^2).\]

In order to prove Theorem 3.2 we need to be able to solve the eikonal, but only to first-order. We apply the operator \(H\) to the function and we obtain the following error terms.

**Lemma 3.7.** We have in the case \(g^{jk}(x, t) \in C^{1,1}(\Omega \times [0, T])\) that

\[|H_0(x, t, \psi_x, \psi_t)| \leq 10M_0|\omega(t)|^2|x-x(t)|^2.\]

**Proof.** This is a refinement of Lemma 3 in [28]. As \(H_0(x, t, \psi_x, \psi_t)\) is positive homogeneous of degree two in \(|\omega(t)|\), we start by showing on the curve defined by (3-22) that

\[\nabla_x H_0(x(t), t, \psi_x(t, x(t)), \psi_t(t, x(t))) = 0,\]

and then refine the o(|x-x(t)|) terms from the Peano form of the remainder. Computing \(\nabla_x H_0(x, t, \psi_x, \psi_t)\), we have that

\[(3-44) \quad \frac{\partial}{\partial x_j}H_0(x, t, \psi_x, \psi_t) = H_{0x_j} + H_{0\xi} \psi_{x_j} + H_{0\tau} \psi_{tx_j}.\]

Dividing (3-44) by 2\(\lambda_1\) or 2\(\lambda_2\) and substituting the equations in (3-22) into the right-hand side, we obtain for the right-hand side of (3-44)

\[(3-45) \quad -\frac{d\omega_j}{dt} + \frac{dx_j}{dt} \psi_{x_j} + \psi_{tx_j}.\]
As \( \psi_{xj}(t, x(t)) = \omega_j(t) \), differentiating \( \omega_j(t) \) with respect to \( t \) we have
\[
\frac{d\omega_j}{dt} = \frac{dx_i}{dt} \psi_{xxj} + \psi_{txj}.
\]
(3-46)
Substituting (3-46) into (3-45) implies (3-45) is 0, which happens if and only if (3-44) vanishes along the Hamiltonian flow defined by (3-22).

We can now write
\[
H_0(x, t, \psi_x(t, x), \psi_t(t, x))
= H_0(x(t), t, \psi_x(t, x(t)), \psi_t(t, x(t))) \\
+ \nabla_x H_0(x(t), t, \psi_x(t, x(t)), \psi_t(t, x(t)))(x - x(t)) \\
+ r_1(t, x)(x - x(t)).
\]
(3-47)
We abbreviate \( H_0(x(t), t, \psi_x, \psi_t) = H_0(x(t)) \). We note that if \( h = x - x(t) \), the remainder \( r_1(t, x) \) is
\[
r_1(t, x) = \frac{H_0(x(t) + h) - H_0(x(t))}{h} - \nabla_x H_0(x(t)).
\]
(3-48)
The result will follow if we can prove
\[
|r_1(t, x)| \leq 10M_0|\omega(t)|^2|h|
\]
(3-49) because we just showed \( \nabla_x H_0(x(t)) = 0 \) and \( H_0(x(t)) = 0 \) by definition of the ODEs (3-22). The mean value theorem applied to (3-48) implies that it suffices to prove for some \( c \in (0, h) \)
\[
|\nabla_x H_0(x(t) + c) - \nabla_x H_0(x(t))| \leq 10M_0|h| |\omega(t)|^2.
\]
(3-50)
But this is follows, as setting \( x(t) + c = \tilde{c} \), we have that
\[
|\nabla_x \left( \sum_{jk} g^{jk} \psi_{xj} \psi_{xk} \right)(\tilde{c}) - \nabla_x \left( \sum_{jk} g^{jk} \psi_{xj} \psi_{xk} \right)(x(t))| \\
\leq 2M_0 |\omega(t)|^2|h|.
\]
(3-51)
In the last inequality we used the fact that we have an explicit form for \( \psi \), so we can expand \( \psi_{xj} \psi_{xk} \) to the correct order, and also the inequality
\[
|\nabla_x (\psi_t^2)(\tilde{c}) - \nabla_x (\psi_t^2)(x(t))| \leq 2M_0 |\omega(t)|^2|h|,
\]
(3-52)
which follows by a short computation writing down the explicit form of \( \psi_t \) and using (3-22), with \( \psi_t(t, x(t)) = \lambda_1(t, x(t)) \).
\( \square \)
Proof of Theorem 3.2. Similarly to the proof of Theorem 3.1, we claim \( U_\lambda + w_\lambda \) is a parametrix solution to (2-5), but the phase functions now have the special form in (3-43). In particular, using the Peano form of the remainder, but this time with only one term in the Taylor expansion, the error term from solving (3-7) is then estimable using the inequalities

