A HEIGHT INEQUALITY FOR RATIONAL POINTS ON ELLIPTIC CURVES IMPLIED BY THE ABC-CONJECTURE

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ABSTRACT. In this short note we show that the uniform abc-conjecture puts strong restrictions on the coordinates of rational points on elliptic curves. For the proof we use a variant of Vojta’s height inequality formulated by Mochizuki. As an application, we generalize a result of Silverman on elliptic non-Wieferich primes.

1. Introduction

If \( E/\mathbb{Q} \) is an elliptic curve in Weierstrass form with point at infinity \( O \) and \( P \in E(\mathbb{Q}) \setminus \{O\} \), then it is well known that we can write

\[
P = \left( \frac{a_P}{d_P^2}, \frac{b_P}{d_P^3} \right),
\]

where \( a_P, b_P, d_P \in \mathbb{Z} \) satisfy \( \gcd(d_P, a_Pb_P) = 1 \) and \( d_P > 0 \).

The structure of the denominators \( d_P \) has been studied, for instance, by Everest-Reynolds-Stevens [ERS07] and Stange [Sta11], and has recently received increasing attention in the context of elliptic divisibility sequences first studied by Ward [War48]. See for instance [EEW01] or [Rey12] and the references therein. In this paper we derive strong conditions on the denominators \( d_P \) from the uniform abc-conjecture over number fields (see Conjecture 2.2 or [GS00]).

If \( n \) is a positive integer, we let \( \text{rad}(n) \) denote the product of distinct prime divisors of \( n \). We call \( n \) powerful if \( \text{ord}_p(n) \neq 1 \) for all prime numbers \( p \). The powerful part of \( n \) is defined to be the largest powerful integer dividing \( n \).

**Theorem 1.1.** Let \( E/\mathbb{Q} \) be an elliptic curve in Weierstrass form and suppose that the uniform abc-conjecture holds. Then, for all \( \varepsilon > 0 \), there exist constants \( c \) and \( c' \), only depending on \( E \) and \( \varepsilon \), such that for all \( P \in E(\mathbb{Q}) \setminus \{O\} \) the following hold:

(i) We have

\[
\max \left\{ \frac{1}{2} \log |a_P|, \log d_P \right\} \leq (1 + \varepsilon) \log \text{rad}(d_P) + c.
\]
(ii) Let $v_P$ be the powerful part of $d_P$ and write $d_P = u_P v_P$; then
\[ \log v_P \leq \varepsilon \log |u_P| + c'. \]

**Remark 1.2.** A strong form of Siegel’s Theorem implies a weaker upper bound (and an analogous lower bound) on $\log |a_P|$: There is a constant $c = c(E, \varepsilon)$ such that
\[ (1 - \varepsilon) \log d_P - c \leq \frac{1}{2} \log |a_P| \leq (1 + \varepsilon) \log d_P + c, \]
see [Sil86, Example IX.3.3].

**Remark 1.3.** Mochizuki [Moc12] has recently announced a proof of the uniform abc-conjecture over number fields.

If, in the notation of Theorem 1.1, $d_P$ is powerful, then $|u_P| = 1$. Hence the following result is an immediate consequence of Theorem 1.1 (ii):

**Corollary 1.4.** Let $E/\mathbb{Q}$ be an elliptic curve in Weierstrass form and suppose that the uniform abc-conjecture holds. Then the set of all $P \in E(\mathbb{Q}) \setminus \{O\}$ such that $d_P$ is powerful is finite.

**Remark 1.5.** In particular, Corollary 1.4 implies that only finitely many $P \in E(\mathbb{Q}) \setminus \{O\}$ have prime power denominator if the uniform abc-conjecture holds. The question of prime power denominators was studied, for instance, in [ERS07]; there it is shown (ERS07 Theorem 1.1) that for a fixed exponent $n > 1$, there are only finitely many $P \in E(\mathbb{Q}) \setminus \{O\}$ such that $d_P$ is an $n$th power. Moreover, it is claimed ([ERS07, Remark 1.2]) that the uniform abc-conjecture over number fields implies that for $n \gg 0$, there are no $P \in E(\mathbb{Q}) \setminus \{O\}$ such that $d_P$ is an $n$th power. Together, these results would also imply that the finiteness of the set of $P \in E(\mathbb{Q}) \setminus \{O\}$ such that $d_P$ is a perfect power is a consequence of the uniform abc-conjecture. However, no proof of [ERS07, Remark 1.2] has been published so far.

Another application of Theorem 1.1 concerns **elliptic non-Wieferich primes**. For a prime $p$ of good reduction for an elliptic curve $E/\mathbb{Q}$, we define $N_p := \#E(\mathbb{F}_p)$. If $P \in E(\mathbb{Q})$ is non-torsion, let
\[ W_{E,P} := \{ p \text{ good prime for } E : N_p P \not\equiv O \mod p^2 \} \]
be the set of elliptic non-Wieferich primes to base $P$.