\[
\lambda^{n/2} \int_{A_\lambda} M_0 \lambda^2 |x - x(t)|^2 \exp(-\lambda |x - x(t)|^2) \, dx \leq C\lambda BM_0
\]

and

\[
\lambda^{n/2} \int_{A_\lambda} \lambda^2 |x - x(t)| \exp(-\lambda |x - x(t)|^2) \, dx \leq C\lambda \text{erfc}(-\sqrt{B}).
\]

The other terms from solving the transport equation (3-8) are bounded in a similar way. From Theorem 3.3 and Corollary 3.5, we can again conclude the desired result after using the asymptotic for the error function (3-35) and minimizing in \( B \).

4. Proof of observability estimates. We write

\[
\|u\|_{L^2(\Omega_0 \times [0,T])}^2 = \lambda^{n/2} \int_0^T \int_{\Omega_0} \left( \sum_{j=1}^2 (a_{0j1} \exp(i\lambda \psi_j)) \right)^2 \, dx \, dt + E_l,
\]

where \( \lambda \) denotes the regularity of the coefficients, with \( l = 1 \) corresponding to \( C^{1,1} \) coefficients and \( l = 2 \) corresponding to \( C^2 \) coefficients. Notice that for \( C^2 \) coefficients we have the higher order \( a_{1j2}, j = 1, 2 \), available for a more accurate construction of the Ansatz, but it is not necessary, for the observability estimates. The term \( E_l, l = 1, 2 \), is bounded by Theorems 3.2 and 3.1 respectively. The cross terms involving \( \exp(i\lambda \psi_1)\exp(i\lambda \psi_2) \) are oscillatory and can be discarded by stationary phase, as by definition, we have that \( \omega_{11}(t) = -\omega_{22}(t) \) and \( \nabla_x (\psi_{11} + \overline{\psi_{22}}) = 2\omega_{11}(t) \neq 0 \). Indeed, separating the phase into real and imaginary parts, applying Lemma A.3 and using the integrals (A-7) gives

\[
\lambda^{n/2} \int a_{01l} \overline{a_{02l}} \exp(i\lambda \psi_1)\exp(i\lambda \psi_2) \, dx \leq C \sqrt{\lambda} \|a_{01l}a_{02l}\|_{C^1(\Omega)}
\]

with \( C \) depending on \( \Omega \) and \( \|g^{jk}\|_{C^{1,1}} \). We use the fact that

\[
(b - d)^2 + (b + d)^2 = 2(b^2 + d^2)
\]
along with the initial conditions to conclude

\begin{equation}
\lambda^{n/2} \int_0^T \int_{\Omega_0} \sigma_l^2(t) \left( f_1^2(x(0)) + \frac{f_2^2(x(0))}{\lambda^2(t, x(0))} \right) \exp(-\lambda \beta(t) |x - x(t)|^2) \, dx \, dt \\
\leq \|u\|_{L^2(\Omega_0 \times [0,T])}^2 + E_1,
\end{equation}

where \( \beta_2(t) = 2 \text{Im } M(t) \) and \( \beta_1(t) = |\omega(t)| \). Now we need a lemma:

\textbf{Lemma 4.1.} We have that \( \sigma_l(t)(\beta_l(t))^{n/2} = C_l(t) > 0 \) with \( l = 1, 2 \) and \( \sigma_l(t) \) given by (3-28).

\textbf{Proof.} We need to solve \( H \psi = 0 \) to first-order around \( x(t) \), as indicated earlier. The function \( \sigma_l(t) \) in (3-28) is strictly positive for both \( l = 1, 2 \) and all finite \( t \). Moreover,

\[ |\omega(t)| > \exp\left(-\frac{M_0}{\alpha_0} t\right) |\omega_0| \]

is also strictly positive by Gronwall’s inequality and \( \text{Im } M(t) \) is positive definite for finite \( t \). Thus we can conclude the existence of such a \( C(t) > 0 \). We remark that the inequality for \( l = 2 \) can also be found as a result of \cite[Lemma 2.58]{16}.

\textbf{Proof of Theorem 2.1.} Whenever \( x \in A_\lambda \) the main term is small, as we know that

\begin{equation}
\lambda^{n/2} \int_{A_\lambda \cap \Omega_0} \exp(-\lambda |x - x(t)|^2) \, dx \leq \text{erfc}(\sqrt{B})
\end{equation}

if \( B = B_{\text{min}}(\epsilon) \).

To find a lower bound for the integral over \( A_\lambda \cap \Omega_0 \), we remark that the set \( A_\lambda \cap \Omega_0 \) contains a ball of volume \( \lambda^{-n/2} \) whenever \( \Omega_0 \) is Lebesgue measurable and \( \lambda \) is sufficiently large, provided \( x(t) \in \Omega_0 \) for some \( t, 0 \leq t \leq T \). We know that \( x(0) \mapsto x(t) \) is connected, as \( x(t) \) traces out a continuous curve. Thus the GCC is automatically satisfied to get any meaningful estimate, as otherwise this ball could be empty.

One necessarily has

\begin{equation}
C_2(T) f_1^2(x(0)) \exp(-1) \leq \|u\|_{L^2(\Omega_0 \times [0,T])}^2 + E_2
\end{equation}
and similarly for \( f_2^2(x(0))/\lambda_1^2(0,x(0)) \), with \( C_2(T) = \int_0^T C_2(t) \, dt \). We now obtain

\[
(4-7) \quad C_2(T) \left( f_1^2(x(0)) + \frac{f_2^2(x(0))}{\lambda_1^2(0,x(0))} \right) (\exp(-1) - \text{erfc}\sqrt{B}) 
\leq \|u\|_{L^2(\Omega_0 \times [0,T])}^2 + C_{21}C_2(\epsilon).
\]

When \( B = B_{\text{min}}(\epsilon) \), then the factor \((\exp(-1) - \text{erfc}\sqrt{B})\) is positive by the asymptotic \((3-35)\), and \( \epsilon \) sufficiently small. Applying Lemma A.1 from the appendix to approximate \( f_1^2(x(0)) \) and \( f_2^2(x(0))/\lambda_1^2(0,x(0)) \) gives the desired result. \(\square\)

**Proof of Theorem 2.3.** The steps of the proof follow exactly the same using the appropriate inequality in Lemma 4.1, except for the fact that the error term \( E_1 \) is now bounded by \( C_1(M_0) \) with \( C_1(M_0) \to 0 \) as \( M_0 \to 0 \). Thus, the parameter \( M_0 \) takes the place of the parameter \( \epsilon \) in the above proof. \(\square\)

**Appendix: Convergence lemmas.**

We prove the following result, similar to [29]:

**Lemma A.1.** Let \( h(t,x) \in C^1((0,T) \times O) \), where \( O \) is an open subset of \( \mathbb{R}^n \) and \( B \) is a symmetric nonsingular matrix such that \( \Re B \geq 0 \). If \( x(t) \) is a continuous curve defined in terms of \( t \) in \( O \), then we have the uniform estimate

\[
(A-1) \quad \left| \left( \frac{\lambda}{\pi} \right)^{n/2} (\det B)^{1/2} \cdot \int_O \exp(\langle -\lambda B(x-x(t)), (x-x(t)) \rangle) h(t,x) \, dx - h(t,x(t)) \right| 
\leq \left( \frac{2\lambda \sigma}{\sqrt{\lambda}} + 4\text{erfc}(-\lambda^{2\sigma}) \right) \|h(t,x)\|_{C^1((0,T) \times O)}
\]

with \( \sigma \in (0,1/6) \).

**Proof.** The assumption that \( h(t,x) \) is in \( C^1([0,T] \times O) \) implies that \( h(t,x) \) is locally uniformly Lipschitz continuous with Lipschitz constant \( \|h(t,x)\|_{C^1((0,T) \times O)} \). We set \( \epsilon = \lambda^{\sigma-1/2} \|h(t,x)\|_{C^1((0,T) \times O)} \). We know
that for $\eta = \lambda^{\sigma - 1/2}$, if $x$ is such that $|x - x(t)| < \eta$, this implies

$$|h(t, x) - h(t, x(t))| < 2\frac{\lambda^{\sigma}}{\sqrt{\lambda}} \|h(t, x)\|_{C^1((0, T) \times \mathcal{O})}.$$ 