**Corollary 1.6.** Let $E/\mathbb{Q}$ be an elliptic curve in Weierstrass form and suppose that the uniform abc-conjecture holds. If $P \in E(\mathbb{Q})$ is non-torsion, then
\[ |\{ p \in W_{E,P} : p \leq X \}| \geq \sqrt{\log(X)} + O_{E,P}(1) \quad \text{as} \quad X \to \infty. \]

**Remark 1.7.** Assuming the abc-conjecture over $\mathbb{Q}$, Silverman has already proved that (1) holds for all non-torsion $P \in E(\mathbb{Q})$ if $j(E) \in \{0, 1728\}$, cf. [Sil88, Theorem 2].
Proof: The only place in Silverman’s proof of [1] where the abc-conjecture and the assumption \( j(E) \in \{0, 1728\} \) are invoked is in the proof of [Sil88, Lemma 13]. In order to deduce the statement of [Sil88, Lemma 13] for arbitrary \( E \), it suffices to show that for all \( \varepsilon > 0 \) there exists a constant \( c = c(E, \varepsilon) \) such that
\[
\log v_{n\cdot P} \leq \varepsilon \log(d_{n\cdot P}) + c
\]
for all \( n \geq 1 \), where \( v_{n\cdot P} \) is the powerful part of \( d_{n\cdot P} \). But this follows at once from part (ii) of Theorem 1.1. □

Corollary 1.6 is the analogue of [Sil88, Theorem 1], giving an asymptotic lower bound (dependent on the abc-conjecture over \( \mathbb{Q} \)) for the number of classical non-Wieferich primes up to a given bound. See [Vol00] for further results concerning elliptic non-Wieferich primes.

In Section 2 we recall work of Mochizuki from [Moc10], which we use in Section 3 for the proof of Theorem 1.1.

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2. The uniform abc-conjecture and Vojta’s height inequality

In this section, we discuss the uniform abc-conjecture and a variant of Vojta’s height conjecture.

Let \( K \) be a number field with ring of integers \( \mathcal{O}_K \), let \( X \) be a smooth, proper, geometrically connected curve over \( K \) and let \( D \) be an effective divisor on \( X \). Extend \( X \) to a proper regular model \( \mathcal{X} \) over \( \text{Spec}(\mathcal{O}_K) \) and let \( D \) be an effective horizontal divisor \( D \in \text{Div}(\mathcal{X}) \).

Suppose that \( P \in X(F) \), where \( F \) is some finite extension of \( K \). We can define the conductor \( \text{cond}_{X,D}(P) \) of \( P \) as follows: Let \( \pi : \mathcal{X}' \to \mathcal{X} \times \text{Spec}(\mathcal{O}_F) \) be the minimal desingularization and let \( \mathcal{P} \in \text{Div}(\mathcal{X}') \) be the Zariski closure of \( P \) in \( \mathcal{X}' \). Then we define
\[
\text{cond}_{X,D}(P) := \prod_{p \in S}Nm(p)^{\frac{1}{Nm(p)} - 1} \in \mathbb{R},
\]
where \( S \) is the set of finite primes \( p \) of \( F \) such that the intersection multiplicity \( (\mathcal{P} \cdot \pi^*D)_p \neq 0 \).

Remark 2.1. For different constructions of the (logarithmic) conductor, see [Moc10, §1] or [BG06, §14.4]. It is easy to see that, up to a bounded function, these constructions are all
equivalent. By [Moc10, Remark 1.5.1] changing the model $X$ only changes $\log \text{cond}_{X,D}$ by a bounded function. Hence, up to a bounded function, $\text{cond}_{X,D}$ only depends on $D$.

If $P \in X(K)$, then we write $k(P)$ for the minimal field of definition of $P$. Mochizuki [Moc10, §2] has rewritten the uniform abc-conjecture over number fields ([GS00]) as follows:

**Conjecture 2.2.** (Uniform abc-conjecture) Let $D = (0) + (1) + (\infty) \in \text{Div}(\mathbb{P}^1)$ and let $h$ denote a Weil height on $\mathbb{P}^1$ with respect to the divisor $(\infty)$. Extend $D$ to an effective horizontal divisor $D$ on $X = \mathbb{P}^1_\mathbb{Q}$.

If $\varepsilon > 0$ and $d \in \mathbb{N}$, then there exists a constant $c = c(\varepsilon, d)$ such that

$$h(P) \leq (1 + \varepsilon) \left( \log \text{disc}(k(P)) + \log \text{cond}_{X,D}(P) \right) + c$$

for all $P \in X(K)$ satisfying $[k(P) : \mathbb{Q}] \leq d$.