Using change of variables, we then obtain the bounds

$$\left| \left( \frac{\lambda}{\pi} \right)^{n/2} (\det B)^{1/2} \int_{\mathcal{O}} \exp \left( \langle -\lambda B(x-x(t)), (x-x(t)) \rangle \right) h(t, x) \, dx \right|$$

$$< \frac{2\lambda^{\sigma}}{\sqrt{\lambda}} \|h(t, x)\|_{C^1((0, T) \times \mathcal{O})} \int_{|y| \leq C_\eta} \left( \frac{\lambda}{\pi} \right)^{n/2} \exp(-\lambda|y|^2) \, dy$$

$$+ 2\|h(t, x)\|_{C^0((0, T) \times \mathcal{O})} \int_{C_\eta < |y| < \infty} \left( \frac{\lambda}{\pi} \right)^{n/2} \exp(-\lambda|y|^2) \, dy$$

$$\leq \left( \frac{2\lambda^{\sigma}}{\sqrt{\lambda}} + 4 \text{erfc}(\lambda^{2\sigma}) \right) \|h(t, x)\|_{C^1((0, T) \times \mathcal{O})}.$$

Here we notice that the normalization factor of $(\det B)^{1/2}$ makes the Gaussian kernel normalized to 1.

We recall the following elementary lemmas.

**Lemma A.2.** Suppose $\Omega \subset \mathbb{R}^n$ is open, $f : \Omega \to \mathbb{R}$ is $C^\infty$, $p \in \Omega$, and $\nabla f(p) \neq 0$. Then there are neighborhoods $U$ and $V$ of 0 and $p$, respectively, and a $C^\infty$ diffeomorphism $G : U \to V$ with $G(0) = p$ and

$$f \circ G(x) = f(p) + x_n. \quad \text{(A-2)}$$

**Lemma A.3.** Let $\phi$ be a real-valued $C^\infty$ function and let $v$ be a $C^\infty_0$ function. Define

$$I(\lambda) = \int \exp(-\pi i \lambda \phi(x)) a(x) v(x) \, dx. \quad \text{(A-3)}$$

Here $\lambda > 0$ is a large scalar and $a(x) \in W^{N,1}(\Omega)$. Suppose $\Omega \subset \mathbb{R}^n$ is open, $\phi : \Omega \to \mathbb{R}$ is $C^\infty$, $p \in \Omega$, and $\nabla \phi(p) \neq 0$. Suppose that $v \in C^\infty_0$ has its support in a sufficiently small neighborhood of $p$. Then

$$\forall N, \exists C_N : |I(\lambda)| \leq C_N \lambda^{-N}. \quad \text{(A-4)}$$

and furthermore, $C_N$ depends only on bounds for $N + 1$ derivatives of $\phi$, the $W^{N,1}(\Omega)$ norm of $a(x)$, and a lower bound for $|\nabla \phi(p)|$. 

Proof. Let $\phi_1 = \phi_2 \circ G$, where $G$ is a smooth diffeomorphism. Then we have
\[
\int \exp(-\pi i \lambda \phi_2(x))a(x)v(x)\,dx = \int \exp(-\pi i \lambda \phi_1(G^{-1}x))a(x)v(x)\,dx \\
= \int \exp(-\pi i \lambda \phi_1(y))a(Gy)v(Gy)\,d(Gy) \\
= \int \exp(-\pi i \lambda \phi_1(y))a(Gy)v(Gy)|J_G(y)|\,dy,
\]
where $J_G$ is the Jacobian determinant. The straightening lemma and the calculation reduce this to the case where $\phi(x) = x_n + c$. In this case, letting $e_n = (0, \ldots, 0, 1)$ we have
\[
I(\lambda) = \exp(-i\pi \lambda c)\hat{a}v(\frac{\lambda}{2}e_n),
\]
and this has the requisite decay as
\[
\hat{D}^{\alpha}f(\xi) = |\xi|^\alpha \hat{f}(\xi).
\]

We recall the following set of 1-dimensional Gaussian integrals:
\[
\int_0^\infty x^{2n} \exp(-ax^2)\,dx = \sqrt{\frac{\pi}{a}} \frac{(2n-1)!!}{a^n 2^{n+1}},
\]
\[
\int_0^\infty x^{2n+1} \exp(-ax^2)\,dx = \frac{n!}{2a^{n+1}},
\]
\[
\int_{-\infty}^\infty x^{2n} \exp(-ax^2)\,dx = \sqrt{\frac{\pi}{a}} \frac{(2n-1)!!}{(2a)^n}.
\]

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