**Remark 2.3.** The abc-conjecture over $\mathbb{Q}$ (see for instance [BG06, Conjecture 12.2.2]) is a special case of Conjecture 2.2. Indeed, let $a$ and $b$ be coprime positive integers, let $c = a + b$ and consider the point $P = [a : c] \in \mathbb{P}^1$. Then, up to a bounded function, we have $h(P) = \log \max\{|a|, |c|\} = \log c$. Moreover, $\text{disc}(k(P)) = 1$ and

$$\text{cond}_{X,D}(P) = \prod_{p \in S} p = \text{rad}(abc),$$

where $S$ is the set of prime numbers $p$ such that $\text{ord}_p(a) > 0$, $\text{ord}_p(b) > 0$ or $\text{ord}_p(c) > 0$.

The following version of Vojta’s conjectured height inequality was stated by Mochizuki [Moc10, §2].

**Conjecture 2.4.** (Vojta’s height inequality) Let $X$ be a smooth, proper, geometrically connected curve over a number field $K$. Let $D \subset X$ be an effective reduced divisor, and $\omega_X$ the canonical sheaf on $X$. Fix a proper regular model $X$ of $X$ over $\text{Spec}(\mathcal{O}_K)$ and extend $D$ to an effective horizontal divisor $D$ on $X$. Suppose that $\omega_X(D)$ is ample and let $h_{\omega_X(D)}$ be a Weil height function on $X$ with respect to $\omega_X(D)$.

If $\varepsilon > 0$ and $d \in \mathbb{N}$, then there exists a constant $c = c(\varepsilon, d, X, D)$ such that

$$h_{\omega_X(D)}(P) \leq (1 + \varepsilon) \left( \log \text{disc}(k(P)) + \log \text{cond}_{X,D}(P) \right) + c$$

for all $P \in X(K) \setminus \text{supp}(D)$ satisfying $[k(P) : \mathbb{Q}] \leq d$.

Obviously Conjecture 2.4 contains Conjecture 2.2 as a special case. In fact, the converse also holds:

**Theorem 2.5.** Conjecture 2.2 and Conjecture 2.4 are equivalent.

**Proof:** See [Moc10, Theorem 2.1], [BG06, Theorem 14.4.16] or [VF02, Theorem 5.1]. □
3. Proof of Theorem 1.1

Proof: We specialize Conjecture 2.4 to the case $K = \mathbb{Q}$, $X = E$, $d = 1$ and $D = (O)$. Let $P \in E(\mathbb{Q}) \setminus \{O\}$; then we have $\omega_E(D) \cong O_E(D)$ and hence

$$h_{\omega_E(D)}(P) = \max \left\{ \frac{1}{2} \log |a_P|, \log d_P \right\} + \mathcal{O}(1),$$

since the function $P \mapsto \max \left\{ \frac{1}{2} \log |a_P|, \log d_P \right\}$ is a Weil height on $E$ with respect to $O_E(D)$.

In order to compute the logarithmic conductor of $P$ we consider the minimal desingularization $X$ of the normal model over Spec($\mathbb{Z}$) determined by the given Weierstrass equation of $E$ and extend $D$ to $D \in \text{Div}(X)$ by taking the Zariski closure. Then a prime number $p$ of good reduction satisfies $(P \cdot D)_p \neq 0$ if and only if $p \mid d_P$; therefore we have

$$|\log \text{cond}_X(D)(P) - \log \text{rad}(d_P)| \leq \sum_{p \text{ bad}} \log p.$$

Hence the functions $P \mapsto \log \text{cond}_X(D)(P)$ and $P \mapsto \log \text{rad}(d_P)$ coincide up to a bounded function and Conjecture 2.4 implies

$$\max \left\{ \frac{1}{2} \log |a_P|, \log d_P \right\} \leq (1 + \varepsilon) \log \text{rad}(d_P) + c.$$

By Theorem 2.5 this finishes the proof of (i).

To prove part (ii), let $\varepsilon > 0$, let $c = c(E, \varepsilon)$ be the corresponding constant from part (i) of the theorem and fix some $\varepsilon' > 0$ such that $\frac{2\varepsilon'}{1 - \varepsilon'} < \varepsilon$.

Let $P \in E(\mathbb{Q}) \setminus \{O\}$. Then (i) implies

$$\log |u_P| + \log v_P \leq (1 + \varepsilon') (\log \text{rad}(u_P) + \log \text{rad}(v_P)) + c \leq (1 + \varepsilon') \left( \log |u_P| + \frac{1}{2} \log v_P \right) + c$$

and hence we conclude

$$\log v_P \leq \frac{2\varepsilon'}{1 - \varepsilon'} \log |u_P| + \frac{2c}{1 - \varepsilon'},$$

which proves (ii). $\square$

References


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